Expansion nets: proof nets for propositional classical logic

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Abstract. We give a calculus of proof-nets for classical propositional logic. These nets improve on a proposal due to Robinson by validating the associativity and commutativity of contraction, and provide canonical representants for classical sequent proofs modulo natural equivalences. We present the relationship between sequent proofs and proof-nets as an annotated sequent calculus, deriving formulae decorated with *expansion/deletion trees*. We then see a subcalculus, *expansion nets*, which in addition to these good properties has a polynomial-time correctness criterion.

1 Introduction

In stark contrast to the well-developed theory of proof-identity for intuitionistic natural deduction (given by interpretation of proofs in a cartesian-closed category), the theory of identity for proofs in classical logic is very poorly understood. Investigations by several researchers over the last ten years [16, 7, 12, 13, 2, 11] have only served to underline the difficulty of the problem. Many of these problems concern proofs with cuts, since the problem of proof-identity must account for the nonconfluence of cut-elimination. Yet even for cut-free proofs, opinions on the "right notion" of proof-identity differ. A reasonable minimal requirement is that proofs differing by *commuting conversions* of noninterfering sequent rules should be equal. Proof-nets [9] provide a tool for providing canonical representants of such equivalence classes of proofs in Linear Logic. A proposal by Robinson [16] gives proof-nets for propositional classical logic, but fails to provide canonical representants for sequent proofs because it contains explicit weakening attachments. The move from sequent proofs to Robinson's nets also fails to validate, among other desirable proof-identities, commutativity/associativity of contraction, a key assumption in the development of abstract models of proofs. In Führmann and Pym's work [7], a categorical model of proofs based on Robinson's nets is built by taking a quotient by equations, ensuring that the structure interpreting the structural rule in the resulting category forms a commutative monoid, and that those monoids are constructed pointwise.

In this paper we take Robinson's nets as a starting point for developing a more abstract notion of proof-net for classical logic; concrete representatives of the equivalence classes used in [7]. We then go on to identify a subcalculus of these nets which has a polynomial-time correctness criterion, and therefore forms a propositional proof system [3].

$$\frac{}{p,\ \bar{p}}\,Ax \qquad\qquad \frac{}{\top}\,Ax_{\top}$$

$$\frac{\Gamma, A,\ B}{\Gamma, A \vee B}\,\vee \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \wedge B}\,\wedge$$

$$\frac{\Gamma, A,\ A}{\Gamma, A}\,C \qquad\qquad \frac{\Gamma}{\Gamma, B}\,W$$

Fig. 1. Cut-free multiplicative **LK** (one-sided)

2 Preliminaries

We assume familiarity with proof-nets for unit free multiplicative linear logic $\mathbf{MLL}^-$ with Mix. In particular, we assume knowledge of the *switching graph* condition for multiplicative proof structures, and how it leads to a proof of sequentialization for $\mathbf{MLL}^-$ + Mix proof nets [5, 6]. We will also assume, without proof, the existence of a *polynomial time* correctness criterion equivalent to the switching criterion; such a criterion is given by attempting to sequentialize by searching for *splitting pars*, a technique first described in [6], and available in English translation in the Linear Logic Primer [4].

3 Proof nets with contraction and weakening.

Robinson's proof-nets for classical logic [16] are based very closely on Girard's proof-nets for **MLL** with units [9]. The basic idea comes from [8]: correctness is given by treating the conjunctions and axioms of classical logic in the same way as the linear logic axiom and tensor, and treating both contraction and disjunction in the same way as the linear logic "par" connective. However, unlike Girard's nets, Robinson's nets are presented in a two-sided form, with multiple premises and multiple conclusions, deriving formulae with an explicit negation connective. We will consider a small variant of this calculus: one-sided nets, over formulae of classical logic in *negation normal form*; we assume a set $\mathcal{A}$ of atomic formulae $p, q, \ldots$ equipped with an involutive function $(\bar{\ })$, such that $p \neq \bar{p}$. Propositional formulae are built from these atomic formulae and the units $\top$ and $\bot$ using the binary connectives $\wedge$ and $\vee$. Negation of general formulae is defined using the De Morgan laws. A cut-free sequent calculus deriving multisets of such formulae, with explicit structural rules and multiplicatively formulated logical rules, is given in Figure 1. Considering one-sided nets allows us to give a more compact presentation of our systems: the one-sidedness is not necessary for the approach, however, and the results of the subsequent sections carry over easily to a two-sided setting. A more important departure from Robinson's setting is the treatment of weakening, as will be explained below.

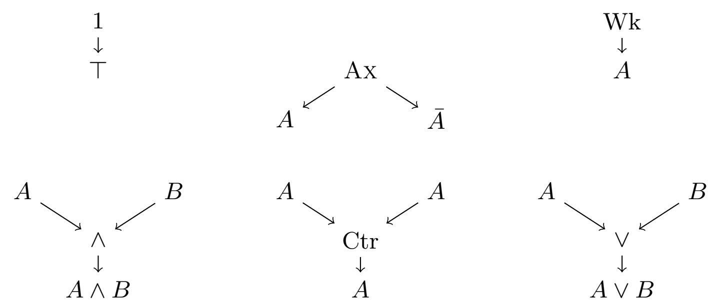


Fig. 2. Proof nets for classical logic with unrestricted contraction and weakening.

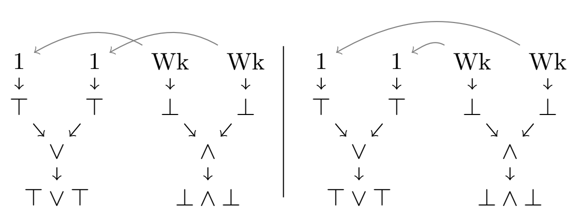
The definition of these nets begins with a notion of *proof structure*: an object which locally has the structure of a proof-net:

Definition 1. *A* Robinson proof-structure *is a directed graph built from the subgraphs in Figure 2 having no incoming edges.*

The edges coming into the $\wedge$ and $\vee$ vertices are *ordered* (alternatively, we can think of them as being labelled "left" and "right"); this is important for distinguishing between the conjunctions $A \wedge B$ and $B \wedge A$. We refer to the vertices of a Robinson structure labelled with formulae of propositional logic as *formula-nodes*. The other vertices are referred to as *rule-nodes*.
In proof-nets, it is typically necessary to anchor each weakening to some other node of the proof. In [16] this anchoring is part of the structure of the weakening node: we instead use the more usual notion of an *attachment*

Definition 2. *An* attachment f *for a Robinson proof-structure* F *is a function mapping each rule node labelled with to some other rule-node of the proof-structure. By an attached proof structure we mean a pair* (F, f) *of a proof structure* F *and an attachment* f *for* F.

Example 1. The grey arrows in the following two proof nets represent two different attachments, assigning a rule node of the structure to each node labelled Wk:



(1)

A proof in LK can be seen as a recipe for building an attached proof structure: each rule of the calculus corresponding to a rule node. This procedure is sometimes referred to as desequentialization, and is described in detail in [16]. We choose the attachment for a weakening from one of the formulae present in the context. This arbitrary choice means that attached proof-nets themselves cannot be the canonical proof objects we seek. For **MLL**, the right notion of canonical proof object is a *quotient* of attached proof-nets by so-called *Trimble rewiring* [17], whereby two proof-nets are equivalent if just one of the attachments of a unit is changed. According to Trimble rewiring, the two attached nets in (1) are not equal; this is important, as the corresponding morphisms are distinguished in some $*$-autonomous categories. We know of no natural model of *classical* proofs (whatever the formulation) where such proofs are distinguished, and so are happy to take *unattached* nets as proof-objects in their own right.

The standard problem in the theory of proof-nets is to give a global *correctness criterion* for identifying, among the proof-structures, those which can be obtained from desequentializing a sequent proof. This then leads to a *sequentialization theorem*, allowing one to reconstruct a sequent proof out of a correct proof-net. We may adapt any of the many equivalent formulations of correctness for $\mathbf{MLL}^-$ nets, taking care to account correctly for the presence of weakenings. For example, the following is the switching graph criterion, suitably altered for our setting:

Definition 3. *Let F be a Robinson proof-structure.*

(a) A rule-node of F is switched *if it is a Ctr or $\vee$ node. A* switching *of a Robinson proof-structure is a choice, for each switched node, of one of its successors.*

(b) Given an attachment f for F, and a switching σ for F, the switching graph $\sigma(F, f)$ *is the graph obtained by deleting from F all edges from a switched node to its successor not chosen by σ, forgetting directedness of edges, and adding an edge from each Wk node to its image under f.*

(c) (F, f) *is* ACC-correct *if, for each switching σ, $\sigma(F, f)$ is acyclic and connected.*

(d) F is a Robinson net *if there is an attachment f such that (F, f) is ACC-correct.*

Thus correctness for (unattached) Robinson nets is in **NP**, since it is necessary to guess an attachment before deciding the switching criterion.

Theorem 1 (Robinson).

(a) Every proof-structure arising from a sequent proof in the system in Figure 1 plus Mix *is a Robinson-net.*

(b) Every Robinson-net can be obtained by desequentializing a sequent proof.

Using the techniques developed in [6, 4], we can capture a larger class of sequent proofs: those using the Mix rule:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\ \textsc{Mix}$$

Definition 4. *Let F be a Robinson proof-structure, and f an attachment for F*

(a) (F, f) *is* AC-correct *if, for each switching σ, $\sigma(F, f)$ is acyclic.*

(b) F *is a* MIXnet *if there is an attachment f such that (F, f) is AC-correct.*

Theorem 2. *(a) Every proof-structure arising from a sequent proof in the system in Figure 1 plus* MIX *is a* MIX*-net.*

(b) Every MIX*-net can be obtained by desequentializing a sequent proof with* MIX*.*

The mix rule will be important later; in its presence, we can give a complete class of proof nets with polynomial-time correctness.

4 Expansion/deletion nets

As a way of canonically representing cut-free proofs, unattached Robinson-style proof-nets are a substantial improvement over **LK** proofs. Two sequent derivations differing by a simple permutation of rule occurrences desequentialize to the same proof-net. However proof identity in classical logic is more complicated than for, for example, $\mathbf{MLL}^{-}$; simple rule permutations are not the only source of non-canonicity in proofs. In the following section we consider sources of non-canonicity arising from the contraction rule, which Robinson's nets suffer from as acutely as the sequent calculus. We will then give a new formulation of proof nets (expansion/deletion nets) which do not exhibit these problems.

4.1 Problems with contraction

Contraction is not associative Given three copies of the conclusion A, there are two ways we can contract them, which should be equivalent.

$$\begin{tikzcd}
A \arrow[dr] & & A \arrow[dl] & & & & A \arrow[dr] & & A \arrow[dl] \\
& \mathrm{Ctr} \arrow[d] & & & \equiv & & & \mathrm{Ctr} \arrow[d] & \\
& A \arrow[dr] & & A \arrow[dl] & & A \arrow[dr] & & A \arrow[dl] & \\
& & \mathrm{Ctr} \arrow[d] & & & & \mathrm{Ctr} \arrow[d] & & \\
& & A & & & & A & &
\end{tikzcd}$$

Girard suggests an obvious fix in [8]: n-ary contraction nodes. Of course, binary contractions are a subcase of n-ary contractions; in addition, we should require that the conclusion of the link is not in turn the premise of another contraction link. We will call contractions of this special kind *expansions*.

Weakening is not a unit for contraction Given a proof-net deriving a formula A, we can weaken to form another copy of A and then immediately apply contraction, to again obtain a proof of A. We would prefer that weakening be a unit to contraction; that these two proofs of A be identified.

Contraction on disjunctions is not pointwise The following two figures contain the same essential information, and two proofs differing by them are essentially the same:

$$
\begin{tikzcd}
A \arrow[dr] & & B \arrow[dl] & & A \arrow[dr] & & B \arrow[dl] \\
& \vee \arrow[drr] & & & & \vee \arrow[dll] & \\
& & & \mathrm{Ctr} \arrow[d] & & & \\
& & & A \vee B & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr] & & B \arrow[drrr] & & A \arrow[dlll] & & B \arrow[dl] \\
& \mathrm{Ctr} \arrow[drr] & & & & \mathrm{Ctr} \arrow[dll] & \\
& & & \vee \arrow[d] & & & \\
& & & A \vee B & & &
\end{tikzcd}
$$

We can ensure that only one of these figures may appear in our nets by forbidding the contraction node to act on disjunctions, this is natural, since the sequent rule introducing disjunction is *invertible*.

4.2 Expansion/deletion trees

In our view a proof net is best seen as a forest together with a relation on the nodes of the forest (representing the axiom links of a proof net as usually presented). For $\mathbf{MLL}^-$ (with or without mix) the forest is built from formula trees, but for classical logic the trees must contain additional structure, to account for contraction and weakening. Our proof nets will be built from typed *expansion/deletion trees* or ed-trees; these can be seen as formula trees where, at a node typed $p, \bar{p}$ or $A \wedge B$, we can *expand* (corresponding to a single n-ary contraction) or *delete* (corresponding to weakening).

Definition 5 (Expansion/deletion trees). *Let* $\mathcal{X} = \mathsf{x}, \mathsf{y}, \ldots$ *be a countable set – the* axiom variables. *An expansion/deletion tree (or ed-tree) over* $\mathcal{X}$ *is of the form* t *below:*

$$t ::= 1 \mid * \mid (w + \cdots + w) \mid (t \vee t) \qquad w ::= \mathsf{x} \mid \bar{\mathsf{x}} \mid t \otimes t$$

where $(w + \cdots + w)$ *denotes a nonempty finite formal sum, and* $*$ *denotes the empty formal sum. We call the empty sum a* deletion*, and a nonempty sum an* expansion*. We call the members of the grammar* w *"witnesses".*

The advantage of using formal sums of witnesses to keep track of contractions is that formal sums are associative and commutative: that $*$ is the unit for the formal sum means that weakening will be the unit for contraction.
Types for ed-trees and witnesses are as follows:

Definition 6. *A* type *is either*
(a) A formula of classical propositional logic;
(b) A witness type *of one of the two following forms:*
- *A* positive witness type*, written* $[p]$*, where* p *is a positive atom;*
- *A* negative witness type*, written* $[\bar{p}]$*, where* $\bar{p}$ *is a negative atom; or*

$$\frac{}{\bar{\mathsf{x}} : [\bar{p}]} \quad \frac{}{1 : \top} \qquad \frac{}{* : A} \quad \frac{}{\mathsf{x} : [p]}$$

$$\frac{t : A \quad s : B}{t \vee s : A \vee B} \quad \frac{t : A \quad s : B}{t \otimes s : A \otimes B}$$

$$\frac{w_1 : [p] \ \cdots \ w_n : [p]}{(w_1 + \cdots + w_n) : p} \quad \frac{w_1 : [\bar{p}] \ \cdots \ w_n : [\bar{p}]}{(w_1 + \cdots + w_n) : \bar{p}} \quad \frac{w_1 : A \otimes B \ \cdots \ w_n : A \otimes B}{(w_1 + \cdots + w_n) : A \wedge B}$$

Fig. 3. Typing derivations for terms

- *A conjunctive witness type, written $A \otimes B$, where A and B are formulae of propositional classical logic.*

Definition 7. *A typed term is a pair $t : A$ of a term t and a type A, derivable in the typing system given in Figure 3.*

The typing rules in Figure 3 ensure that the conclusion of an expansion is never the premise of another expansion.
Having found the right notion of tree, a proof-structure is just a forest of those trees. We will refer to these forests as *annotated sequents*, since we will later give sequent calculi deriving them.

Definition 8. *An* annotated sequent *is a forest F of typed ed-trees in which*

(a) there is at most one occurrence of each axiom variable x, and
(b) there is an occurrence of $\bar{\mathsf{x}}$ in F if and only if there is an occurrence of x.

The type of an annotated sequent F is the ordinary sequent comprising the multiset of types of the the ed-trees making up F.

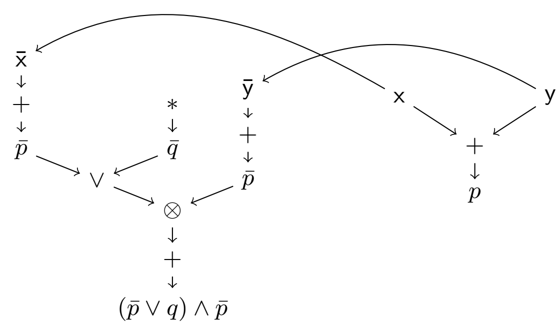
To see an annotated sequent as a proof structure in the more usual sense we can consider its graph, in which we add axiom links to the forest:

Definition 9. *The* graph *of an annotated sequent F is a directed graph with vertices given by instances of subtrees of F; we call these the* nodes *of F. The edges of the graph are given by the forest structure (with edges directed toward the root), plus an edge from x to $\bar{\mathsf{x}}$ for each variable x appearing in F. The edges above a $\otimes$ or $\vee$ node are ordered; edges above expansion nodes are unordered.*

Example 2. The following annotated sequent represents a proof of Pierce's law

$$(((\bar{\mathsf{x}}) \vee *) \otimes (\bar{\mathsf{y}})) : (\bar{p} \vee q) \wedge \bar{p}, \quad (\mathsf{x} + \mathsf{y}) : p \tag{2}$$

The graph of this annotated sequent is



(3)

Correctness for annotated sequents is analogous to that for Robinson structures:

Definition 10. *Let F be an annotated sequent. An attachment for F is a function assigning, to each deletion $*$ of F, some other subterm of F. An attached annotated sequent is a pair (F, f) of an annotated sequent F and an attachment f for F.*

Definition 11. *Let (F, f) be an attached annotated sequent.*
(a) A switching σ for (F, f) is a choice of successor for each expansion node and each $\vee$ node.
(b) The switching graph $\sigma(F, f)$ is obtained from the graph of F by
  1: deleting all incoming edges to each expansion and $\vee$ node, other than those coming from the nodes chosen by the switching,
  2: forgetting the directedness of edges,
  3: adding an edge between each deletion and its image under the attachment f.
(c) (F, f) is an ed-net if, for every switching σ of F, $\sigma(F, f)$ is acyclic.

4.3 Expansion nets

We consider now a subclass of ed-nets, *expansion nets*, which are interesting because they have a *default attachment*; this allows a polynomial time correctness criterion.

Definition 12. *(a) An expansion/deletion tree t is an* expansion tree *if every deletion $*$ of t occurs as the left or right disjunct of a disjunctive subterm.*
(b) An ed-net F is an expansion net *if every term appearing in F is an expansion tree.*

Definition 13. *Let F be an forest of typed expansion trees. The* default attachment *of F is a function assigning to each subterm of the form $*$, the subterm t with which it forms $t \vee *$ or $* \vee t$.*

This default attachment can then be used to check correctness:

$$\frac{}{\vdash 1 : \top}\,\mathrm{Ax}_\top \qquad \frac{}{(\bar{\mathsf{x}}) : \bar{p},\ (\mathsf{x}) : p}\,\mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\,\vee \qquad \frac{F,\ t : A \qquad G,\ s : B}{F, G,\ (t \otimes s) : A \wedge B}\,\wedge$$

$$\frac{F}{F,\ * : A}\,\mathrm{W} \qquad \frac{F \quad G}{F,\ G}\,\textsc{Mix}$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\,\mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\,\mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\,\mathrm{C}_{\bar{p}}$$

Fig. 4. $\mathbf{LK}_{ed}$

Proposition 1. *An forest of typed expansion trees F is an expansion-net if and only if, for every switching σ, $\sigma(F, f)$ is acyclic, where f is the default attachment.*

Since the acyclicity of the switching graphs can be decided polynomially, correctness for expansion-nets is polynomial time.

Remark 1. Let $\mathbf{MLL}^*$ be the subset of binary **MLL** formulae (in which each atom occurs at most once) having no subformula of the form $(\bot \otimes A)$ or $(A \otimes \bot)$. A similar argument to the one above shows that provability for this fragment of **MLL** is polynomial time: the formula itself defines a proof-net with a default attachment for each $\bot$. By replacing switched nodes with par and unswitched nodes with tensor, an expansion net gives rise to a binary **MLL** formula; this formula belongs to $\mathbf{MLL}^*$, and so its provability can be checked in polynomial time.

What remains to see is that we have not lost any theorems of propositional logic by restricting contraction and weakening: that the system of expansion nets is *complete.* To see this, we consider the relationship between sequent proofs and expansion nets; specifically, we give an *annotated* sequent calculus deriving annotated sequents. We first give a sequent calculus such that every annotated sequent derivable in this system is an ed-net, and then give a system deriving expansion nets.

5 Decorating sequent derivations with terms

In Figure 4 we give a sequent-style calculus for deriving annotated sequents. One should think of this calculus in the same way as a lambda-term-annotated sequent system for intuitionisitic logic; the annotated sequents themselves are proof objects, with the sequent proof giving their inductive buildup. The annotated system plays, for $\mathbf{LK}_{ed}$, the role of desequentialization.

$$\frac{F,\ t:A,\ s:B}{F,\ \ t\vee s:A\vee B}\vee \qquad \frac{F,\ t:A}{F,\ t\vee *:A\vee B}\vee_L \qquad \frac{F,\ s:B}{F,\ *\vee s:A\vee B}\vee_R$$

Fig. 5. The three disjunction rules of $\mathbf{LK}_e$

Example 3. The following annotated sequent proof illustrates how contractions at the level of proofs are interpreted by expansions at the level of the assignment.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\bar{\mathsf{x}}):\bar{a},\ (\mathsf{x}):a}\text{Ax} \qquad \dfrac{}{(\bar{\mathsf{y}}):\bar{a},\ (\mathsf{y}):a}\text{Ax}}{(\bar{\mathsf{x}}):\bar{a},\ (\bar{\mathsf{y}}):\bar{a}, (\mathsf{x}\otimes\mathsf{y}):a\wedge a}\wedge \qquad \dfrac{}{(\bar{\mathsf{z}}):\bar{a},\ (\mathsf{z}):a}\text{Ax}}{(\bar{\mathsf{x}}):\bar{a},\ (\bar{\mathsf{y}}):\bar{a},\ \bar{\mathsf{z}}:\bar{a},\ ((\mathsf{x}\otimes\mathsf{y})\otimes\mathsf{z}):(a\wedge a)\wedge a}\wedge}{\dfrac{(\bar{\mathsf{x}}):\bar{a},\ (\bar{\mathsf{y}}+\bar{\mathsf{z}}):\bar{a},\ ((\mathsf{x}\otimes\mathsf{y})\otimes\mathsf{z}):(a\wedge a)\wedge a}{(\bar{\mathsf{x}}+\bar{\mathsf{y}}+\bar{\mathsf{z}}):\bar{a},\ ((\mathsf{x}\otimes\mathsf{y})\otimes\mathsf{z}):(a\wedge a)\wedge a}\text{C}}\text{C} \tag{4}$$

The particular way in which the contractions are carried out does not affect the annotated endsequent: any commutation or association of the contractions gives rise to the same term assignment.

Applying the standard sequentialization techniches to ed-nets, we obtain the following statement of the surjectivity of desequentialization:

Proposition 2. *An annotated sequent F is an ed-net if and only if it can be derived in* $\boldsymbol{LK}_{ed}$.

Given a proof in $\mathbf{LK}_{ed}$, we can recover an ordinary sequent proof by forgetting the annotations: this yields a proof in $\mathbf{LK}$. This forgetful projection of $\mathbf{LK}_{ed}$ is a *subcalculus* of $\mathbf{LK}$, since it only has contractions for conjunctions and atoms. If we can show all the missing rules admissible in $\mathbf{LK}_{ed}$, then we have shown that $\mathbf{LK}_{ed}$ (and, by extension, expansion/deletion nets) are complete. In fact, we will show an even more restricted calculus, $\mathbf{LK}_e$, complete: this calculus derives expansion nets. Completeness of $\mathbf{LK}_{ed}$ will then follow as a corollary.

5.1 A calculus deriving expansion nets

Let $\mathbf{LK}_e$ be derived from $\mathbf{LK}_{ed}$ as follows; $\mathbf{LK}_e$ consists of all the rules of $\mathbf{LK}_{ed}$ except W, and has in addition the two rule $\vee_L$ and $\vee_R$ shown in Figure 5. The term $*$ is only introduced by these disjunction rules, and so the conclusion of an $\mathbf{LK}_e$ derivation consists of expansion trees. We show now that $\mathbf{LK}_e$ and $\mathbf{LK}_{ed}$ are equivalent with respect to provability – that is, they prove the same theorems. The easier direction is the following:

Proposition 3. *If* $\boldsymbol{LK}_e \vdash t : A$, *then* $\boldsymbol{LK}_{ed} \vdash t : A$.

Proof. By induction on the length of proofs. The property clearly holds for the axioms. For every rule in $\mathbf{LK}_e$ other than $\vee_L$ and $\vee_R$, there is a corresponding rule in $\mathbf{LK}_{ed}$, and the proof is easy. We need only show the admissibility of $\vee_L$ and $\vee_R$. But these can be easily simulated by one application of weakening followed by one of the $\mathbf{LK}_{ed}$ $\vee$ rule.

For the opposite direction, we will need the following easy lemma:

Lemma 1. *(a) If $\boldsymbol{LK}_e \vdash F, t \vee s : A \vee B$, and $s, t \neq *$, then $\boldsymbol{LK}_e \vdash t : A, s : B$.*
(b) If $\boldsymbol{LK}_e \vdash F, t \vee * : A \vee B$ or $\boldsymbol{LK}_e \vdash F, * \vee t : B \vee A$, then $\boldsymbol{LK}_e \vdash F, t : A$.

Since $\mathbf{LK}_e$ is not complete for sequents, but only for (annotated) formulae, we cannot directly prove that if $\mathbf{LK}_{ed}$ proves a sequent of type Γ, so does $\mathbf{LK}_e$. Instead, we prove that, if $\mathbf{LK}_{ed}$ proves a sequent of type Γ, there is a term t such that $\mathbf{LK}_e$ proves $t : \bigvee \Gamma$.

Proposition 4. *If F has type $\Gamma = A_1, \ldots, A_n$, and $\boldsymbol{LK}_{ed} \vdash F$, then there are terms $t_i : A_i$ such that, if $t = (((t_1 \vee t_2) \vee \ldots t_{n-1}) \vee t_n)$, then$\boldsymbol{LK}_e \vdash t : (((A_1 \vee A_2) \vee \ldots A_{n-1}) \vee A_n)$.*

Proof. For any proof in $\mathbf{LK}_{ed}$ of an annotated sequent $s_1 : A_1, \ldots s_n : A_n$, we give a sequence $t_1 : A_1, \ldots t_n : A_n$ such that $t = (((t_1 \vee t_2) \vee \ldots t_{n-1}) \vee t_n) : \bigvee \Gamma$ is provable in $\mathbf{LK}_e$, by induction on the height of a proof in $\mathbf{LK}_{ed}$.
For the axioms of $\mathbf{LK}_{ed}$, the claim is clearly true. For the inductive step, we proceed by case analysis on the last rule ρ of the $\mathbf{LK}_{ed}$ proof. We assume that this rule introduces the last formula in the sequent. In each case, we assume that the proposition holds for the premisses of ρ.

- $\rho =$ W By the induction hypothesis, we have terms $t_1 : A_1 \ldots t_{n-1} : A_{n-1}$ with $\mathbf{LK}_e \vdash ((t_1 \vee t_2) \vee \ldots t_{n-1}) : ((A_1 \vee A_2) \vee \cdots \vee A_{n-1})$ apply the rule $\vee_L$ in $\mathbf{LK}_e$, to add a new disjunct of type A_n to the conclusion.
- $\rho = \vee$ In the case of the $\vee$ rule, applied to an annotated sequent $F, s_1 : A, s_2 : B$, consider the subterms $t_{n-1} : A$ and $t_n : B$ of t. If neither t_{n-1} nor t_n are $*$, we can use Lemma 1 and the $\vee$ rule to obtain the required proof. If both t_{n-1} and t_n are $*$, then apply Lemma 1 twice, followed by $\vee_L$, to obtain a proof of the correct term. If $t_n = *$ but $t_{n-1} \neq *$, then apply Lemma 1, once to remove the deletion, again to isolate t_{n-1}, and then followed by $\vee_L$ and $\vee$, gives a proof of the correct term.
- $\rho =$ C Similar to the treatment of disjunction.
- $\rho =$ Mix Suppose that $\mathbf{LK}_{ed} \vdash F$ and $\mathbf{LK}_{ed} \vdash G$, where F has type $A_1, \ldots, A_n$ and G has type $B_1, \ldots B_m$, and that we have corresponding $\mathbf{LK}_e$-provable terms t and s. The result of applying Mix to F and G has a corresponding term of type $(((((A_1 \vee A_2) \ldots, \vee A_n) \vee B_1) \vee \ldots) \vee B_m)$; we leave details to the reader.
- $\rho = \wedge$ Given annotated sequents $F_1, s_1 : A_1$ and $F_2, s_2 : A_2$, with corresponding $\mathbf{LK}_e$-provable terms t and t', let t_{A_1} and t_{A_2} be the disjuncts of t and t' corresponding to s_1 and s_2:
  - If neither t_{A_1} nor t_{A_2} is $*$, then we may apply Lemma 1 twice, followed by $\wedge$ and then $\vee$, to obtain a proof of the correct shape.

- If both $t_{A_1} = *$ and $t_{A_2} = *$ are $*$ then apply Lemma 1 to remove the deletions. By applying Mix and then $\vee_L$, we obtain a provable term of the required shape.
- The final case is where exactly one $t_{A_i} = *$; without loss of generality, let it be t_{A_1}. We treat this much like a cut against weakening in **LK**. We know that $\mathbf{LK}_e \vdash (((t_1 \vee t_2) \vee \dots t_m) \vee *)$. By Lemma 1, $\mathbf{LK}_e \vdash (((t_1 \vee t_2) \vee \dots t_m)$. Now "weaken" the conclusion once for $A \wedge B$ and once for each member of F_2: that is, apply $\vee_L$ once for each of those formulae. The result is an $\mathbf{LK}_e$ provable term $((((t_1 \vee t_2) \vee \dots t_m) \vee *) \vee \dots \vee *)$ of the correct type.

The content of the above result is that, at the theorem level, the rules of conjunction, disjunction, weakening and Mixare admissible in $\mathbf{LK}_e$; the contraction rule is also admissible when restricted to atoms and conjunctions. In the following section we demonstrate the general admissibility of contraction in $\mathbf{LK}_e$, which is enough to see that it is a complete calculus for classical propositional logic.

5.2 Cut-free completeness of $\mathbf{LK}_e$

By cut-free completeness of $\mathbf{LK}_e$, we mean the following:

Theorem 3. *For every formula A of classical propositional logic such that $\vdash A$ in* **LK***, there is an expansion tree t such that $\mathbf{LK}_e \vdash t : A$.*

To show this, we need only show that the contraction rule of **LK** is admissible for theorems of $\mathbf{LK}_e$, in the sense that, if $t : B \vee (A \vee A)$ is provable, then there is a term t' so that $t' : B \vee A$ is provable ; the remaining cases to check are disjunctions and the unit $\top$. The following lemma will be essential:

Lemma 2. *(a) If $\mathbf{LK}_e \vdash F, 1 : \top$, then $\mathbf{LK}_e \vdash F$.*
(b) If $\mathbf{LK}_e \vdash t \vee 1 : A \vee \top$, then either $t = *$ or $\mathbf{LK}_e \vdash t : A$.

Proof. By induction on the length of proofs. For example, in case the last rule proving $F,\ 1 : \top$ is a conjunction

$$\frac{\vdash G_1,\ t : A,\ 1 : \top \qquad G_2,\ s : B}{\vdash G_1,\ G_2,\ (t \otimes s) : A \wedge B,\ 1 : \top} \wedge$$

$G_1,\ t : A$ is is provable, and so we may prove $G_1, G_2,\ (t \otimes s) : A \wedge B$ $\square$

Lemma 3. *If $\mathbf{LK}_e \vdash t \vee (s_1 \vee s_2) : B \vee (A \vee A)$, then there is a term s such that $\mathbf{LK}_e \vdash t \vee s : B \vee A$.*

Proof. If either one or both s_i is $*$, this can be easily shown using Lemma 1. Similarly, if A is a conjunction or atom, we can use Lemma 1 and the relevant contraction rule of $\mathbf{LK}_e$. If A is the unit $\top$, then s_1 and s_2 are equal to 1, and by Lemma 2, $\mathbf{LK}_e \vdash t \vee 1 : B \vee \top$. Finally, suppose that the claim holds for all formulae of size n, and let $A = B_1 \vee B_2$ of size $n+1$. Apply Lemma 1 four times to obtain a proof of

$$t : B,\ t_a : B_1,\ t_b : B_2,\ t_c : B_1,\ t_d : B_2$$

using $\vee$ and the induction hypothesis, we obtain a proof of $(t \vee u) \vee v : (B \vee B_1) \vee B_2$ and by rearranging the order of the disjunctions using Lemma 1, we obtain a proof of $t \vee (u \vee v) : B \vee (B_1 \vee B_2)$ $\square$

Corollary 1. *If A is provable in $\boldsymbol{LK}$, then there is an expansion/deletion tree t such that $\boldsymbol{LK}_e \vdash t : A$.*

6 Conclusions and further work

We have given a calculus of proof-nets which identifies more sequent proofs than Robinson's proposal, while maintaining a connection with the sequent calculus. Other researchers have given abstract notions of proof-net for classical logic; these make the identifications we wish to make but lack a strong connection to the sequent calculus. Lamarche and Strassburger [12] give two notions of proof-net, both of which validate more identities than Robinson. The $\mathbb{B}$-nets are nothing more than binary linkings on a sequent forest: they possess sequentialization into an additive sequent calculus, but checking correctness of such a net is no more efficient than checking the truth-table of the conclusion. The same paper introduces $\mathbb{N}$-nets, which give a better account of proof-identity, but for which no correctness criterion/sequentialization theorem is known. Hughes's combinatorial proofs [10, 11] also make more identifications than Robinson's nets, and have a polynomial-time correctness criterion. However, the mapping from sequent proofs to combinatorial proofs is not surjective; there are correct combinatorial proofs which do not correspond to a sequent-calculus proof. Moreover, Hughes's approach does not deal directly with the units $\top$ and $\bot$. Hughes's system can be seen as a kind of "Herbrand's theorem for propositional logic", reducing provability in unit-free propositional logic (coNP) to provability in the binary fragment of unit free **MLL** + Mix (P-time). Seen in this light, our result extends this connection to the classical units; we reduce provability in propositional logic to provability in a (polytime decidable) fragment of **MLL** + Mix (with units).
In both of the cases above, there is a mismatch between sequent calculus and the proposed proof nets: the nets we present here are, we believe, the first sufficiently abstract nets to maintain a good correspondence to sequent calculus proofs.
We mention now some further work.

Garbage collection Given a subterm of the form $s = * \otimes t$ or $s = t \otimes *$ in an ed-net, we can view the subproof introducing t as *garbage*; garbage collection would be an algorithm taking a net with garbage and returning a garbage free net: i.e. an expansion net. For similar situations in MAL + Mix nets [1] and combinatorial proofs, there is a confluent garbage collection algorithm; unfortunately, attempts to apply those methods to ed-nets yield annotated sequents which fail to satisfy correctness. This opens up two directions for further research: to find a garbage collection procedure which stays within correct proof nets, or to find a good generalization of correctness so that the existing algorithms work.

Cut-elimination We can easily add cuts to ed-nets by adding a new constructor $\bowtie$ for terms, with typing rule

$$\frac{t : A \qquad s : \bar{A}}{t \bowtie s : \textsc{Cut}}$$

It is then possible to define a weakly normalizing cut-elimination procedure based on Gentzen's original procedure; the definition of the reductions requires the notion of *subnet*, which for ed-nets is rather tricky to define. Since cut-elimination depends on the calculation of subnets (either kingdoms or empires) it is not local; this is somewhat alien to the spirit of proof nets, but it is not clear if a cut-reduction theory for such proof-nets can be local and retain a close correspondence to the sequent calculus. One way to improve the cut-reduction theory of the nets is to *asymmetrize* all the cuts, by insisting that, for each dual pair A and $\bar{A}$, contraction is admissible for one of the pair. This is only a challenge for the atoms, where we need contraction on both p and $\bar{p}$ for completeness. Nevertheless, this is possible, by treating the atoms in the same way as universal/existential quantifiers, leading to a calculus in which the contraction/contraction and weakening/weakening critical pairs cannot be formed.

Classical quantifiers The terminology *expansion/deletion* recalls Miller [15], whose *expansion tree proofs* can be seen as a prototype notion of proof-net for classical logic. The paper [14] makes this connection explicit in the case of first-order prenex formulae; the paper introduces a notion of *Herbrand net* using Girard's notion of a quantifier jump, in which provability at the propositional level is treated as trivial — propositional axioms are replaced by arbitrary propositional tautologies. We foresee no major obstacles in combining Herbrand nets with the results of the current paper to capture nets for first- or higher-order classical quantifiers.

Nets for additively formulated classical logic The correctness/sequentialization results for our nets are heavily tied to the multiplicatively formulated sequent calculus. It is, of course, possible to extract an ed-net from a proof in an additively formulated calculus, but there are natural identities in those calculi which are not validated by our nets.Taking the view that the additive classical connectives are essentially different operations (that hapen to coincide at the level of provability), we look for natural notions of proof net for additively formulated classical logic.

References

1. Gianluigi Bellin. Two paradigms of logical computation in affine logic? *Logic for concurrency and synchronisation*, pages 111–144, 2003.
2. Gianluigi Bellin, Martin Hyland, Edmund Robinson, and Christian Urban. Categorical proof theory of classical propositional calculus. *Theor. Comput. Sci.*, 364(2):146–165, 2006.
3. S. A. Cook and R. A. Reckhow. The relative efficiency of propositional proof systems. *J. Symb. Logic*, 44(1):36–50, 1979.
4. V Danos and R Di Cosmo. The linear logic primer. lecture notes, available from `http://www.dicosmo.org/CourseNotes/LinLog/`.
5. V. Danos and L. Regnier. The structure of multiplicatives. *Archive for Mathematical Logic*, 28:181–203, 1989.
6. Vincent Danos. *La Logique Linéaire Appliquée l'étude de Divers Processus de Normalisation.* PhD thesis, University of Paris VII, 1990.
7. Carsten Führmann and David Pym. Order-enriched categorical models of the classical sequent calculus. *Journal of Pure and Applied Algebra*, 204(1):21 – 78, 2006.
8. Jean-Yves Girard. A new constructive logic: Classical logic. *Mathematical Structures in Computer Science*, 1(3):255–296, 1991.
9. Jean-Yves Girard. Proof-nets: The parallel syntax for proof-theory. In *Logic and Algebra*, pages 97–124. Marcel Dekker, 1996.
10. Dominic J. D. Hughes. Towards Hilbert's 24th problem: Combinatorial proof invariants. *Electron. Notes Theor. Comput. Sci.*, 165:37–63, 2006.
11. Dominic J.D. Hughes. Proofs Without Syntax. *Annals of Mathematics*, 143(3):1065–1076, November 2006.
12. F Lamarche and L Strassburger. Naming proofs in classical logic. In *TLCA '05*, 2005.
13. Francois Lamarche and Lutz Strassburger. Constructing free boolean categories. In *LICS '05: Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science*, pages 209—218, Washington, DC, USA, 2005. IEEE Computer Society.
14. Richard McKinley. Proof nets for Herbrand's theorem. Preprint, available as `http://arxiv.org/abs/1005.3986v1`.
15. Dale Miller. A compact representation of proofs. *Studia Logica*, 46(4):347–370, 1987.
16. Edmund Robinson. Proof nets for classical logic. *Journal of Logic and Computation*, 13(5):777–797, 2003.
17. Todd H. Trimble. *Linear logic, bimodules, and full coherence for autonomous categories.* PhD thesis, Rutgers University, 1994.