Self-Referentiality in Contraction-free Fragments of Modal Logic S4

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1 Introduction

In this Master's thesis we study sequent systems without structural rules, i.e., without weakening, contraction, and cut. After reviewing how such systems are constructed for classical propositional logic and modal logic **S4**, we introduce such a sequent system for the Logic of Proofs **LP**. Further, we investigate how the presence of contraction affects self-referentiality in modal logic **S4**. By eliminating all forms of contraction in the sequent calculus, we construct a fragment of **S4** that is free from self-referentiality.

Modal logic and justification logic. The basic modal language is the language of propositional logic with an additional unary modal operator $\Box$, called "box." A modal operator dual to $\Box$ in the same sense in which the existential and universal quantifiers are dual in first-order logic is denoted $\Diamond$ and called "diamond," i.e., $\Diamond A := \neg\Box\neg A$. Modal operators can been interpreted in a variety of ways: as necessity in *alethic modal logic*, as obligation in *deontic modal logic*, as future necessity in *temporal modal logic*. Multimodal logics can be used to describe behavior of multiagent systems, for instance in distributed computing. The readings we are most interested come from

- *epistemic logic* where the basic modal language is used to reason about knowledge and $\Box A$ stands for "the agent knows that A." It is customary to write KA instead of $\Box A$ in this setting;
- *provability logic* where $\Box A$ is read as "A is provable in a suitable formal theory."

In [Goe33] Gödel introduced a modal calculus of provability, basically equivalent to the modal logic **S4** of Lewis from [LL32], and showed that (propositional) intuitionistic logic could be translated into **S4** in a theorem-preserving way. Gödel's provability calculus is an extension of classical propositional logic by modal axioms and rules. However, unlike Gödel–Löb logic **GL**, this provability calculus, which encodes intuitionistic reasoning, cannot be directly interpreted into Peano Arithmetic. Indeed, let $\perp$ be the Boolean constant "false," then $\Box\perp\to\perp$ corresponds to the statement that expresses consistency of Peano Arithmetic **PA**. Since this formula $\Box(\Box\perp\to\perp)$ is a theorem of **S4**, a direct translation of **S4** into **PA**, whereby $\Box A$ is interpreted as "there is a an x such that x is a proof of the formula A," would result in the consistency of **PA** being provable in **PA**, which contradicts the second Gödel Incompleteness Theorem. Thus, Gödel left the problem of creating a provability semantics for **S4** and for intuitionistic logic [Goe33] open.
A solution was found by Artemov [Art95]. Instead of the modality $\Box$, justification logics use constructs of the form $t : F$ with the meaning "justification term t is a proof of the formula F." The first justification logic, the *Logic of Proofs* **LP**, was introduced by Artemov in [Art95]. Artemov connected **S4** and **LP** by proving the Realization Theorem:

- Replacing each justification term in an **LP**-theorem by $\Box$ yields an **S4**-theorem. The **S4**-formula obtained by such a replacement is called the *forgetful projection*.
- Vice versa, it is possible to *realize* all occurrences of $\Box$ in an **S4**-theorem by justification terms in such a way that the resulting justification formula is valid. This process of replacing boxes by justification terms is called *realization*.

Therefore, **LP** is called the justification counterpart of **S4**. Such correspondences have also been proved for all normal modal logics formed by the modal axioms d, t, b, 4, and 5 (see [Art08, BGK10]).

Proof Theory can be roughly divided into two parts: *structural* proof theory and *interpretational* proof theory. In the latter, the tools are syntactical translations of one formal theory into another, which are often semantically motivated. An example of such a translation is the embedding of propositional intuitionistic logic into the propositional fragment of modal logic **S4** due to Gödel [Goe33]. Hilbert's program, which called for a complete formalization of the relevant parts of mathematics, including the logical steps in mathematical arguments, has been the driving force behind the development of proof theory. Interest in proofs as combinatorial structures in their own right has been awakened, and is the subject of structural proof theory. Its true beginnings may be dated from the publication of Gentzen's *Untersuchungen über das logische Schliessen* in 1935, [Gen35], as a contrast to the old axiomatic proof theory by Hilbert. For Hilbert, the aim was to prove the consistency of axiomatizations of the essential parts of mathematics by methods that might be considered as evident because of their elementary character, an aim in which proof theory failed because of Gödel's Incompleteness Theorems. The first Incompleteness Theorem provides a counterexample to completeness by offering an arithmetic statement that is true in the standard model but neither provable nor refutable in **PA**. The second Incompleteness Theorem, which follows from the first, states that the consistency of **PA** cannot be proved in **PA** itself.

For Gentzen, on the other hand, the aim was to understand the structure of mathematical proofs. The use of sequent calculus permits the analysis of proofs with profound results. Today, proof theory is applied in automated theorem proving, which requires studying proofs as combinatorial structures, and in connection with computer science, e.g., in verification of correctness of computer programs.

Sequent systems can be used in systems of automated proof search, as well as in logic programming. In this thesis, we are interested in sequent systems without structural rules for classical propositional logic, modal logic **S4**, and the Logic of Proofs. *Structural rules* are rules such as Weakening, Contraction, and Cut:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\text{ Left Weakening} \qquad\qquad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\text{ Left Contraction}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{ Cut}$$

The rules for right weakening and right contraction are similar to the left weakening and left contraction respectively.

The presence of structural rules in a sequent system is critical for automated proof search, that is searching for a proof of a given formula in a specific logical system. If we consider an instance of the cut rule above, the formula A appears in both premises of the rule but not necessarily in the conclusion. Thus, while applying the rules in reverse during a proof search, it is impossible to know how to choose a suitable A among the infinitely many formulas. Formally speaking, the cut rule violates the *subformula property* since the formulas that occur in the premises of the rule do not necessarily occur as subformulas in the conclusion. The weakening rules allow to add arbitrary formulas on both sides of the arrow. The same derivable sequent may be obtained by several different applications of a weakening rule, and the premises of these weakening rules need not be derivable themselves. Formally speaking, the weakening rules are not *invertible*. The rules for contraction, when applied backward during a proof search, produces a premise that is more complex than the conclusion, rather than making it simpler. Sequent calculi free of these structural rules are powerful tools for analyzing formal derivations; they permit control over the structure of proofs.

Purpose of this thesis. While such sequent systems without structural rules are well known for propositional logic (e.g. [TS00, section 3.5]) and the logic **S4** (e.g. [TS00, section 9.1]), a similar system has not been developed for the Logic of Proofs yet. The first goal of this thesis is to construct a weakening-, contraction-, and cut-free sequent system for **LP**. The second aim is to find out whether self-referentiality occurs in contraction-free fragments of modal logic **S4**. It should be noted that, while the contraction rule can be eliminated from the sequent calculus for **S4**, certain forms of contraction must be retained to ensure completeness of the system. We investigate the self-referential properties of fragments where these forms of contraction are also eliminated by applying the machinery of *prehistoric phenomena*, originally introduced by Junhua Yu in [Yu10] for the complete structural-rule-free sequent calculus **G3s** for modal logic **S4**, to two incomplete but sound sequent systems **G3s'** and **G3s*** with eliminated weak forms of contraction.

Overview. We start this thesis by describing various sequent systems for classical propositional logic in Section 1. We recall the main properties of the sequent systems of interest and show how a weakening-, contraction- and cut-free sequent system can be obtained from the system that contains structural rules. This section about propositional logic will serve as a basis for the sections about **S4** and the Logic of Proofs, which are extensions of propositional logic. In Section 2 we recall how these methods extend from propositional logic to **S4**. In Section 3, we apply the methods developed for propositional logic and **S4** to introduce a sequent system without structural rules for the Logic of Proofs. In the last section, we present Yu's prehistoric phenomena in a Gentzen-style formulation of **S4** and study the occurrence of self-referentiality in various contraction-free fragments for **S4**.

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2 Classical propositional logic

Before we start defining systems for classical propositional logic, we make some general remarks about the various formalization styles of a logic and how we present deductions in a specific system.

2.1 Preliminary notes

Trees

Deductions in any system will be presented as labeled trees. Trees are partially ordered sets $(X, \leq)$ with a least element and all sets $\{y : y \leq x\}$ for $x \in X$ linearly ordered. The elements of X are called the *nodes* of the tree. *Branches* are maximal linearly ordered subsets of X. Trees grow upwards, the least node at the bottom is called the *root* of the tree. All the branches in our considerations will be finite and they end in a *leaf*, that is, a maximal element of the tree. The *length* of a branch is defined as the number of nodes in the branch minus 1; the *depth* of a tree is the maximum length of the branches in the tree.

Types of formalism

There are many ways to formalize a logic, such as Hilbert systems, Natural deduction, Resolution systems, Gentzen systems, Tableau systems and so on. In this work, we will concentrate on two different formalization styles for classical propositional logic, and later on for **S4** (section 3) and the Logic of Proofs **LP** (section 4). The formalization we are especially interested in is the Gentzen style calculus. However, for reasons of simplicity, we typically introduce logics via Hilbert systems. Proofs or deductions in Hilbert systems as well as in Gentzen systems will be presented as trees, whose nodes are labeled with entities or deduction elements of the same type, which will be described more precisely in the following. A deduction in such a system is a finite tree whose leaves are labeled with *special entities*, each internal node-label is connected with the labels of the successor nodes according to one of the rules of the system, and the single root of the tree is labeled with the entity which is going to be derived by the tree. An *n-premise rule* R is a set of sequences $S_0, S_1, ..., S_{n-1}, S$ of length $n+1$, where S_i, S are deduction elements. An element of R is said to be an *instance* or *application* of R. An instance is usually written as

$$\frac{S_0 \quad S_1 \quad S_2 \quad ... \quad S_{n-1}}{S}\, R$$

where $S_0, S_1, ..., S_{n-1}$ are *the premises* and S is *the conclusion* of the rule-application. A calculus is defined through a finite set of rules of the form described above. Rules of the form

$$\frac{}{S}\, R$$

are called *axioms*. In other words, an axiom is a rule without premises. An axiom is typically written as S. In *Hilbert systems* axioms are one of the two special entities,

from which we can start a prooftree. The other special entities are *assumptions*. Assumptions are entities we are allowed to use in addition to the axioms of the system. A very distinctive property of Hilbert systems is that they usually consist of many axioms and only few rules. Deductions in Hilbert systems are often written in linear format, a definition of a deduction in linear format for classical propositional logic is given later in Definition 2.9. Hilbert systems are widely used in the logical literature, but they are not very useful to do proof search.
Compared to Hilbert systems, *Gentzen systems* consist of few axioms and many rules. An introduction to Natural deduction systems is for example given in [TS00, section 2] and [Ind10, sections 2, 4], to Resolution in [TS00, section 7] and [Ind10, section 3.2]. An elementary introduction to tableau systems with further references can be found in [Ind10, section 3.1.2].

2.2 A Hilbert system for classical propositional logic

Definition 2.1. The *standard language* $\mathcal{L}$ for classical propositional logic contains

1. a set of *atomic propositions* Φ whose elements are usually denoted P, Q, and so on;
2. the constant $\perp$;
3. the Boolean connectives $\vee, \wedge, \rightarrow$;
4. parentheses.

Atomic formulas of $\mathcal{L}$ are atomic propositions. *Prime formulas* are formulas which are either atomic or $\perp$.
The *formulas* (A, B, C, A_1, B_1,..., F, G,...) of $\mathcal{L}$ are inductively defined as follows:

1. Every prime formula is a formula.
2. If A and B are formulas, then $(A \vee B), (A \wedge B)$ and $(A \rightarrow B)$ are formulas.

In addition we set

$$\begin{aligned} (A \leftrightarrow B) \quad &:= \quad ((A \rightarrow B) \wedge (B \rightarrow A)) \\ \neg A \quad &:= \quad (A \rightarrow \perp) \end{aligned}$$

Notational conventions: In writing formulas we save on parentheses by assuming that $\neg$ binds stronger than $\vee, \wedge$, and that in turn $\vee, \wedge$ bind stronger than $\rightarrow, \leftrightarrow$. Outermost parentheses are also usually dropped.

Definition 2.2. The set of *subformulas* of a formula A, denoted by Sub(A), is inductively defined as

- $\mathrm{Sub}(P) = P$, for prime formulas P;
- $\mathrm{Sub}(B \circ C) = \mathrm{Sub}(B) \cup \mathrm{Sub}(C) \cup \{B \circ C\}$, for arbitrary formulas B, C and $\circ \in \{\vee, \wedge, \rightarrow\}$.

The set of *strict subformulas* of a formula A is defined as Sub(A)\{A}.

Definition 2.3. The *depth* or *complexity* of a formula A, denoted by $|A|$, is defined recursively:

- $|P| = 0$ for prime formulas P;
- $|A \circ B| = max(|A|, |B|) + 1$ for $\circ \in \{\vee, \wedge, \rightarrow\}$.

In the following it is defined how the "truth" or "falsity" of a propositional formula depends on the "truth" or "falsity" of its propositions. This is called the semantics of propositional logic.

Definition 2.4. Let T and F be distinct new symbols, thought of as "true" and "false". A *truth assignment* for a set $S \subseteq \Phi$ of atomic propositions is, by definition, a function

$$v : S \rightarrow \{T, F\}.$$

For each truth assignment v we define its extension $\bar{v}$ to the set of all propositional formulas formed by propositions from S as follows:

$$\begin{array}{rcl} \bar{v}(P) & = & v(P) \text{ if } P \text{ is atomic;} \\ \bar{v}(\perp) & = & F; \\ \bar{v}(A \wedge B) & = & \begin{cases} T, & \text{if } \bar{v}(A) = \bar{v}(B) = T, \\ F, & \text{otherwise;} \end{cases} \\ \bar{v}(A \vee B) & = & \begin{cases} T, & \text{if } \bar{v}(A) = T \text{ or } \bar{v}(B) = T \text{ or both,} \\ F, & \text{otherwise;} \end{cases} \\ \bar{v}(A \rightarrow B) & = & \begin{cases} F, & \text{if } \bar{v}(A) = T \text{ and } \bar{v}(B) = F, \\ T, & \text{otherwise.} \end{cases} \end{array}$$

This definition can be summarized by means of the following *truth table*:

A	B	A∧B	A∨ B	A→ B	¬A	A↔ B
T	T	T	T	T	F	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Definition 2.5. A propositional formula A built from propositions of $S \subseteq \Phi$ is *valid*, or called a *tautology*, if $\bar{v}(A) = T$ for all truth assignments $v : S \rightarrow \{T, F\}$.

Example 2.6.

1. The formula $\neg P \vee Q \rightarrow R$ has the following truth table:

P	Q	R	$\neg P$	$\neg P \vee Q$	$\neg P \vee Q \rightarrow R$
T	T	T	F	T	T
F	T	T	T	T	T
T	F	T	F	F	T
F	F	T	T	T	T
T	T	F	F	T	F
F	T	F	T	T	F
T	F	F	F	F	T
F	F	F	T	T	F

2. The formula $A \vee \neg A$ has the following truth table:

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

Since the truth value of the formula $A \vee \neg A$ is always T, no matter if the truth value of A is T or F, the formula $A \vee \neg A$ (law of the excluded middle) is a tautology. Other simple examples of tautologies are

$$
\begin{aligned}
\neg(A \wedge \neg A) & \qquad \text{(law of contradiction)},\\
A \leftrightarrow \neg\neg A & \qquad \text{(law of double negation)},\\
A \wedge B \rightarrow A, &\\
A \rightarrow A \vee B. &
\end{aligned}
$$

Definition 2.7. [TS00, Definition 2.4.1][1] The Hilbert system **Hcp** for classical propositional logic is defined by the following axioms, for arbitrary formulas A, B, C in $\mathcal{L}$:

$$
\begin{array}{ll}
(1) & A \to (B \to A) \\
(2) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
(3) & A \to A \vee B \\
(4) & B \to A \vee B \\
(5) & (A \to C) \to ((B \to C) \to (A \vee B \to C)) \\
(6) & A \wedge B \to A \\
(7) & A \wedge B \to B \\
(8) & A \to (B \to A \wedge B) \\
(9) & \bot \to A \\
(10) & \neg\neg A \to A
\end{array}
$$

and the rule called *modus ponens*

$$\frac{A \quad A \to B}{B}\, MP,$$

for all formulas A,B.

Definition 2.8. A *proof* or *deduction* in **Hcp** of a formula A from assumptions Γ is a tree where instances of axioms and assumptions from Γ appear at the top nodes, lower nodes are formed from their successors by the single rule MP, and the root is labeled by A. Notation: $\Gamma \vdash_{\mathbf{Hcp}} A$, or when it is clear that we mean a proof in **Hcp**: $\Gamma \vdash A$. If Γ is the empty set, we simply write $\vdash_{\mathbf{Hcp}} A$ or $\vdash A$, respectively. In this case, A is called *theorem* of **Hcp**.

Proofs in Hilbert systems are often written in linear format, here is the definition of such a proof:

Definition 2.9. Let Γ be an arbitrary set of formulas. A finite sequence $A_1, ..., A_n$ of formulas is called a **Hcp**-*proof from T* if for each i, $1 \leq i \leq n$ one of the following three conditions is satisfied:

1. A_i is an axiom of **Hcp**;
2. A_i is an element of Γ;
3. A_i is the conclusion of (MP) whose premises belong to the sequence $A_1, ..., A_{i-1}$.

A formula A is derivable from Γ in **Hcp**, if there exists a **Hcp**-proof $A_1, ..., A_n$ from Γ such that A_n is the formula A.

Remark 2.10. A linear proof is a linearization of the partial order in the corresponding proof presented as a tree.

[1] Our definition presented is a restriction of the definition of **Hc**, a Hilbert system for classical predicate logic, to the propositional fragment of classical logic.

Theorem 2.11. *(Soundness and completeness) Let A be a formula of $\mathcal{L}$, then*

A is valid iff A is a theorem of **Hcp**.

Proof. "⇐": See e.g. [Men97, Proposition 1.12].
"⇒": See e.g. [Men97, Proposition 1.14]. □

2.3 Gentzen systems for classical propositional logic

In the current and the following subsection we will present various Gentzen systems for propositional logic and discuss the properties of the systems. The notation of the systems **G1c, G2c, G3c**[2] is due to [TS00, chapter 3]. Our aim is to define the system without structural rules in section 2.4 and we show how we obtain it from the system **G1c**. Before we introduce a first Gentzen system, we have to define what sequents are, since, unlike Hilbert systems, Gentzen systems do not derive formulas, but sequents.

Definition 2.12. *Sequents* are expressions of the form $\Gamma \Rightarrow \Delta$, with Γ, Δ finite multisets of formulas. *Multisets* are sets with multiplicity, i.e. elements can occur more than once. A set is a special case of a multiset: a multiset in which each element occurs only once is a set. The denotational interpretation of $\Gamma \Rightarrow \Delta$ is that the conjunction of the elements in Γ implies the disjunction of the formulas in Δ, that is $A_1 \wedge ... \wedge A_n \rightarrow B_1 \vee ... \vee B_m$ (should be read as $(...((A_1 \wedge A_2) \wedge A_3) \wedge ... \wedge A_n) \rightarrow (...((B_1 \vee B_2) \vee B_3) \vee ... \vee B_m))$, if $\Gamma = \{A_1, ..., A_n\}$ and $\Delta = \{B_1, ..., B_m\}$. The short form of the conjunction (disjunction) is $\bigwedge \Gamma$ ($\bigvee \Delta$) for Γ, Δ sets or multisets. For the multiset union $\Gamma \cup \Delta$ of Γ and Δ we write Γ, Δ, while Γ, A designates a multiset which is the union of Γ and the singleton multiset containing only A. In a sequent $\Gamma \Rightarrow \Delta$, the multiset Γ is called the *antecedent* and Δ the *succedent*.

Definition 2.13. [TS00][Definition 3.1.1] The Gentzen system **G1c** for classical propositional logic is defined by the axioms and rules, listed in Figure 1.

The axioms and those rules with the digit '1' in their labeling are going to be modified in either the system **G2c** or **G3c**.
In the rules listed in Figure 1, the elements of Γ, Δ are called *side formulas*. The *active formulas* are those formulas in the premise(s), which are not side formulas. The *principal formula* is the formula that is not a side formula in the conclusion. The right and left weakening (RW, LW) as well as the right and left contraction (RC, LC) rules are so called *structural rules*. The weakening rules allow us to add arbitrary formulas to the antecedent and the succedent, while the contraction rules allow us to remove duplicate formulas in the antecedent and the succedent. The remaining rules are called *logical rules*, they introduce logical connectives.

Example 2.14.

1. Let us consider the rule L$\rightarrow$ from the system **G1c**: In the premises, Γ, Δ are the side formulas and A, B are both active formulas. In the conclusion of this rule, $A \rightarrow B$ is the principal formula and the side formulas from Γ and Δ remain unchanged.

[2] In the following, we will use the notations **G[123]c** and **G[12]c** if we want to say something about the three, two respectively, systems in common.

$$A \Rightarrow A \text{ (Ax1)} \qquad \bot \Rightarrow \quad \text{(L}\bot\text{1)}$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge 1 \text{ , (i=0,1)} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee 1 \text{ , (i=0,1)}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} L \rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} R \rightarrow$$

Figure 1: Gentzen system **G1c** for propositional logic

2. In the (LW)-rule from **G1c** in Figure 1 there are only side formulas Γ, Δ but no active formula in the premise. In the conclusion, the side formulas from Γ and Δ remain unchanged and the principal formula A is the formula weakened by the rule.

Definition 2.15. Let **T** be a logical system, whose proofs are trees. We write $\mathcal{D} \vdash_n S$ if a prooftree $\mathcal{D}$ derives S and has depth at most n. We write $\vdash_n S$ or $\mathbf{T} \vdash_n S$ if for some $\mathcal{D}$ in the system **T** we have $\mathcal{D} \vdash_n S$.

Definition 2.16. Let **T** be a logical system, R an arbitrary rule.

- An n-premise rule R is called a *derivable rule* in **T** if for each instance $S_0, ..., S_{n-1}, S$ there is a deduction of S from all S_i by means of the rules of **T**. That is to say, in this deduction the S_i are treated as additional axioms.
- An n-premise rule R is said to be *admissible* for **T**, if for all instances $S_0, S_1, ..., S_{n-1}, S$ of R it is the case that
$$\text{if for all } i < n \ \vdash S_i, \text{ then } \vdash S.$$
- An n-premise rule R is said to be *depth-preserving* admissible (dp-admissible) for **T** if for all m
$$\text{if for all } i < n \ \vdash_m S_i, \text{ then } \vdash_m S.$$
- An n-premise rule R of **T** is said to be *i-invertible* [*i-dp-invertible* for **T**], $i = 1, ..., n-1$, if the rule
$$R_i \equiv \{(S, S_i) : (S_0, ..., S_{n-1}, S) \in R\}$$
is admissible [dp-admissible].
- An n-premise rule R is *invertible* [*dp-invertible*] if R is i-invertible [i-dp-invertible] for all $0 \leq i < n$.

Definition 2.17. A rule of a sequent system has the *(strict) subformula property*, if the active formulas are (strict) subformulas of the principal formula. A sequent system has the *(strict) subformula property*, if each rule has it.

Example 2.18.

1. The contraction rules of **G1c** have the subformula- but not the strict subformula property. The active formulas are subformulas of the principal one, but not strict subformulas.
2. All the rules introducing logical connectives in **G1c** have the strict subformula property.

The (strict) subformula property is very nice to have for sequent systems. If a sequent system has the strict subformula property, we have that not only the active formulas of each rule are built out of subformulas of the principal formula, but also these active formulas are strictly simpler than the principal formula. If we think about proof search this is exactly what we want. The cut-rule that we define in the following, does not enjoy the subformula property.

Definition 2.19. The following rule is called *Cut*:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; Cut$$

The cut-rule is a structural rule. A is called the *cut formula*. The *rank* of a cut is $|A| + 1$. The *cutrank* of a deduction $\mathcal{D}$, is the maximum of the ranks of the cut formulas occurring in $\mathcal{D}$.

The cut formula A in the two premises is in general not a subformula of a formula in $\Gamma, \Gamma', \Delta, \Delta'$, hence the cut rule violates the subformula property. Although the rule is not contained in the system **G1c**, it can be proved that cut is admissible for **G1c**. That is, the conclusion of cut is derivable in **G1c**, if the premises are. This is a very important property. which all the following systems also have.
For the proof of the following theorem we refer to [TS00, Theorem 3.2.1].

Theorem 2.20. *Cut is admissible for* **G1c***.*

Theorem 2.21. *The Gentzen system* **G1c** *and the Hilbert system* **Hcp** *are equivalent.*

Proof. The equivalence of the two systems follows from [TS00, Theorem 3.3.3], which states that

$$\mathbf{G1c} + Cut \vdash \Gamma \Rightarrow A \quad \text{iff} \quad \mathbf{Nc} \vdash \Gamma \Rightarrow A,$$

where **G1c** is the sequent system for full classical logic defined in [TS00, Definition 3.1.1], **Nc** is the natural deduction system for classical logic, and from [TS00, Theorem 2.4.2], which states the equivalence of the systems **Hc** and **Nc**. The theorems we refer to in [TS00] are proved for full classical logic, not only for the propositional fragment, but the statements also hold for propositional logic. □

On our way to define a sequent system without structural rules for classical propositional logic, the first step is to define a weakening-free sequent system.

Definition 2.22. [TS00, Definition 3.1.6] The Gentzen system **G2c** for classical propositional logic, is the system obtained from **G1c** by leaving out the weakening rules (LW, RW) and taking the more general axioms:

$$\Gamma, A \Rightarrow A, \Delta \quad (\text{Ax2}) \quad \text{and} \quad \bot, \Gamma \Rightarrow \Delta \quad (\text{L}\bot 2).$$

In **G2c**, the left conjunction rule and the right disjunction rule are denoted by L$\wedge$2, R$\vee$2, respectively. Even though there is no difference between the corresponding rules L$\wedge$1 and R$\vee$1 from system **G1c**. The reason therefore is that it is clear from which systems axioms and rules we are talking in the following. All the axioms and those

rules with an additional digit '1', '2' or '3' in their labeling get modified at least in one of the systems **G[123]c**.
It is easy to prove, that weakening is depth-preserving admissible in **G2c**, that is

Lemma 2.23.

$$\textit{If} \quad \mathbf{G2c} \vdash_n \Gamma \Rightarrow \Delta \quad \textit{then} \quad \mathbf{G2c} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta'.$$

Proof. By induction on the depth n of the proof $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$:
Case 1. If $\Gamma \Rightarrow \Delta$ is an instance of the axiom (Ax2) or (L⊥2), then $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is an axiom-instance, too.
Case 2. If the last rule of $\mathcal{D}$ is (LC), $\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LC$, the derivation of the premise $A, A, \Gamma \Rightarrow \Delta$ has depth $\leq n{-}1$ and by induction hypothesis we get $\vdash_{n-1} A, A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. Applying (LC) again we obtain a proof of depth $\leq n$ of $A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$:

$$\frac{A, A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, LC.$$

Case 3. If the last rule of $\mathcal{D}$ is R∧

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, R\wedge$$

The premises have deductions of smaller depth, thus induction hypothesis can be applied to the premises to obtain

$$\vdash_{n-1} \Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \quad \text{and} \quad \vdash_{n-1} \Gamma, \Gamma' \Rightarrow \Delta, \Delta', B.$$

Now we use R∧ to get $\mathbf{G2c} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \wedge B$.
The remaining cases are similar. □

The dp-admissibility of weakening in **G2c** implies the fact that all the sequents which are derivable in **G1c** can be also proved in **G2c** and vice versa. The two systems are equivalent, that is exactly what the following theorem says:

Theorem 2.24.

$$\mathbf{G1c} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \mathbf{G2c} \vdash \Gamma \Rightarrow \Delta.$$

Proof. "⇒": Let $\mathcal{D}$ be a **G1c**-proof of depth at most n, $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. By an induction on the depth n of $\mathcal{D}$, we show that there is **G2c**-derivation of $\Gamma \Rightarrow \Delta$:
Case 1. If $\Gamma \Rightarrow \Delta$ is one of the axioms (Ax1) or (L⊥1), $\Gamma \Rightarrow \Delta$ is a **G2c**-axiom too.
Case 2. If the last rule of $\mathcal{D}$ is (LW) or (RW)

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LW, \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\, RW$$

we apply the induction hypothesis to the premise, which has a deduction of depth $\leq n{-}1$, and have that $\mathbf{G2c} \vdash_{n-1} \Gamma \Rightarrow \Delta$. Since weakening is depth-preserving admissible in **G2c** we obtain $\mathbf{G2c} \vdash_{n-1} A, \Gamma \Rightarrow \Delta$ and $\mathbf{G2c} \vdash_{n-1} \Gamma \Rightarrow \Delta, A$.

The remaining rules of **G1c** are exactly the same as in **G2c**, hence we apply the induction hypothesis to the premise of the last rule used in $\mathcal{D}$ and apply the same rule in **G2c** to get the desired **G2c**-proof. The described translation from a **G1c**-proof to a **G2c**-proof does not increase the depth of the proofs.

"$\Leftarrow$": Let $\mathcal{D}$ be a **G2c**-proof of depth at most n, $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. By an induction on the depth n of $\mathcal{D}$, we show that there is **G1c**-derivation of $\Gamma \Rightarrow \Delta$. To prove this direction, it suffices to show that the more general axioms from **G2c** are derivable in **G1c**, since the rules of **G2c** are contained in **G1c**.

Case 1. Consider (Ax2) $A, \Gamma \Rightarrow \Delta, A$: we obtain a **G1c**-proof of this sequent if we apply k instances of (LW), where k is the number of formulas in Γ and l instances of (RW), where l is the number of formulas in Δ, to the corresponding axiom $A \Rightarrow A$:

$$\cfrac{\cfrac{A \Rightarrow A}{A, \Gamma_1, ..., \Gamma_k \Rightarrow A}\ LW}{A, \Gamma_1, ..., \Gamma_k \Rightarrow A, \Delta_1, ..., \Delta_l}\ RW$$

Where the double lines in the prooftree stand for several rule applications. Obviously, the (LW)- and (RW)-rules can be applied in arbitrary order.

Case 2. The same way we proceed to obtain a **G1c**-proof of (L$\bot$2) $\bot, \Gamma \Rightarrow \Delta$: we take the corresponding axiom in **G1c** and apply left and right weakening as much as we have formulas in Γ and Δ

$$\cfrac{\cfrac{\bot \Rightarrow}{\bot, \Gamma_1, ..., \Gamma_k \Rightarrow}\ LW}{\bot, \Gamma_1, ..., \Gamma_k \Rightarrow \Delta_1, ..., \Delta_l}\ RW$$

$\square$

Corollary 2.25.

$$\textit{If} \quad \mathbf{G1c} \vdash_n \Gamma \Rightarrow \Delta \quad \textit{then} \quad \mathbf{G2c} \vdash_n \Gamma \Rightarrow \Delta.$$

Proof. From the proof of the previous lemma direction "$\Rightarrow$" it follows that whenever **G1c** $\vdash_n \Gamma \Rightarrow \Delta$ then **G2c** $\vdash_n \Gamma \Rightarrow \Delta$. But the converse does not hold: we have for example

$$\mathbf{G2c} \vdash_0 P \wedge Q, R \Rightarrow P \wedge Q, S$$

since the sequent $P \wedge Q, R \Rightarrow P \wedge Q, S$ is an instance of (Ax2), and

$$\mathbf{G1c} \nvdash_0 P \wedge Q, R \Rightarrow P \wedge Q, S$$

since (Ax1) allows only axiom instances of the form $A \Rightarrow A$, but of course the sequent $P \wedge Q, R \Rightarrow P \wedge Q, S$ derivable in **G1c**:

$$\cfrac{\cfrac{P \wedge Q \Rightarrow P \wedge Q}{P \wedge Q, R \Rightarrow P \wedge Q}\ LW}{P \wedge Q, R \Rightarrow P \wedge Q, S}\ RW$$

$\square$

Corollary 2.26. *Cut is admissible for* **G2c**.

Proof. The statement follows from Theorem 2.20 (cut admissible for **G1c**) and Theorem 2.24 (equivalence of the systems **G1c** and **G2c**). □

Corollary 2.27. *The systems* **Hcp** *and* **G2c** *are equivalent.*

Proof. The claim follows from the equivalence of the systems **Hcp** and **G1c** (Theorem 2.21), and the systems **G1c** and **G2c** (Theorem 2.24). □

In this subsection we have seen that if we replace the axioms of **G1c** by more general ones and leave out the weakening rules, the resulting system **G2c** is still strong enough to derive exactly the same sequents like the original one. But the system **G2c** is just an intermediate system on our way to the system where all the structural rules are absorbed in the axioms and rules. In the next subsection we obtain such a system without structural rules from **G2c**.

2.4 A contraction-free Gentzen system for classical propositional logic

We start by defining the weakening- and contraction-free system **G3c** and proceed by proving some distinctive properties such as dp-admissibility of weakening, contraction and cut, invertibility of the rules and the equivalence of the systems **G1c** and **G3c**.

Definition 2.28. [TS00, Definition 3.5.1] The Gentzen system **G3c** is obtained from **G2c** by dropping the contraction rules (LC, RC) and taking

$$P, \Gamma \Rightarrow \Delta, P \text{ (P atomic) (Ax3)} \quad \text{instead of} \quad A, \Gamma \Rightarrow \Delta, A \text{ (Ax2)},$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\, L\wedge 3 \quad \text{instead of} \quad \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\, L\wedge 2 \text{ , (i=0,1)},$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\, R\vee 3 \quad \text{instead of} \quad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\, R \vee 2, \text{ (i=0,1)}.$$

Remark 2.29.

1. The reason for exchanging the two rules will become more intelligible later in this subsection. Here is just a try to direct the reader's focus on the next paragraph: the two **G2c**-rules (L∧2) and (R∨2) are not invertible, but the modified **G3c**-rules are. The invertibility of the rules is necessary for the method we will use to prove admissibility of contraction. For a similar reason we have to modify the axiom (Ax3): if we allow axioms of the form $A, \Gamma \Rightarrow \Delta, A$, we cannot prove depth-preserving invertibility of all the rules from **G3c**.

2. Since **G3c** does not contain any structural rules and the logical rules all enjoy the strict subformula property, the system has the strict subformula property.

Lemma 2.30. *Weakening is dp-admissible in* **G3c***, that is*

$$\textit{if} \quad \mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta \quad \textit{then} \quad \mathbf{G3c} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta'.$$

Proof. This property can be proved by induction on the depth of the proof of $\Gamma \Rightarrow \Delta$, like we did it for **G2c**. □

An essential property of the system **G3c** is that all the rules of the system are depth-preserving invertible: if the conclusion of a rule R of **G3c** can be derived in **G3c** by a deduction of depth n, then there is a deduction of depth $\leq n$ in **G3c** of the premise(s) of R.

Lemma 2.31. *(dp-invertibility of the rules) Let $\vdash$ denote deducibility in* **G3c***.*

1. *If $\vdash_n A \wedge B, \Gamma \Rightarrow \Delta$, then $\vdash_n A, B, \Gamma \Rightarrow \Delta$.*
2. *If $\vdash_n \Gamma \Rightarrow \Delta, A \vee B$, then $\vdash_n \Gamma \Rightarrow \Delta, A, B$.*
3. *If $\vdash_n A \vee B, \Gamma \Rightarrow \Delta$, then $\vdash_n A, \Gamma \Rightarrow \Delta$ and $\vdash_n B, \Gamma \Rightarrow \Delta$.*
4. *If $\vdash_n \Gamma \Rightarrow \Delta, A \wedge B$, then $\vdash_n \Gamma \Rightarrow \Delta, A$ and $\vdash_n \Gamma \Rightarrow \Delta, B$.*
5. *If $\vdash_n \Gamma \Rightarrow A \to B, \Delta$, then $\vdash_n \Gamma, A \Rightarrow \Delta, B$.*
6. *If $\vdash_n \Gamma, A \to B \Rightarrow \Delta$ then, $\vdash_n \Gamma \Rightarrow \Delta, A$ and $\vdash_n \Gamma, B \Rightarrow \Delta$.*

Proof. By induction on the depth n of the derivation. As a typical example we prove statement 4: assume 4. to have been proved for n and all Γ, Δ, A, B. Let $\vdash_{n+1} \Gamma \Rightarrow \Delta, A \wedge B$ by a deduction $\mathcal{D}$, then we have to consider the following cases:
Case 1. If $\Gamma \Rightarrow \Delta, A \wedge B$ is an axiom, then $A \wedge B$ is not principal and $\Gamma \Rightarrow \Delta, A$ as well as $\Gamma \Rightarrow \Delta, B$ are axioms too.
Case 2. If $\Gamma \Rightarrow \Delta, A \wedge B$ is not an axiom and $A \wedge B$ is not principal, then we have:

$$\frac{\Gamma' \Rightarrow \Delta', A \wedge B \quad (\Gamma'' \Rightarrow \Delta'', A \wedge B)}{\Gamma \Rightarrow \Delta, A \wedge B} R$$

we apply the induction hypothesis to the premise(s), which have deductions of depth at most n, to get:

$$\vdash_n \Gamma' \Rightarrow \Delta', A \tag{1}$$
$$\vdash_n \Gamma' \Rightarrow \Delta', B \tag{2}$$
$$\vdash_n (\Gamma'' \Rightarrow \Delta'', A) \tag{3}$$
$$\vdash_n (\Gamma'' \Rightarrow \Delta'', B). \tag{4}$$

An inspection of all the rules from **G3c** shows that we can apply the same rule R to (1) and (3) to obtain a deduction of $\vdash_{n+1} \Gamma \Rightarrow \Delta, A$ and to (2) and (4) to obtain a deduction of $\vdash_{n+1} \Gamma \Rightarrow \Delta, B$. The reason this works is that no rule has any restrictions on the side formulas in the premise, and what we changed by applying the induction hypothesis were only the side formulas in the premise(s).

Case 3. If $A \wedge B$ is principal, the last rule of $\mathcal{D}$ is

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} R\wedge$$

and we can take the immediate subdeductions, which are of depth at most n, of the premises.

□

In the following we discuss the properties of those axioms and rules from the systems **G[123]c**, which are of particular interest:

1. Two easy counterexamples show that the weakening rules

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW, \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

of the system **G1c** are not invertible, since we lose the information content of the formula A:

Example 2.32.

(a) If P, Q are atomic formulas, we have an easy derivation of the sequent $P, Q \Rightarrow Q$:

$$\frac{Q \Rightarrow Q}{Q, P \Rightarrow Q} LW.$$

Invertibility of (LW) would imply that **G1c** $\vdash P \Rightarrow Q$, which is obviously wrong. Thus **G1c** $\nvdash P \Rightarrow Q$ and **G1c** $\vdash Q, P \Rightarrow Q$.

(b) For the right weakening we have for example **G1c** $\vdash P \Rightarrow P, Q$ and **G1c** $\nvdash P \Rightarrow Q$.

The non-invertibility of (LW) and (RW) is the reason for our ambition to get rid of them - but to still have dp-admissibility of weakening.

2. The contraction rules of **G[12]c**

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC, \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

are invertible in **G[12]c**, that is

(a) if $\vdash A, \Gamma \Rightarrow \Delta$, then $\vdash A, A, \Gamma \Rightarrow \Delta$ and

(b) if $\vdash \Gamma \Rightarrow \Delta, A$, then $\vdash \Gamma \Rightarrow \Delta, A, A$.

In the weakening-free system **G2c**, we can prove dp-invertibility of the contraction rules, since weakening is depth-preserving admissible. In the system **G1c** we can only prove invertibility of contraction, but not depth-preserving invertibility. The reason is that weakening is present as a rule, or more precisely as two rules. For instance, if we have **G1c** $\vdash_n \Gamma \Rightarrow \Delta, A$, we apply (RW) to obtain **G1c** $\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A$. But it is not possible to have a deduction for $\Gamma \Rightarrow \Delta, A, A$ with the same depth as for $\Gamma \Rightarrow \Delta, A$: we have for example that **G1c** $\vdash_0 B \Rightarrow B$ for any formula B, since the sequent is an instance of (Ax1). On the other hand $B \Rightarrow B, B$ is no instance of (Ax1), thus **G1c** $\nvdash_0 B \Rightarrow B, B$, but of course still derivable in **G1c**:

$$\frac{B \Rightarrow B}{B \Rightarrow B, B} RW.$$

3. We refer to Remark 2.29 and show in more detail, why we need axioms of the form $P, \Gamma \Rightarrow \Delta, P$, where P is atomic, instead of $A, \Gamma \Rightarrow \Delta, A$: consider the sequent $C \to D \Rightarrow C \to D$ and we try to prove dp-invertibility of R$\to$. In the system where $A, \Gamma \Rightarrow \Delta, A$ is an axiom, $C \to D \Rightarrow C \to D$ is obviously an instance of the axiom (Ax2) and has a proof of depth 0. However, the sequent $C \to D, C \Rightarrow D$, is not an axiom-instance and therefore not derivable with depth 0.

4. We still refer to Remark 2.29: it was not only the axioms we modified to obtain **G3c** from **G2c**, we also had to adjust the left conjunction (L$\wedge$2) and the right disjunction (R$\vee$2) rule. To be precise, in **G[12]c** there are even two (L$\wedge$)[3] and (R$\vee$) rules each, namely

$$\frac{A_0, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge[12], \qquad \frac{A_1, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge[12],$$

$$\frac{\Gamma \Rightarrow \Delta, A_0}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee[12], \qquad \frac{\Gamma \Rightarrow \Delta, A_1}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee[12].$$

The four rules are not invertible, but the corresponding **G3c**-rules are. We give an example for the (L$\wedge$[12])-rule: the sequent $P \wedge Q \Rightarrow P \wedge Q$ is an axiom-instance in **G2c** and derivable in **G[13]c**. To prove dp-invertibility of (L$\wedge$), the premises of the following rule-applications have to be derivable with the same or lower depth as the conclusions are:

$$\frac{P \Rightarrow P \wedge Q}{P \wedge Q \Rightarrow P \wedge Q} L\wedge[12], \qquad \frac{P, Q \Rightarrow P \wedge Q}{P \wedge Q \Rightarrow P \wedge Q} L\wedge 3.$$

It is easy to see, that the premise of (L$\wedge$[12]) is not derivable in **G2c** for all P, Q, because **G2c** is sound and $P \to P \wedge Q$ is not a valid formula (truth table). Thus (L$\wedge$[12]) is not invertible. A similar reasoning implies the non-invertibility of (R$\vee$[12]). On the other hand, we proved invertibility of (L$\wedge$3) and (R$\vee$3) in **G3c** in the previous lemma.

5. The proof of Lemma 2.31 applies to the systems **G[12]c**, too, with the exception of the rules mentioned in the current enumeration.

[3] In the following, by L$\wedge$[12] we denote the identic rules L$\wedge$1 from **G1c** and L$\wedge$2 from **G2c**. The same applies to R$\vee$[12].

Lemma 2.33. *(dp-admissibility of contraction)*

1. *If* $\quad \mathbf{G3c} \vdash_n A, A, \Gamma \Rightarrow \Delta \quad$ *then* $\quad \mathbf{G3c} \vdash_n A, \Gamma \Rightarrow \Delta.$

2. *If* $\quad \mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, A, A \quad$ *then* $\quad \mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, A.$

Proof. We prove the lemma by a simultaneous induction on n for both statements. Assume the statements to be true for derivations of depth $\leq n$, and let $\mathcal{D}$ be a proof of depth $n+1$, such that $\mathcal{D} \vdash_{n+1} A, A, \Gamma \Rightarrow \Delta$.

Case 1. If $A, A, \Gamma \Rightarrow \Delta$ is an instance of one of the axioms (Ax3), (L⊥3) and

1. A is principal, then the sequent is of the form $P, P, \Gamma \Rightarrow \Delta', P$ or $\bot, \bot, \Gamma \Rightarrow \Delta$. In this case, the sequents $P, \Gamma \Rightarrow \Delta', P$ and $\bot, \Gamma \Rightarrow \Delta$ are also instances of the axioms (Ax3), and (L⊥3), respectively.

2. A is not principal, then $A, \Gamma \Rightarrow \Delta$ is an instance of an axiom, too.

Case 2. If A is not principal in the last rule applied in $\mathcal{D}$, then the last rule of $\mathcal{D}$ is of the form:

$$\frac{\vdash_n A, A, \Gamma' \Rightarrow \Delta' \quad (\vdash_n A, A, \Gamma'' \Rightarrow \Delta'')}{\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta} R$$

We apply the induction hypothesis to the premise(s) to get $\vdash_n A, \Gamma' \Rightarrow \Delta'$ (and $\vdash_n A, \Gamma'' \Rightarrow \Delta''$, if R is a two-premise rule) and if we use R again, we have that

$$\mathbf{G3c} \vdash_{n+1} A, \Gamma \Rightarrow \Delta.$$

Case 3. If A is principal in the last rule applied in $\mathcal{D}$ we have to consider the following subcases:

1. The last rule of $\mathcal{D}$ is L∧:

$$\frac{\vdash_n B, C, B \wedge C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \wedge C, B \wedge C, \Gamma \Rightarrow \Delta} L\wedge 3$$

If we apply the inversion lemma to the premise, we find a proof of $\vdash_n B, C, B, C, \Gamma \Rightarrow \Delta$. Now we can use the induction hypothesis twice to get $\vdash_n B, C, \Gamma \Rightarrow \Delta$ and by applying L∧3 we get that $\vdash_{n+1} B \wedge C, \Gamma \Rightarrow \Delta$.

2. The last rule of $\mathcal{D}$ is L∨:

$$\frac{\vdash_n B, B \vee C, \Gamma \Rightarrow \Delta \qquad \vdash_n C, B \vee C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \vee C, B \vee C, \Gamma \Rightarrow \Delta} L\vee$$

We apply the inversion lemma to the premises, which have deductions of depth at most n, and find proofs of $\vdash_n B, B, \Gamma \Rightarrow \Delta$ (from the left premise) and $\vdash_n C, C, \Gamma \Rightarrow \Delta$ (from the right premise). By induction hypothesis we have $\vdash_n B, \Gamma \Rightarrow \Delta$ and $\vdash_n C, \Gamma \Rightarrow \Delta$. To obtain the desired sequent we apply L∨.

3. The last rule of $\mathcal{D}$ is L→:

$$\frac{\vdash_n B \to C, \Gamma \Rightarrow \Delta, B \qquad \vdash_n B \to C, C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \to C, B \to C, \Gamma \Rightarrow \Delta} L \to$$

We apply the inversion lemma to the premises and find proofs of $\vdash_n \Gamma \Rightarrow B, B, \Delta$ (from the left premise) and $\vdash_n \Gamma, C, C \Rightarrow \Delta$ (from the right premise). By induction hypothesis we obtain $\vdash_n \Gamma \Rightarrow B, \Delta$ (induction hypothesis for the right contraction) and $\vdash_n \Gamma, C \Rightarrow \Delta$ (induction hypothesis for the left contraction) from which we can derive $\vdash_{n+1} B \rightarrow C, \Gamma \Rightarrow \Delta$ by one application of L$\rightarrow$.

Let $\mathcal{D}$ be a derivation of depth $n+1$, such that $\mathcal{D} \vdash_{n+1} \Gamma \Rightarrow \Delta, A, A$.

Case 4. If the sequent $\Gamma \Rightarrow \Delta, A, A$ is an instance of an axiom, and

1. A is principal, then the sequent is of the form $\Gamma', P \Rightarrow \Delta, P, P$ and $\Gamma', P \Rightarrow \Delta, P$ is an instance of (Ax3) too.

2. A is not principal, then $\Gamma \Rightarrow \Delta, A$ is an axiom-instance too.

Case 5. The case where A is not principal in the last rule applied in $\mathcal{D}$, can be treated similarly to case 2.

Case 6. If A is principal in the last rule applied in $\mathcal{D}$ we have to consider the following subcases:

1. The last rule of $\mathcal{D}$ is R$\rightarrow$:

$$\frac{\vdash_n \Gamma, B \Rightarrow \Delta, C, B \rightarrow C}{\vdash_{n+1} \Gamma \Rightarrow \Delta, B \rightarrow C, B \rightarrow C} R \rightarrow$$

We apply the inversion lemma to the premise and find a proof $\vdash_n \Gamma, B, B \Rightarrow \Delta, C, C$. Applying the induction hypothesis twice, leads to $\vdash_n \Gamma, B \Rightarrow \Delta, C$ and with one instance of R$\rightarrow$ we have a proof of the desired sequent.

2. The case where the last rule applied in $\mathcal{D}$ is R$\wedge$ can be treated similarly to case 3.2.

3. The case where the last rule applied in $\mathcal{D}$ is R$\vee$ can be treated similarly to case 3.1.

□

Lemma 2.34. *The sequent $A \Rightarrow A$ is derivable in* **G3c** *for all A.*

Proof. Induction on the complexity of the formula A. If A is a prime formula (A atomic or $A \equiv \perp$): $P \Rightarrow P$ and $\perp \Rightarrow \perp$ are both instances of the axioms in **G3c**. Let A be a formula of depth $k+1$, and assume that the statement holds for formulas of smaller depth.

Case 1. A$\equiv A_0 \wedge A_1$:

By induction hypothesis, $A_0 \Rightarrow A_0$ and $A_1 \Rightarrow A_1$ are derivable sequents. By dp-admissibility of weakening we have **G3c** $\vdash A_0, A_1 \Rightarrow A_0$ and **G3c** $\vdash A_0, A_1 \Rightarrow A_1$. We get a proof of the desired sequent as follows:

$$\frac{\dfrac{A_0, A_1 \Rightarrow A_0 \quad A_0, A_1 \Rightarrow A_1}{A_0, A_1 \Rightarrow A_0 \wedge A_1} R\wedge}{A_0 \wedge A_1 \Rightarrow A_0 \wedge A_1} L\wedge 3$$

Case 2. A$\equiv A_0 \vee A_1$: By induction hypothesis, $A_0 \Rightarrow A_0$ and $A_1 \Rightarrow A_1$ are derivable sequents. By dp-admissibility of weakening we have **G3c** $\vdash A_0 \Rightarrow A_0, A_1$ and **G3c** $\vdash A_1 \Rightarrow A_0, A_1$. We get a proof of the desired sequent as follows:

$$\dfrac{\dfrac{A_0 \Rightarrow A_0, A_1 \quad A_1 \Rightarrow A_0, A_1}{A_0 \vee A_1 \Rightarrow A_0, A_1} L\vee}{A_0 \vee A_1 \Rightarrow A_0 \vee A_1} R\vee 3$$

Case 3. $A \equiv A_0 \rightarrow A_1$: By induction hypothesis, $A_0 \Rightarrow A_0$ and $A_1 \Rightarrow A_1$ are derivable sequents. By dp-admissibility of weakening we have **G3c** $\vdash A_0 \Rightarrow A_0, A_1$ and **G3c** $\vdash A_0, A_1 \Rightarrow A_1$. We get a proof of the desired sequent as follows:

$$\dfrac{\dfrac{A_0 \Rightarrow A_0, A_1 \quad A_0, A_1 \Rightarrow A_1}{A_0, A_0 \rightarrow A_1 \Rightarrow A_1} L\rightarrow}{A_0 \rightarrow A_1 \Rightarrow A_0 \rightarrow A_1} R\rightarrow$$

□

The dp-admissibility of weakening and contraction for **G3c** implies that the systems **G3c** and **G1c** are equivalent:

Theorem 2.35.

$$\mathbf{G1c} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \mathbf{G3c} \vdash \Gamma \Rightarrow \Delta.$$

Proof. In both directions, the proof of this equivalence proceeds by induction on the depth of the deductions.
"$\Rightarrow$": Let $\mathcal{D}$ be a **G1c**-proof of depth n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. It suffices to treat the cases where $\Gamma \Rightarrow \Delta$ is an axiom or the last rule of $\mathcal{D}$ is one of the omitted or modified rules:

Case 1. If $\Gamma \Rightarrow \Delta$ is an instance of the axiom $A \Rightarrow A$. we know by the previous lemma that **G3c** $\vdash A \Rightarrow A$. If $\Gamma \Rightarrow \Delta$ is an instance of $\perp\Rightarrow$, it is an **G3c**-axiom too.

Case 2. If the last rule of $\mathcal{D}$ is (LW)

$$\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW,$$

we apply the induction hypothesis to the premise, which has a deduction of depth $n-1$, and get **G3c** $\vdash \Gamma \Rightarrow \Delta$. By dp-admissibility of weakening, we have **G3c** $\vdash A, \Gamma \Rightarrow \Delta$. The case where the last rule of $\mathcal{D}$ is (RW) can be treated similar.

Case 3. The last rule of $\mathcal{D}$ is (RC)

$$\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

we apply the induction hypothesis to the premise again to get **G3c** $\vdash \Gamma \Rightarrow \Delta, A, A$ and by dp-admissibility of contraction we have **G3c** $\vdash \Gamma \Rightarrow \Delta, A$. The case where the last rule of $\mathcal{D}$ is (LC) can be treated similar.

Case 4. If the last rule of $\mathcal{D}$ is (L∧1)

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge 1, \text{(i=0,1)}$$

we apply the induction hypothesis to the premise and have **G3c** $\vdash A_i, \Gamma \Rightarrow \Delta$. By dp-admissibility of weakening we can add A_0, A_1 respectively, depending on which of the formulas A_i is, to get **G3c** $\vdash A_0, A_1, \Gamma \Rightarrow \Delta$. It remains to use (L∧3) and we have **G3c** $\vdash A_0 \wedge A_1, \Gamma \Rightarrow \Delta$.

Case 5. The last case we have to consider is, if the last rule of $\mathcal{D}$ is (R∨1)

$$\frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee 1, \text{(i=1,0)}$$

we apply the induction hypothesis to the premise and weaken the sequent by the formula we need to get **G3c** $\vdash \Gamma \Rightarrow \Delta, A_0, A_1$. Applying (R∨3) leads to

$$\mathbf{G3c} \vdash \Gamma \Rightarrow \Delta, A_0 \vee A_1.$$

"⇐": Let $\mathcal{D}$ be a **G3c**-deduction of depth n, such that **G3c** $\vdash_n \Gamma \Rightarrow \Delta$. Again, to prove this direction it suffices to treat the axioms and the modified rules:

Case 1. If $\Gamma \Rightarrow \Delta$ is an instance of $P, \Gamma' \Rightarrow \Delta', P$, this sequent can be derived from the instance $P \Rightarrow P$ of the **G1c**-axiom (Ax1) by applying weakening (LW) and (RW) as much as there are formulas in Γ', Δ'. If $\Gamma \Rightarrow \Delta$ is an instance of $\perp, \Gamma' \Rightarrow \Delta'$, we can derive this sequent in **G1c** from the corresponding axiom $\perp\Rightarrow$ by applying weakening again.

Case 2. If the last rule of $\mathcal{D}$ is (L∧3)

$$\frac{A_0, A_1, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge 3$$

we apply the induction hypothesis to the premise, which has a deduction of depth $n-1$, and get **G1c** $\vdash A_0, A_1, \Gamma \Rightarrow \Delta$. Now we can use (L∧1) twice to have

$$\mathbf{G1c} \vdash A_0 \wedge A_1, A_0 \wedge A_1, \Gamma \Rightarrow \Delta.$$

Now we can contract the two occurrences of $A_0 \wedge A_1$ by one LC-application and have the desired derivation in **G1c**.

Case 3. If the last rule of $\mathcal{D}$ is (R∨3)

$$\frac{\Gamma \Rightarrow \Delta, A_0, A_1}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee 3$$

by induction hypothesis we have **G1c** $\vdash \Gamma \Rightarrow \Delta, A_0, A_1$. Now we use (R∨1) twice and contract the two occurrences of $A_0 \vee A_1$ by one instance of RC.

$\square$

Remark 2.36. From the equivalence-proof of the systems **G1c** and **G2c** we even obtained that if **G1c** $\vdash_n \Gamma \Rightarrow \Delta$ then **G2c** $\vdash_n \Gamma \Rightarrow \Delta$. In [TS00, remark to proposition 3.5.9] this corollary is stated for **G1c** and **G3c** as well, which turns out to be a mistake. The source of the problem is the axiom (Ax3) and its restriction to atomic propositions: Since $B \Rightarrow B$ is an instance of (Ax1), we have that **G1c** $\vdash_0 B \Rightarrow B$ for any formula B. But in **G3c** the same statement **G3c** $\vdash_0 B \Rightarrow B$ is restricted to prime formulas B. If B is an atomic proposition then $B \Rightarrow B$ is an instance of (Ax3), if $B \equiv \perp$ then $B \Rightarrow B$ is an instance of (L$\perp$3). But if we take B to be $B \equiv P \to Q$, then $P \to Q \Rightarrow P \to Q$ is not an instance of (Ax3) for sure, thus we have

$$\mathbf{G1c} \vdash_0 P \to Q \Rightarrow P \to Q \text{ and } \mathbf{G3c} \nvdash_0 P \to Q \Rightarrow P \to Q,$$

which is a counterexample for the mentioned remark in [TS00].

The following property of **G3c** follows from the equivalence of the systems **G1c** and **G3c**, and from Theorem 2.20:

Corollary 2.37. *Cut is admissible for* **G3c**.

Remark 2.38. The previous statement can also be proved directly, see for example [TS00, Theorem 4.1.5].

Corollary 2.39. *The Hilbert system* **Hcp** *and the Gentzen systems* **G1c**, **G2c** *and* **G3c** *are equivalent.*

Proof. The claim follows from the equivalence of **Hcp** and **G1c** (Theorem 2.21), the equivalence of **G1c** and **G2c** (Theorem 2.24), and the equivalence of **G1c** and **G3c** (Theorem 2.35). $\square$

We summarize how we obtained the system **G3c** from the system **G1c**: first, we got rid of the weakening rules. Instead of the two rules for weakening, we have more general axioms in **G2c** and we proved that weakening is still dp-admissible in the weakening-free system **G2c**, which implies that **G1c** and **G2c** are equivalent.
In a second step we also absorbed the contraction rules into the remaining axioms and rules of the system: therefore we had to restrict the axiom (Ax2) to $\Gamma, P \Rightarrow P, \Delta$, such that P is atomic. But this was not the only modification we had to make. Since we want the system without structural rules to be equivalent to the original system **G1c**, contraction has to be depth-preserving admissible. To prove this property, we need all the rules of the system **G3c** to be depth-preserving invertible. Since (L$\wedge$1) and (R$\vee$1), as they are defined in **G[12]c**, are not invertible, we have to modify them for the system **G3c**.
This strategy will be used again when we define contraction-free sequent systems for the modal logic **S4** and for the Logic of Proofs **LP**.

3 Modal logic S4 and its formalizations

In this section, our aim is to define a weakening-, contraction- and cut-free sequent system for the logic **S4**. We start by introducing a Hilbert system for **S4** in the first subsection, and in the second subsection we change to the semantic perspective and introduce the canonical model for **S4**. The canonical model will be used later on in this section, when we will prove that one of the contraction-free sequent systems we introduce, is incomplete.

3.1 A Hilbert system for S4

Modal logic **S4** is one of five axiom systems introduced first in Lewis and Langford's joint book *Symbolic Logic* [LL32], published in 1932, which contains a detailed development of Lewis' earlier ideas. Lewis was not the first who considered modal reasoning, but the link between his work and contemporary modal logic is more straightforward done than for other mathematicians work. But Lewis' work seems strange to modern eyes. For example, his axiomatic systems are not modular. Instead of extending a base system of propositional logic with specifically modal axioms, Lewis defines his axioms directly in terms of the binary modality he introduced. The modular approach to modal Hilbert systems is due to Kurt Gödel. Instead of using the Lewis and Langford axiomatization, in [Goe33] Gödel took $\Box$ as a primitive and formulated **S4** in the way that has become standard.

Definition 3.1. The *language of* **S4**, we denote it by $\mathcal{L}^+$, is obtained by adding to the language of classical propositional logic $\mathcal{L}$ a unary operator $\Box$.
The *modal formulas* are given by the grammar

$$A ::= \quad P \quad | \quad \bot \quad | \quad (A_1 \wedge A_2) \quad | \quad (A_1 \vee A_2) \quad | \quad (A_1 \to A_2) \quad | \quad \Box A.$$

$\Box A$ may be read as A *is provable* or A *is known*. The formulas $\neg A$ and $A_1 \leftrightarrow A_2$ are defined as in Definition 2.1. We shall use our conventions for eliminating parentheses. In addition, we set

$$\Diamond A := \neg\Box\neg A.$$

The operator $\Diamond$ is the dual of $\Box$. $\Diamond A$ may be read as *possibly* A or *diamond* A.

Definition 3.2. To define the *depth* or *complexity* of a modal formula $\Box A$, we refer to the Definition 2.3 of the depth of a propositional formula and add the following case: $|\Box A| = |A| + 1$, for all formulas A.

Definition 3.3. [TS00, Definition 9.1.1] A Hilbert style system **Hs** for **S4** is obtained by adding to the axioms and rules for classical propositional logic **Hcp**, the following axioms, where A, B denote any $\mathcal{L}^+$-formulas:

(**T**)	$\Box A \to A$
(**K**)	$\Box(A \to B) \to (\Box A \to \Box B)$
(**4**)	$\Box A \to \Box\Box A$

and the *necessitation rule*: $\dfrac{A}{\Box A}\,N,$ for all $\mathcal{L}^+$-formulas A.

The notion of a *deduction in* **Hs**, may be defined as follows. A deduction is a tree constructed starting from

$$A \ (A \text{ axiom})$$

by means of rules

$$\frac{A}{\Box A}\,N \qquad \frac{A \to B \quad A}{B}\,MP.$$

Remark 3.4. **S4** belongs to the family of *normal modal logics*. A modal logic is normal, if it contains the formulas:

$$\begin{array}{rl} (\mathbf{K}) & \Box(A \to B) \to (\Box A \to \Box B), \\ (\mathit{Dual}) & \Diamond A \leftrightarrow \neg\Box\neg A \end{array}$$

and is closed under *necessitation*, that is, if A is derivable, then so is $\Box A$. An introduction to normal modal logic is given in [BRV01, section 1.6].

3.2 The canonical model for S4

In this subsection we introduce relational semantics for **S4**, and therefore refer to parts of section 1 and 4 from [BRV01]. Relational semantics is often called Kripke semantics, since Kripke's work was crucial in establishing the relational approach.
Our aim is to state soundness and completeness for **S4** with respect to its canonical model. For the proofs of the theorems stated in this subsection, we will refer to the corresponding theorem in [BRV01].

Definition 3.5. An *n-place relation* (or a *relation with n arguments*) on a set X is a subset of X^n - that is, a set of ordered n-tupels of elements of X. A 2-place relation is called a *binary* relation. Given a binary relation R on a set X, we simply write uRv instead of $(u, v) \in R$, for $u, v \in X$. A binary relation R on X is said to be *reflexive* if xRx for all $x \in X$; R is *transitive* if xRy and yRz imply xRz, for $x, y, z \in X$.

Definition 3.6. A *relational structure* is a tuple $\mathcal{F}$ whose first component is a non-empty set W called the *universe (or domain)* of $\mathcal{F}$, and whose remaining components are relations on W. We assume that every relational structure contains at least one relation. The elements of W have a variety of names, including: *points, states, nodes, worlds, times, instants* and *situations*.

Definition 3.7. A *frame* for the language $\mathcal{L}^+$ is a pair $\mathcal{F} = (W, R)$ such that

1. W is a non-empty set.
2. R is an accessibility relation on $W \times W$.

In other words, a frame for $\mathcal{L}^+$ is simply a relational structure bearing a single binary relation. A *model* for $\mathcal{L}^+$ is a pair $\mathcal{M} = (\mathcal{F}, V) = ((W, R), V)$, where $\mathcal{F}$ is a frame for $\mathcal{L}^+$ and V is a function assigning to each proposition letter P a subset $V(P)$ of W. We will often write $\mathcal{M} = (W, R, V)$ to denote a model $\mathcal{M} = ((W, R), V)$. Formally, V is a map: $\Phi \to P(W)$, where $P(W)$ denotes the power set of W. Informally we think of $V(P)$ as the set of worlds in our model where P is true. The function V is called a *valuation*. Given a model $\mathcal{M} = (\mathcal{F}, V)$, we say that $\mathcal{M}$ is *based on* the frame $\mathcal{F}$, or that $\mathcal{F}$ is the frame *underlying* $\mathcal{M}$.

Definition 3.8. [BRV01, Definition 1.20][4] Suppose w is a world in a model $\mathcal{M} = (W, R, V)$, that is $w \in W$. Then we inductively define the notion of a formula A being *satisfied* or *true* in $\mathcal{M}$ at world w as follows:

$$\begin{array}{rll} \mathcal{M}, w \Vdash P & \text{iff} & w \in V(P), \text{ where } P \in \Phi, \\ \mathcal{M}, w \Vdash \bot & & \text{never }, \\ \mathcal{M}, w \Vdash A \vee B & \text{iff} & M, w \Vdash A \text{ or } M, w \Vdash B, \\ \mathcal{M}, w \Vdash A \wedge B & \text{iff} & M, w \Vdash A \text{ and } M, w \Vdash B, \\ \mathcal{M}, w \Vdash A \to B & \text{iff} & M, w \Vdash A \text{ implies } M, w \Vdash B, \\ \mathcal{M}, w \Vdash \Box A & \text{iff} & \text{for all } v \in W \text{ such that } wRv, \text{ we have } \mathcal{M}, v \Vdash A. \end{array}$$

If follows from this definition that $\mathcal{M}, w \Vdash \Diamond A$ if and only if for some $v \in W$ with wRv we have $\mathcal{M}, v \Vdash A$. Finally, we say that a *set* Σ of formulas is true at a world w of a model $\mathcal{M}$, if all members of Σ are true at w. Notation: $\mathcal{M}, w \Vdash \Sigma$.

Notational conventions:
If $\mathcal{M}$ does not satisfy A at w we write $\mathcal{M}, w \nVdash A$, and say that A is *false* at w. When $\mathcal{M}$ is clear from the context, we write $w \Vdash A$ for $\mathcal{M}, w \Vdash A$ and $w \nVdash A$ for $\mathcal{M}, w \nVdash A$.

Definition 3.9. A formula A is *globally* or *universally true* in a model $\mathcal{M}$ (notation: $\mathcal{M} \Vdash A$) if it is satisfied at all worlds in $\mathcal{M}$ (that is, if $\mathcal{M}, w \Vdash A$, for all $w \in W$). A formula A is *satisfiable* in a model $\mathcal{M}$ if there is some world in $\mathcal{M}$ at which A is true; a formula is *falsifiable* or *refutable* in a model if its negation is satisfiable.
A *set* Σ is globally true (satisfiable, respectively) in a model $\mathcal{M}$ if $\mathcal{M}, w \Vdash \Sigma$ for all worlds w in $\mathcal{M}$ (some world w in $\mathcal{M}$).

Definition 3.10. A formula A is *valid at a world w in a frame $\mathcal{F}$* if A is true at w in every model $(\mathcal{F}, V)$ based on $\mathcal{F}$. Notation: $\mathcal{F}, w \Vdash A$. A is *valid in a frame $\mathcal{F}$* if it is valid at every world in $\mathcal{F}$. Notation: $\mathcal{F} \Vdash A$. A formula A is *valid on a class of frames* $\mathbf{F}$ if it is valid on every frame $\mathcal{F}$ in $\mathbf{F}$. Notation: $\mathbf{F} \Vdash A$. A formula A is *valid* if it is valid on the class of all frames. Notation: $\Vdash A$. The set of all formulas that are valid in a class of frames $\mathbf{F}$ is called the *logic* of $\mathbf{F}$. Notation: $\Lambda_{\mathbf{F}}$.

Definition 3.11. Let $\mathbf{S}$ be a class of models (a class of frames), and Σ, A be a set of formulas and a single formula. A is a *local semantic consequence of* Σ *over* $\mathbf{S}$ (notation: $\Sigma \Vdash_S A$) if for all models $\mathcal{M}$ from $\mathbf{S}$, and all worlds w in $\mathcal{M}$, if $\mathcal{M}, w \Vdash \Sigma$ then $\mathcal{M}, w \Vdash A$.

[4]Note that we adapt the definition to the definitions we chose for the language $\mathcal{L}^+$ and its well-formed formulas.

Definition 3.12. If $\Gamma \cup \{A\}$ is a set of formulas then A is *deducible in the logic* Λ *from* Γ if $\vdash_\Lambda A$ or there are formulas $B_1, ..., B_n \in \Gamma$ such that

$$\vdash_\Lambda (B_1 \wedge ... \wedge B_n) \to A.$$

If this is the case we write $\Gamma \vdash_\Lambda A$, if not, $\Gamma \nvdash_\Lambda A$. A set of formulas Γ is Λ-*consistent* if $\Gamma \nvdash_\Lambda \bot$, and Λ-*inconsistent* otherwise. A formula A is Λ-consistent if $\{A\}$ is Λ-consistent; otherwise A is Λ-inconsistent.

Now we define the two fundamental concepts *soundness* and *completeness* linking the syntactic and semantic perspectives:

Definition 3.13. [BRV01, Definition 4.9] Let S be a class of frames (or models). A normal modal logic Λ is *sound* with respect to S if $\Lambda \subseteq \Lambda_S$. (Equivalently: Λ is *sound* with respect to S if for all formulas A , and all structures $\mathcal{S} \in$ S, $\vdash_\Lambda A$ implies $\mathcal{S} \Vdash A$.) If Λ is sound with respect to S we say that S *is a class of frames* (or models) *for* Λ.

Theorem 3.14. S4 *is sound with respect to the class of reflexive, transitive frames.*

For the proof of the soundness of **S4** one shows that the axioms (**T**, **K**, **4**) are valid, and that the rules of proof preserve validity.

Definition 3.15. [BRV01, Definition 4.10] Let S be a class of frames (or models). A logic Λ is *strongly complete* with respect to S if for any set of formulas $\Gamma \cup \{A\}$, if $\Gamma \Vdash_S A$ then $\Gamma \vdash_\Lambda A$. That is, if Γ semantically entails A on S then A is deducible in Λ from Γ. A logic is *weakly complete* with respect to S if for any formula A, if S $\Vdash A$ then $\vdash_\Lambda A$.

Proposition 3.16. *[BRV01, Proposition 4.12] A logic* Λ *is strongly complete with respect to a class of structures S iff every* Λ*-consistent set of formulas is satisfiable on some* $\mathcal{S} \in S$. Λ *is weakly complete with respect to a class of structures S iff every* Λ*-consistent formula is satisfiable on some* $\mathcal{S} \in S$.

The content of the proposition above is, that completeness theorems are essentially model existence theorems. Given a normal logic Λ, we prove its strong completeness with respect to some class of structures by showing that every Λ-consistent set of formulas can be satisfied in some suitable model. Thus the fundamental question is: how are this suitable satisfying models built? The answer to this question presented in [BRV01, section 4.2] is, to build models out of maximal consistent sets of formulas, and in particular, build canonical models.

Definition 3.17. A set of formulas Γ is *maximal* Λ*-consistent* if Γ is Λ-consistent, and any set of formulas properly containing Γ is Λ-inconsistent. If Γ is a maximal Λ-consistent set of formulas then we say it is a Λ-MCS.

It is good to know that any consistent set of formulas can be extended to a maximal consistent one. This is what Lindenbaum's Lemma is about:

Lemma 3.18. *(Lindenbaum's Lemma)[BRV01, Lemma 4.17] If* Σ *is a* Λ*-consistent set of formulas then there is a* Λ*-MCS* Σ^+ *such that* $\Sigma \subseteq \Sigma^+$.

To see why MCSs are used in completeness proofs, we first have to note that every world w in every model $\mathcal{M}$ for a logic Λ is associated with a set of formulas, namely $\{A : \mathcal{M}, w \Vdash A\}$. It can be shown that this set of formulas is actually a Λ-MCS. That is: if A is true in some model for Λ, then A belongs to Λ-MCS. Second, if w is related to v in some model $\mathcal{M}$, then it is clear that the information embodied in the MCSs associated with w and v is "coherently related".
The idea behind the canonical model construction is to turn these observations about MCSs and models around: that is, to work backwards from collections of coherently related MCSs to the desired model. By building a special model, the *canonical model*, whose worlds are all MCSs of the logic of interest, it is possible to prove the so called *Truth Lemma* (cp. [BRV01, Lemma 4.21]), which states that "A belongs to an MCS" is actually equivalent to "A is true in some model."

Definition 3.19. [BRV01, Definition 4.18][5] The *canonical model* $\mathcal{M}^{\mathbf{S4}}$ for the modal logic **S4** is the triple $(W^{\mathbf{S4}}, R^{\mathbf{S4}}, V^{\mathbf{S4}})$ where:

1. $W^{\mathbf{S4}}$ is the set of all **S4**-MCSs;
2. $R^{\mathbf{S4}}$ is the binary relation on $W^{\mathbf{S4}}$ defined by $wR^{\mathbf{S4}}v$ if $\Box A \in w$ implies $A \in v$, for all formulas A. $R^{\mathbf{S4}}$ is called the *canonical relation*;
3. $V^{\mathbf{S4}}$ is the valuation defined by $V^{\mathbf{S4}}(P) = \{w \in W^{\mathbf{S4}} : P \in w\}$. $V^{\mathbf{S4}}$ is called the *canonical (or natural) valuation.*

The pair $\mathcal{F}^{\mathbf{S4}} = (W^{\mathbf{S4}}, R^{\mathbf{S4}})$ is called the *canonical frame* for **S4**.

In the following we will omit the superscripts in $W^{\mathbf{S4}}, R^{\mathbf{S4}}, V^{\mathbf{S4}}$ if it is clear that we are talking about the canonical model for **S4**.
Here are some comments about the three clauses: First, the canonical valuation equates the truth of a propositional symbol at w with its membership in w. It is possible to lift this "truth=membership" equation to arbitrary formulas in $\mathcal{L}^{+}$ (cp. [BRV01, Truth Lemma 4.2.1]). Second, it should be noted that the worlds of $\mathcal{M}^{\mathbf{S4}}$ consist of *all* **S4**-consistent MCSs. The significance of this is that by Lindenbaum's Lemma, *any* **S4**-consistent set of formulas is a subset of some world in $\mathcal{M}^{\mathbf{S4}}$ - hence, by the Truth Lemma, any **S4**-consistent set of formulas is true at some world in this model. In short, the single structure $\mathcal{M}^{\mathbf{S4}}$ is a "universal model" for the logic **S4**, which is why it is called "canonical". Finally, we consider the canonical relation: a world w is related to a world v precisely when for $\Box A$ in w, v contains the information A, for each formula A. Intuitively, this captures what is meant by MCSs being "coherently related." The canonical model $\mathcal{M}^{\mathbf{S4}}$ is reflexive and transitive, that is, the canonical relation R on W is reflexive and transitive. In fact, the canonical frame of any normal logic containing the axiom **4** is transitive (cp. [BRV01, Theorem 4.27]). A similar statement can be proved for the axiom **T**: the canonical frame of any normal logic containing **T** is reflexive (cp. [BRV01, Theorem 4.28]).

[5] We restrict ourselves to the definition of the canonical model for **S4**. While the canonical model can be defined for any normal modal logic, it does not always work, that is, the canonical model may not belong to the desired class of models.

Theorem 3.20. *[BRV01, Theorem 4.22]*[6] *Modal logic* **S4** *is strongly complete with respect to its canonical model* $\mathcal{M}^{\mathbf{S4}}$.

Proof. Suppose Σ is a consistent set of the logic **S4**. By Lindenbaum's Lemma there is a **S4**-MCS Σ^+ extending Σ. By the Truth Lemma, $\mathcal{M}^{\mathbf{S4}}, \Sigma^+ \Vdash \Sigma$. □

Theorem 3.21. *[BRV01, Theorem 4.29]* **S4** *is strongly complete with respect to the class of reflexive, transitive frames.*

3.3 Sequent systems G1s and G2s

In this section we define two sequent systems for **S4**. We proceed by the same strategy we used to define the **G3**-system for classical propositional logic. First, we define a Gentzen system with structural rules, and then, we omit the structural rules and modify the remaining axioms and rules, such that the obtained system is still equivalent to the original one.

Definition 3.22. [TS00, Definition 9.1.3][7] The axioms and rules defining the Gentzen system **G1s** for modal logic **S4** are listed in Figure 2.

Remark 3.23.

1. Like we did it in the definition of the system **G1c**, we add the digit '1' to the name of those rules and axioms, which are going to be modified either in **G2s** or in **G3s**. The corresponding axiom- and rule-names in **G2s**, **G3s** respectively, will have the number '2', '3' respectively, even if they are not modified.

2. The system **G1c** is a subsystem of **G1s** or in other words, **G1s** is **G1c** with two additional rules, L□1 and R□1, thus the properties of the propositional fragment **G1s** are the same as for the rules in **G1c**. Especially we would like to mention the non-invertibility of the weakening rules and the rules R∨1, L∧1 (cp. observation 4. on page 24).

3. The R□1-rule is only restrictively applicable. The sequent, to which the R□1-rule is applied, has to satisfy the following conditions:

   - the antecedent contains only boxed formulas, and
   - the only formula present in the succedent, is the formula which is going to be boxed by the R□1-rule.

   The R□1-rule is the only rule in **G1s** which has this restrictive conditions.

4. The modal rule L□1 is not invertible: we have for example **G1s** $\vdash \Box P \Rightarrow \Box P$ for atomic propositions P, since $\Box P \Rightarrow \Box P$ is an instance of (Ax1) and **G1s** $\nvdash P \Rightarrow \Box P$, since $P \to \Box P$ is not valid.

[6] In the theorem we refer to, the *Canonical Model Theorem* is stated for any normal modal logic. But since in this case, we are only interested in **S4**, we restrict the statement to the logic of interest.

[7] In the definition we refer to, there are four modal rules, $L\Box, L\Diamond, R\Box$, and $R\Diamond$, added to **G1c** to obtain **G1s**, but as the remark in definition 9.1.3 says, it suffices to add L□ and R□ (or equivalently, L◇ and R◇), since L□ and R◇, such as R□ and L◇ are dual rules.

$$A \Rightarrow A \text{ (Ax1)} \qquad \bot \Rightarrow \quad \text{(L}\bot\text{1)}$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge 1 \text{ , (i=0,1)} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee 1\text{, (i=0,1)}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} L \rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} R \rightarrow$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 1 \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} R\Box 1$$

where $\Box\Gamma = \{\Box\Gamma_1, ..., \Box\Gamma_n\}$ is a finite multiset of boxed formulas only.

Figure 2: Gentzen system **G1s** for modal logic **S4**

5. The modal rule R$\Box$1 is invertible (cp. Corollary 3.25), but not dp-invertible: if $\Box\Gamma \Rightarrow \Box A$ is an instance of (Ax1), then the only element of $\Box\Gamma$ is $\Box A$ and the instance of (Ax1) is $\Box A \Rightarrow \Box A$, thus **G1s** $\vdash_0 \Box A \Rightarrow \Box A$. If R$\Box$1 were dp-invertible, $\Box A \Rightarrow A$ would have to be an axiom too, but obviously we have **G1s** $\nvdash_0 \Box A \Rightarrow A$.

As it can be proved for the propositional sequent systems we presented in section 2, cut is admissible for **G1s**. For the proof of the following theorem, we refer to [TS00, Theorem 9.1.5].

Theorem 3.24. *Cut is admissible for the system* **G1s***.*

Corollary 3.25. *The rule $R\Box 1$ from the system* **G1s** *is invertible.*

Proof. What we have to show is that whenever **G1s** $\vdash \Box\Gamma \Rightarrow \Box A$, then **G1s** $\vdash \Box\Gamma \Rightarrow A$, for all Γ, A. Let **G1s** $\vdash \Box\Gamma \Rightarrow \Box A$, and since $A \Rightarrow A$ is an instance of (Ax1) we have

$$\frac{A \Rightarrow A}{\Box A \Rightarrow A}\, L\Box 1.$$

Together we have **G1s** $\vdash \Box\Gamma \Rightarrow \Box A$ and **G1s** $\vdash \Box A \Rightarrow A$. By Cut-admissibility for **G1s** we obtain **G1s** $\vdash \Box\Gamma \Rightarrow A$. $\Box$

Theorem 3.26. *The sequent system* **G1s** *and the Hilbert system* **Hs** *are equivalent.*

Proof. By an induction on the depth of the **G1s**-proof, **Hs**-proof respectively, it can be shown that $\vdash_{\mathbf{Hs}} \bigwedge\Gamma \to A$ if and only if **G1s** $\vdash \Gamma \Rightarrow A$.
For the equivalence of a similar Gentzen system for **S4** and the Hilbert system **Hs** see for example [Kan57]. $\Box$

We define the weakening-free sequent system **G2s** next. As for **G2c**, weakening is depth-preserving admissible for **G2s**.

Definition 3.27. The system **G2s** for modal logic **S4** is obtained from **G1s** by leaving out the weakening rules (LW), (RW), replacing the axioms by the following two general ones

$$\Gamma, A \Rightarrow A, \Delta \quad \text{(Ax2)} \qquad \text{and} \qquad \bot, \Gamma \Rightarrow \Delta \quad \text{(L}\bot\text{2)},$$

and taking

$$\frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'}\, R\Box 2 \qquad \text{instead of} \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R\Box 1.$$

The system **G2s** is an extension of **G2c** by the two modal rules L$\Box$2 and R$\Box$2, where L$\Box$2 is exactly the same rule as L$\Box$1.

Lemma 3.28. *Weakening is depth-preserving admissible for* **G2s***, that is*

$$\text{if} \quad \mathbf{G2s} \vdash_n \Gamma \Rightarrow \Delta \quad \text{then} \quad \mathbf{G2s} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta'.$$

Proof. In section 2, we already proved the lemma for **G2c**, by induction on the depth n of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$, so it remains to show the two cases, where the last rule of the deduction $\mathcal{D}$ is a modal one:

Case 1. If the last rule of $\mathcal{D}$ is L□2, then we have:

$$\frac{\vdash_{n-1} A, \Gamma \Rightarrow \Delta}{\vdash_n \Box A, \Gamma \Rightarrow \Delta} L\Box 2$$

and we apply the induction hypothesis to the premise and get **G2s** $\vdash_{n-1} A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. Using L□2 we have **G2s** $\vdash_n \Box A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Case 2. If the last rule of $\mathcal{D}$ is R□2 then we have:

$$\frac{\vdash_{n-1} \Box\Gamma \Rightarrow A}{\vdash_n \Psi, \Box\Gamma \Rightarrow \Box A, \Phi} R\Box 2$$

and we just apply the following instance of R□2 to the premise

$$\frac{\vdash_{n-1} \Box\Gamma \Rightarrow A}{\vdash_n \Gamma', \Psi, \Box\Gamma \Rightarrow \Box A, \Phi, \Delta'} R\Box 2$$

to get the desired derivation.

□

Remark 3.29. In the proof of the second case it becomes clear, why we have to modify the rule introducing a □ in the succedent for the weakening-free system **G2s**: without doing it, we would not be able to prove dp-admissibility of weakening for this system. By the way, in opposition to R□1, R□2 is no longer invertible. As an easy counterexample we have **G2s** $\vdash \Box P \Rightarrow \Box P$ for any proposition P, since this sequent is an instance of (Ax2) and **G2s** $\nvdash \Rightarrow P$.

Theorem 3.30. *The systems* **G1s** *and* **G2s** *are equivalent, that is*

$$\textbf{G1s} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \textbf{G2s} \vdash \Gamma \Rightarrow \Delta.$$

Proof. Again, we refer to the proof of the corresponding theorem in the previous section (equivalence of **G1c** and **G2c**) and only show the modal cases. Therefore, the dp-weakening admissibility lemma for **G2c** is replaced in the proof by the previous dp-weakening admissibility lemma for **G2s**. For both directions, the proof proceeds by an induction on the depth of the deduction. "⇒": Let $\mathcal{D}$ be a **G1s**-derivation of $\Gamma \Rightarrow \Delta$ of depth n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$:

Case 1. If the last rule of $\mathcal{D}$ is L□1:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 1$$

we apply the induction hypothesis to the premise and use the same rule in **G2s**, denoted by L□2, to have a desired proof.

Case 2. If the last rule of $\mathcal{D}$ is R$\Box$1:

$$\frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R\Box 1$$

by induction hypothesis we get **G2s** $\vdash_{n-1} \Box\Gamma \Rightarrow A$. We can use R$\Box$2 in **G2s**:

$$\frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'}\, R\Box 2$$

with $\Gamma' = \Delta' = \emptyset$, to have the desired **G2s**-proof of the sequent.

It should be noted that also for the modal cases, the depth of a **G2s**-derivation of $\Gamma \Rightarrow \Delta$ does not exceed the depth of the **G1s**-derivation of the same sequent.

"$\Leftarrow$": Let $\mathcal{D}$ be a **G2s**-derivation of $\vdash_n \Gamma \Rightarrow \Delta$:

Case 1. If the last rule of $\mathcal{D}$ is L$\Box$2:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\, L\Box 2$$

we apply the induction hypothesis to the premise and use the same rule, denoted by L$\Box$1, in **G1s** to have a desired proof.

Case 2. If the last rule of $\mathcal{D}$ is R$\Box$2:

$$\frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'}\, R\Box 2$$

by induction hypothesis we get **G1s** $\vdash_{n-1} \Box\Gamma \Rightarrow A$. We can use R$\Box$1 in **G1s** to have **G1s** $\vdash_n \Box\Gamma \Rightarrow \Box A$, and applying some instances of weakening we obtain the desired **G1s**-proof of the sequent.

□

Corollary 3.31.

$$\textit{If} \quad \mathbf{G1s} \vdash_n \Gamma \Rightarrow \Delta \quad \textit{then} \quad \mathbf{G2s} \vdash_n \Gamma \Rightarrow \Delta.$$

Proof. The statement follows from the proof of the equivalence of **G1c** and **G2c** (Theorem 2.24), and the previous theorem. □

Corollary 3.32. *Cut is admissible for* **G2s**.

Proof. The statement follows from Theorem 3.24 (cut-admissibility for **G1s**) and the equivalence of the systems **G1s** and **G2s** (Theorem 3.30). □

Corollary 3.33. *The Hilbert system* **Hs** *and the Gentzen system* **G2s** *are equivalent.*

Proof. The corollary follows from the equivalence of the systems **Hs** and **G1s** (Theorem 3.26), and the systems **G1s** and **G2s** (Theorem 3.30). □

3.4 A contraction-free sequent system for S4

In this subsection we define a Gentzen system without weakening- and contraction rules for **S4** and prove that it is equivalent to the system containing structural rules.

Definition 3.34. [TS00, Definition 9.1.4] The Gentzen system **G3s** for **S4** is obtained from **G2s** by leaving out the contraction rules (LC, RC) and taking

$P, \Gamma \Rightarrow \Delta, P$ (P atomic) (Ax3) instead of $A, \Gamma \Rightarrow \Delta, A$ (Ax2),

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} L\wedge 3 \quad \text{instead of} \quad \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta} L\wedge 2 \text{ , (i=0,1),}$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} R\vee 3 \quad \text{instead of} \quad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1} R\vee 2 \text{ , (i=0,1),}$$

$$\frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 3 \quad \text{instead of} \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 2.$$

The system **G3s** extends **G3c** by the two modal rules.

Lemma 3.35. *Weakening is depth-preserving admissible for* **G3s***, that is*

$$\text{if} \quad \mathbf{G3s} \vdash_n \Gamma \Rightarrow \Delta \quad \text{then} \quad \mathbf{G3s} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta'.$$

Proof. We already proved the lemma for **G3c** and **G2s** by an induction on the depth n of the derivation $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. The case where the last rule of $\mathcal{D}$ is R□3 can be checked up in the proof of the corresponding lemma for **G2s**, since R□3 is the same rule as R□2. If the last rule of $\mathcal{D}$ is L□3,

$$\frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 3$$

we apply the induction hypothesis to the premise, which has a derivation of depth $\leq n-1$, and obtain **G3s** $\vdash_{n-1} \Gamma', \Gamma, A, \Box A \Rightarrow \Delta, \Delta'$. Now, we apply L□3 to have the desired derivation. □

Lemma 3.36. *(dp-invertibility of the rules) Let* $\vdash$ *be deducibility in* **G3s***.*

1. *If* $\vdash_n A \wedge B, \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, B, \Gamma \Rightarrow \Delta$.
2. *If* $\vdash_n \Gamma \Rightarrow \Delta, A \vee B$, *then* $\vdash_n \Gamma \Rightarrow \Delta, A, B$.
3. *If* $\vdash_n A \vee B, \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, \Gamma \Rightarrow \Delta$ *and* $\vdash_n B, \Gamma \Rightarrow \Delta$.
4. *If* $\vdash_n \Gamma \Rightarrow \Delta, A \wedge B$, *then* $\vdash_n \Gamma \Rightarrow \Delta, A$ *and* $\vdash_n \Gamma \Rightarrow \Delta, B$.
5. *If* $\vdash_n \Gamma \Rightarrow A \to B, \Delta$, *then* $\vdash_n \Gamma, A \Rightarrow \Delta, B$.

6. *If* $\vdash_n \Gamma, A \to B \Rightarrow \Delta$*, then* $\vdash_n \Gamma \Rightarrow \Delta, A$ *and* $\vdash_n \Gamma, B \Rightarrow \Delta$.

7. *If* $\vdash_n \Gamma, \Box A \Rightarrow \Delta$*, then* $\vdash_n \Gamma, A, \Box A \Rightarrow \Delta$.

Proof. We proved dp-invertibility of the rules for the system **G3c** by an induction on the depth n of the derivation, in Lemma 2.31. Since **G3s** in an extension of **G3c** by the rules R□3 and L□3, we have to show that the proof still works with this two additional rules. As a representative statement, we prove the first statement. Assume 1. to have been proved for n and all Γ, Δ. Let $\vdash_{n+1} A \wedge B, \Gamma \Rightarrow \Delta$ by a deduction $\mathcal{D}$. It suffices to treat the cases where the last rule of the deduction $\mathcal{D}$ is one of the modal ones, since the other cases work exactly the same as in the proof of Lemma 2.31.

Case 1. If the last rule of $\mathcal{D}$ is L□3

$$\frac{A \wedge B, \Box D, D, \Gamma \Rightarrow \Delta}{A \wedge B, \Box D, \Gamma \Rightarrow \Delta}\, L\Box 3$$

we apply the induction hypothesis to the premise and obtain $\vdash_n A, B, \Box D, D, \Gamma \Rightarrow \Delta$. To this sequent we apply L□3 and have $\vdash_{n+1} A, B, \Box D, \Gamma \Rightarrow \Delta$.

Case 2. If the last rule of $\mathcal{D}$ is R□3

$$\frac{\Box\Gamma \Rightarrow D}{A \wedge B, \Gamma', \Box\Gamma \Rightarrow \Box D, \Delta'}\, R\Box 3$$

the premise of the last rule-application has to be of the form $\Box\Gamma \Rightarrow D$, otherwise the R□3-rule is not applicable. Thus we just apply another R□3-instance to $\Box\Gamma \Rightarrow D$:

$$\frac{\Box\Gamma \Rightarrow D}{A, B, \Gamma', \Box\Gamma \Rightarrow \Box D, \Delta'}\, R\Box 3$$

to obtain the desired proof.

The remaining cases for the statements 2.-4. work similar.
dp-invertibility of the L□3-rule 7., is the only rule, for which we did not proved the statement yet. But if we have **G3s** $\vdash_n \Gamma, \Box A \Rightarrow \Delta$, then the desired derivation of **G3s** $\vdash_n \Gamma, \Box A, A \Rightarrow \Delta$ follows from the dp-admissibility of weakening for **G3s**. □

Remark 3.37. We only stated dp-invertibility for L□3 but not for the R□3-rule. Since R□3 is the same rule as R□2, the rule is not invertible for a similar reason. On one hand we have **G3s** $\vdash \Box P \Rightarrow \Box P$: $P \Rightarrow P$ is an instance of (Ax3) and by dp-weakening admissibility we obtain **G3s** $\vdash P, \Box P \Rightarrow P$. The desired sequent can be obtained as follows:

$$\cfrac{\cfrac{P, \Box P \Rightarrow P}{\Box P \Rightarrow P}\, L\Box 3}{\Box P \Rightarrow \Box P}\, R\Box 3$$

On the other hand we have **G3s** $\nvdash \Rightarrow P$. But fortunately it will turn out that the non-invertibility of R□3 is no problem, even to prove dp-admissibility of contraction.

Lemma 3.38. *(dp-admissibility of contraction for* **G3s***)*

1. If $\mathbf{G3s} \vdash_n A, A, \Gamma \Rightarrow \Delta$ *then* $\mathbf{G3s} \vdash_n A, \Gamma \Rightarrow \Delta$.

2. If $\mathbf{G3s} \vdash_n \Gamma \Rightarrow \Delta, A, A$ *then* $\mathbf{G3s} \vdash_n \Gamma \Rightarrow \Delta, A$.

Proof. We prove the lemma by a simultaneous induction on the depth n for both statements. We already proved this lemma for **G3c** (Lemma 2.33), thus it remains to treat the cases where the last rule of the deductions is a modal one.
Let $\mathcal{D}$ be a deduction of depth n+1, such that $\mathcal{D} \vdash_{n+1} A, A, \Gamma \Rightarrow \Delta$.

Case 1. If neither occurrence of A is principal and the last rule applied in $\mathcal{D}$ is R□3, then, depending on the formula A and the rule instance of R□3, we have the following possibilities:

1. If A is not a boxed formula, or $A \equiv \Box B$ is a boxed formula but has been weakened in R□3

$$\frac{\Box\Gamma \Rightarrow D}{\Gamma', A, A, \Box\Gamma \Rightarrow \Box D, \Delta'} R\Box 3$$

then we take the premise of the last R□3-application and use another instance of R□3 to obtain $\mathbf{G3s} \vdash_{n+1} \Gamma', A, \Box\Gamma \Rightarrow \Box D, \Delta'$.

2. If A is a boxed formula $A \equiv \Box B$ and only one occurrence of $\Box B$ has been weakened by the R□3-rule

$$\frac{\Box B, \Box\Gamma \Rightarrow D}{\Gamma', \Box B, \Box B, \Box\Gamma \Rightarrow \Box D, \Delta'} R\Box 3$$

then we just apply the following R□3-instance to obtain a proof of the desired sequent:

$$\frac{\Box B, \Box\Gamma \Rightarrow D}{\Gamma', \Box B, \Box\Gamma \Rightarrow \Box D, \Delta'} R\Box 3.$$

3. If A is a boxed formula $A \equiv \Box B$ and both occurrences have not been weakened in R□3

$$\frac{\Box B, \Box B, \Box\Gamma \Rightarrow D}{\Gamma', \Box B, \Box B, \Box\Gamma \Rightarrow \Box D, \Delta'} R\Box 3$$

we apply the induction hypothesis to the premise to obtain $\mathbf{G3s} \vdash_n \Box B, \Box\Gamma \Rightarrow D$. To this sequent we can apply R□3 and we have that $\mathbf{G3s} \vdash_{n+1} \Gamma', \Box B, \Box\Gamma \Rightarrow \Box D, \Delta'$.

Case 2. If A is not principal and the last rule applied in $\mathcal{D}$ is L□3,

$$\frac{A, A, B, \Box B, \Gamma \Rightarrow \Delta}{A, A, \Box B, \Gamma \Rightarrow \Delta} L\Box 3$$

we apply the induction hypothesis to the premise to obtain $\mathbf{G3s} \vdash_n A, B, \Box B, \Gamma \Rightarrow \Delta$. To this sequent we can apply L□3 and we have that $\mathbf{G3s} \vdash_{n+1} A, \Box B, \Gamma \Rightarrow \Delta$.

Case 3. If A is principal and the last rule applied in $\mathcal{D}$ is L□3, then A is of the form A$\equiv \Box B$

$$\frac{\Box B, B, \Box B, \Gamma \Rightarrow \Delta}{\Box B, \Box B, \Gamma \Rightarrow \Delta} L\Box 3$$

We apply the induction hypothesis to the premise, and get $\mathbf{G3s} \vdash_n B, \Box B, \Gamma \Rightarrow \Delta$. To this sequent we can apply L□3 and we have that $\mathbf{G3s} \vdash_{n+1} \Box B, \Gamma \Rightarrow \Delta$.

Let $\mathcal{D}$ be a deduction of depth n+1 of the sequent $\Gamma \Rightarrow \Delta, A, A$.

Case 4. If A is not principal and the last rule applied in $\mathcal{D}$ is L□3

$$\frac{B, \Box B, \Gamma \Rightarrow \Delta, A, A}{\Box B, \Gamma \Rightarrow \Delta, A, A} L\Box 3$$

we apply the induction hypothesis to the premise to obtain **G3s** $\vdash_n B, \Box B, \Gamma \Rightarrow \Delta, A$. To this sequent we can apply L□3 and we have that **G3s** $\vdash_{n+1} \Box B, \Gamma \Rightarrow \Delta, A$.

Case 5. If A is not principal and the last rule applied in $\mathcal{D}$ is R□3,

$$\frac{\Box\Gamma \Rightarrow B}{\Gamma', \Box\Gamma \Rightarrow \Box B, A, A, \Delta'} R\Box 3$$

we just take the immediate subdeduction of depth n of the premise and apply the following instance of R□3

$$\frac{\Box\Gamma \Rightarrow B}{\Gamma', \Box\Gamma \Rightarrow \Box B, A, \Delta'} R\Box 3$$

to obtain **G3s** $\vdash_{n+1} \Gamma', \Box\Gamma \Rightarrow \Box B, A, \Delta'$.

Case 6. If A is principal and the last rule applied in $\mathcal{D}$ is R□3, then A is of the form A≡ $\Box B$

$$\frac{\Box\Gamma \Rightarrow B}{\Gamma', \Box\Gamma \Rightarrow \Box B, \Box B, \Delta'} R\Box 3$$

we take the immediate subdeduction of the premise and apply the following instance of R□3

$$\frac{\Box\Gamma \Rightarrow B}{\Gamma', \Box\Gamma \Rightarrow \Box B, \Delta'} R\Box 3$$

to obtain **G3s** $\vdash_{n+1} \Gamma', \Box\Gamma \Rightarrow \Box B, \Delta'$.

□

Remark 3.39.

1. The depth-preserving invertibility of the L□3-rule is not needed in the proof above. The reason for not necessarily using this property of the L□3-rule is the additional copy of $\Box B$ in the premise of the rule, in other words, the embedded contraction in L□3. Obviously, if L□3 was not dp-invertible, for example if we take the left box-rule to be the rule L□[12] defined in **G[12]s**, then this particular proof of case 3 would not be possible.

2. The R□3-rule is the only rule from the system **G3s**, which does not need to be invertible to prove dp-admissibility of contraction. The depth-preserving invertibility of any **G3s**-rule R, with the exception of L□3, in the proof of dp-admissibility of contraction is needed in the specific case, where the last rule of the deduction is the rule R and one of the formulas A, which are going to

be contracted, is the principal formula in R. In all the rules R, except for R□3, there is an occurrence of the second copy of the formula A, the non-principal A, in the premise of R. We try to make the difference visible for the case of dp-admissibility of the right contraction:

$$\frac{\Box\Gamma \Rightarrow B}{\Gamma', \Box\Gamma \Rightarrow \Box B, \Box B, \Delta'}\, R\Box 3 \qquad \frac{\Gamma \Rightarrow \mathbf{A_0} \lor \mathbf{A_1}, A_0, A_1, \Delta}{\Gamma \Rightarrow A_0 \lor A_1, A_0 \lor A_1, \Delta}\, R\lor$$

$$\frac{\Gamma \Rightarrow \mathbf{A_0} \land \mathbf{A_1}, A_0, \Delta \quad \Gamma \Rightarrow \mathbf{A_0} \land \mathbf{A_1}, A_1, \Delta}{\Gamma \Rightarrow A_0 \land A_1, A_0 \land A_1, \Delta}\, R\land 3 \qquad \frac{\Gamma, A_0 \Rightarrow \mathbf{A_0} \to \mathbf{A_1}, A_1, \Delta}{\Gamma \Rightarrow A_0 \to A_1, A_0 \to A_1, \Delta}\, R\to$$

In the cases, where the last rule is R∧3, R∨ and R→ there is still one of the formulas, which are going to be contracted, in the succedent of the premise (the formula written in bold face). With the dp-invertibility of this rules, we find a derivation of $\Gamma \Rightarrow A_0, A_1, A_0, A_1, \Delta$ (in the case for R∨) on which we can finally apply the induction hypothesis of the proof. But if the last rule applied is R□3, this second copy of $\Box B$ has been weakened by R□3 and we do not need invertibility of the rule.

Lemma 3.40. *The sequent $A \Rightarrow A$ is derivable in* **G3s** *for all formulas A.*

Proof. Induction on the complexity of the formula A. The corresponding property has been proved for **G3c**, thus it suffices to consider the case where $A \equiv \Box B$. Therefore, the dp-weakening admissibility lemma for **G3c** is be replaced in the proof by the dp-weakening admissibility lemma proven for **G3s**.
Assume that the statement holds for sequents $A \Rightarrow A$ with depth $|A| = k$. Let $A \equiv \Box B$, by induction hypothesis we know that $B \Rightarrow B$ is a derivable sequent. By dp-admissibility of weakening it follows that **G3s** $\vdash B, \Box B \Rightarrow B$. We get a proof of the desired sequent as follows:

$$\cfrac{\cfrac{B, \Box B \Rightarrow B}{\Box B \Rightarrow B}\, L\Box 3}{\Box B \Rightarrow \Box B}\, R\Box 3$$

□

Now we are ready to prove that the weakening- and contraction-free system **G3s** is equivalent to the original system **G1s**:

Theorem 3.41.

$$\textbf{G1s} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \textbf{G3s} \vdash \Gamma \Rightarrow \Delta.$$

Proof. In both directions, the proof proceeds by an induction on the depth n of the deductions. Since we already proved the corresponding theorem for **G1c** and **G3c**, it remains to treat the cases where the last rule of the deduction $\mathcal{D}$ is L□3 and R□3.
"⇒": Let $\mathcal{D}$ be a **G1s**-deduction of depth at most n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$.

Case 1. If the last rule of $\mathcal{D}$ is L□1:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\, L\Box 1$$

we can apply the induction hypothesis to the premise to obtain **G3s** $\vdash \Gamma, A \Rightarrow \Delta$. By the dp-admissibility of weakening for **G3s** we get **G3s** $\vdash \Gamma, A, \Box A \Rightarrow \Delta$, now L□3 is applicable in **G3s** and we have a derivation of the desired sequent.

Case 2. If the last rule of $\mathcal{D}$ is R□1:

$$\frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ R\Box 1$$

we apply the induction hypothesis to the premise and have **G3s** $\vdash \Box\Gamma \Rightarrow A$, R□3 is directly applicable and we have **G3s** $\vdash \Box\Gamma \Rightarrow \Box A$.

"$\Leftarrow$": Let $\mathcal{D}$ be a **G3s**-deduction of depth at most n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$.

Case 1. If the last rule of $\mathcal{D}$ is L□3:

$$\frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ L\Box 3$$

Applying the induction hypothesis to the premise implies **G1s** $\vdash \Gamma, A, \Box A \Rightarrow \Delta$ and by the following rule-applications we get a **G1s**-proof of the desired sequent as follows:

$$\cfrac{\cfrac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A, \Box A \Rightarrow \Delta}\ L\Box 1}{\Gamma, \Box A \Rightarrow \Delta}\ LC$$

Case 2. If the last rule of $\mathcal{D}$ is R□3:

$$\frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'}\ R\Box 3$$

we apply the induction hypothesis to the premise and get **G1s** $\vdash \Box\Gamma \Rightarrow A$, where we can directly apply R□1 to obtain **G1s** $\vdash \Box\Gamma \Rightarrow \Box A$. Applying some instances of weakening we obtain a proof of the desired sequent.

□

The equivalence of the systems **G1s**, **G3s**, and cut-admissibility for **G1s** (Theorem 3.24) imply the following corollary:

Corollary 3.42. *Cut is admissible for* **G3s**.

Remark 3.43. The previous statement can also be proven directly, see for example [TS00, Theorem 9.1.5].

Corollary 3.44. *The Gentzen systems* **G1s***,* **G2s***,* **G3s** *and the Hilbert system* **Hs** *for* **S4** *are equivalent.*

Proof. This follows from the equivalence of the systems **Hs** and **G1s** (Theorem 3.26), **G1s** and **G2s** (Theorem 3.30), and **G1s** and **G3s** (Theorem 3.41). □

3.5 G3s' - a modification of the system G3s

In the current subsection, we slightly modify the weakening- and contraction-free system **G3s**, and it turns out that this slight modification has a really bad consequence: the resulting system is incomplete. With the help of this example, we show how important the properties, that we proved for **G3s**, actually are.

Definition 3.45. The sequent style calculus **G3s'** is the system **G3s** with an alternative box rule for the left side. The original rule L□3 is replaced by the rule

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\, L\Box 1.$$

Lemma 3.46. *Weakening is depth-preserving admissible for* **G3s'**.

Proof. We already proved the lemma for **G3s** (Lemma 3.35) by an induction on the depth n of the derivation $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. If the last rule of $\mathcal{D}$ is L□1,

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\, L\Box 1$$

we apply the induction hypothesis to the premise, which has a derivation of depth $\leq n-1$, and obtain $\mathbf{G3s'} \vdash_{n-1} \Gamma', \Gamma, A \Rightarrow \Delta, \Delta'$. Now, we apply L□1 and have

$$\mathbf{G3s'} \vdash_n \Gamma', \Gamma, \Box A \Rightarrow \Delta, \Delta'.$$

□

Lemma 3.47. *The sequent $A \Rightarrow A$ is derivable in* **G3s'** *for all formulas A.*

Proof. Induction on the complexity of the formula A. Assume that the statement holds for sequents $A \Rightarrow A$ with depth $|A| = k$ and show that the statement holds for formulas A such that $|A| = k + 1$. Since we already proved this lemma for **G3s**, the only case it remains to consider is the case where $A \equiv \Box B$:

$$\cfrac{\cfrac{B \Rightarrow B}{\Box B \Rightarrow B}\, L\Box 1}{\Box B \Rightarrow \Box B}\, R\Box$$

By induction hypothesis, $B \Rightarrow B$ is derivable in **G3s'**. □

In Theorem 3.48 and 3.50, if we use the expression *incomplete (sound) with respect to* **S4**, we mean incomplete (sound) with respect to the class of Kripke models for **S4** introduced in subsection 3.2.

Theorem 3.48. *The system* **G3s'** *is incomplete with respect to* **S4**.

There are two different proofs we would like to give for the incompleteness of the system **G3s'**. In the first proof we introduce a formula, which is valid but not provable in **G3s'**, and in the second proof we show that the systems **G3s** and **G3s'** are not equivalent, which will bring to light the reason for **G3s'** being incomplete.

First proof of the Theorem. The useful formula $\phi = \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$, where P is an atomic proposition, is due to Rajeev P. Goré [Gor92, page 47]. First, we prove that this formula is not derivable in **G3s'**:
Starting with the sequent $\Rightarrow \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$ at the root of a tree, the first five steps backwards are determined:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}{\Rightarrow \neg\Box\neg P \to \Box\neg\Box\neg P}\,R\to}{\Rightarrow\bot, \Box(\neg\Box\neg P \to \Box\neg\Box\neg P)}\,R\Box 3 \qquad \bot\Rightarrow\bot}{\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow\bot}\,L\to}{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow\bot}\,L\Box 1}{\Rightarrow \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)}\,R\to \tag{5}$$

Considering the sequent $\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$ we have two possibilities to continue: we could apply the rule R$\Box$ or the rule L$\to$. If we apply R$\Box$ backwards, the following steps are fix again and we get the tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow P, \bot \qquad \bot\Rightarrow\bot}{\neg P \Rightarrow\bot}\,L\to}{\Box\neg P \Rightarrow\bot}\,L\Box 1}{\Rightarrow \neg\Box\neg P}\,R\to}{\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}\,R\Box 3 \tag{6}$$

The sequent $\Rightarrow P \perp$ is not an instance of an axiom, thus this is no proof for our sequent. But what happens if we continue with L$\to$ instead of R$\Box$?

$$\dfrac{\Rightarrow \Box\neg P, \Box\neg\Box\neg P \qquad \bot\Rightarrow \Box\neg\Box\neg P}{\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}\,L\to \tag{7}$$

Going one step backwards with L$\to$ we get an instance of (L$\perp$) on the right side and the sequent $\Rightarrow \Box\neg P, \Box\neg\Box\neg P$ on the left. Since both formulas in the succedent are boxed, the only rule we can apply backwards is the right box rule. It does not depend on which formula we apply the rule first, in both trees we get if we proceed the backward proof search, we do not have axioms at the top nodes:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow P, \bot \qquad \bot\Rightarrow\bot}{\neg P \Rightarrow\bot}\,L\to}{\Box\neg P \Rightarrow\bot}\,L\Box 1}{\Rightarrow \neg\Box\neg P}\,R\to}{\Rightarrow \Box\neg P, \Box\neg\Box\neg P}\,R\Box 3 \tag{8}$$

$$\dfrac{\dfrac{P \Rightarrow\bot}{\Rightarrow \neg P}\,R\to}{\Rightarrow \Box\neg P, \Box\neg\Box\neg P}\,R\Box 3 \tag{9}$$

Since our possibilities to derive a sequent like $\Rightarrow \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$ from our axioms by means of the available rules are limited, and we checked all the possibilities, we can be sure that there is no proof for this sequent in our system. Thus

$$\mathbf{G3s'} \nvdash\Rightarrow \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P).$$

It remains to show that the formula $\neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$ is valid: Let $\mathcal{M} = (W, R, V)$ be an arbitrary reflexive and transitive model. If we want to check whether or not our formula $\neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$ is valid in the model $\mathcal{M}$, we have to write out the definition of validity first:

$$\begin{array}{rll}
\mathcal{M} \Vdash \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) & iff & \text{for all } w \in W, w \Vdash \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P). \\
 & iff & \text{for all } w \in W, w \nVdash \Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \\
 & iff & \exists u \in W \text{ s.t. } wRu \text{ and } u \nVdash \neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \\
 & iff & \exists u \in W \text{ s.t. } wRu \text{ and } u \Vdash \Box(\neg\Box\neg P \to \Box\neg\Box\neg P). \\
u \Vdash \Box(\neg\Box\neg P \to \Box\neg\Box\neg P) & iff & \forall v \in W(uRv \to v \Vdash \neg\Box\neg P \to \Box\neg\Box\neg P) \\
 & iff & \forall v \in W(uRv \to (v \nVdash \neg\Box\neg P \text{ or } v \Vdash \Box\neg\Box\neg P)) \\
 & iff & \forall v \in W(uRv \to (v \Vdash \Box\neg P \text{ or } v \Vdash \Box\neg\Box\neg P)). \\
v \Vdash \Box\neg P \text{ or } v \Vdash \Box\neg\Box\neg P & iff & \forall r(vRr \to r \Vdash \neg P) \text{ or } \forall r(vRr \to r \Vdash \neg\Box\neg P) \\
 & iff & \forall r(vRr \to r \nVdash P) \text{ or } \forall r(vRr \to r \nVdash \Box\neg P) \\
 & iff & \forall r(vRr \to r \nVdash P) \text{ or } \forall r(vRr \to \exists s(rRs \text{ and } s \nVdash \neg P)) \\
 & iff & \forall r(vRr \to r \nVdash P) \text{ or } \forall r(vRr \to \exists s(rRs \text{ and } s \Vdash P)).
\end{array}$$

We have to show that for all $w \in W$ there exists a world u such that the listed conditions hold. Let w be an arbitrary world in W. We define the set

$$R^P(w) = \{u : wRu \text{ and } u \Vdash P\}$$

of worlds, accessible from w, where P is valid. If wRv, then the corresponding set $R^P(v)$ of worlds accessible from v satisfying P is a subset of $R^P(w)$ by transitivity of R: let p be an element of $R^P(v)$, so what we know about p is vRp and $p \Vdash P$. Since we have wRv (by assumption) and vRp the transitivity of R leads to wRp and therefore to $p \in R^P(w)$, thus $R^P(v) \subseteq R^P(w)$. To complete the proof it remains to distinguish the following two cases:

Case 1. If $R^P(u) = \emptyset$ for some u with wRu, by definition of $R^P(u)$ we have found a world u such that for any accessible world r (uRr), $r \nVdash P$.

Case 2. If for all worlds u accessible from w, $R^P(u) \neq \emptyset$, we pick any r with $wRuRvRr$. By transitivity of R we have wRr, so $R^P(r) \neq \emptyset$. Thus there is an s such that rRs and $s \Vdash P$.

□

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                              \dfrac{\Box\neg P, P \Rightarrow P, \bot}{\Box\neg P \Rightarrow P, \neg P}\,R\to
                              \qquad \bot, \Box\neg P \Rightarrow \neg P
                            }{\Box\neg P, \neg P \Rightarrow \neg P}\,L\to
                          }{\Box\neg P \Rightarrow \neg P}\,L\Box 3
                        }{\Box\neg P \Rightarrow \Box\neg P, \Box\neg\Box\neg P}\,R\Box 3
                        \qquad \bot, \Box\neg P \Rightarrow \Box\neg\Box\neg P
                      }{\Box\neg P, \neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}\,L\to
                    }{\Box\neg P \Rightarrow \neg\Box\neg P \to \Box\neg\Box\neg P}\,R\to
                  }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \Box\neg P \Rightarrow \Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \bot}\,R\Box 3
                  \qquad \bot, \Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \Box\neg P \Rightarrow \bot
                }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \Box\neg P \Rightarrow \bot}\,L\to
              }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \neg\Box\neg P}\,R\to
            }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \neg\Box\neg P}\,L\Box 3
          }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}\,R\Box 3
        }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \neg\Box\neg P \to \Box\neg\Box\neg P}\,R\to
      }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot, \Box(\neg\Box\neg P \to \Box\neg\Box\neg P)}\,R\Box 3
      \qquad \bot, \Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot
    }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P), \neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot}\,L\to
  }{\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot}\,L\Box 3
}{\Rightarrow \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)}\,R\to
$$

Figure 3: **G3s**-derivation of $\neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$

Since the system **G3s** is sound an complete with respect to **S4**, and ϕ is valid in any model for **S4**, the formula ϕ has to be derivable in **G3s**. In Figure 3 there is a derivation of ϕ in **G3s**.

Second proof of the Theorem. In this proof of the incompleteness of **G3s'** it becomes clear, for which reason the system is incomplete. But actually, we already know it: it has to be the non-invertibility of the L□1-rule, the only rule which makes the difference between **G3s'** and **G3s**. A counterexample proves that the rule $L\Box 1$ is not invertible:

$$(1) \qquad \mathbf{G3s'} \vdash \Box\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$$
$$(2) \qquad \mathbf{G3s'} \nvdash \neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$$

(1) follows from Lemma 3.47, the sequent $A \Rightarrow A$ is derivable in **G3s'** for any formula A, and (2) follows from the first proof of the incompleteness theorem. All attempts to derive the sequent $\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$ in **G3s'** were without success. The invertibility of L□ would state that if $\mathbf{G3s'} \vdash \Box\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$ then $\mathbf{G3s'} \vdash \neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$. Of course, it would be nice to have a system with invertible rules, but why is it so bad, that L□1 is not invertible? The problem is that with the non-invertible L□1-rule, we cannot prove dp-admissibility of contraction: the standard proof (see for example the proof of dp-admissibility of contraction for **G3s**, Lemma 3.38) proceeds by an induction on the depth of the derivation $\mathcal{D}$, such that $\mathcal{D} \vdash_n A, A, \Gamma \Rightarrow \Delta$. In the case, where the last rule of $\mathcal{D}$ is L□1, and one of the formulas A is the principal formula, thus A is of the form $A \equiv \Box B$, we would need the depth-preserving invertibility of L□1 to prove the statement:

$$\frac{B, \Box B, \Gamma \Rightarrow \Delta}{\Box B, \Box B, \Gamma \Rightarrow \Delta}\, L\Box 1.$$

Thus, we already found the reason for **G3s'** not being equivalent to the original system **G3s**. In the previous subsection we have shown that contraction is dp-admissible for **G3s**, and it is clear from the observations above, that it is not dp-admissible for **G3s'**. In fact, we can even state more, namely, that contraction is not admissible for **G3s'** in general. The following counterexample proves the statement:

$$(1) \qquad \mathbf{G3s'} \vdash \Box\neg(P \to \Box P), \Box\neg(P \to \Box P) \Rightarrow \bot$$
$$(2) \qquad \mathbf{G3s'} \nvdash \Box\neg(P \to \Box P) \Rightarrow \bot\ .$$

(1) follows from the following deduction in **G3s'**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \Rightarrow \Box P, P}{\Rightarrow P \to \Box P, P}\,R\to \qquad \bot \Rightarrow P}{\neg(P \to \Box P) \Rightarrow P}\,L\to}{\Box\neg(P \to \Box P) \Rightarrow P}\,L\Box 1}{\Box\neg(P \to \Box P), P \Rightarrow \Box P, \bot}\,R\Box 3}{\Box\neg(P \to \Box P) \Rightarrow P \to \Box P, \bot}\,R\to \qquad \bot, \Box\neg(P \to \Box P) \Rightarrow \bot}{\neg(P \to \Box P), \Box\neg(P \to \Box P) \Rightarrow \bot}\,L\to}{\Box\neg(P \to \Box P), \Box\neg(P \to \Box P) \Rightarrow \bot}\,L\Box 1$$

Statement (2) follows from the following observations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow P}{P \Rightarrow \Box P, \bot}\,R\Box 3}{\Rightarrow P \to \Box P, \bot}\,R\to \qquad \bot \Rightarrow \bot}{\neg(P \to \Box P) \Rightarrow \bot}\,L\to}{\Box\neg(P \to \Box P) \Rightarrow \bot}\,L\Box 1$$

Looking for a **G3s'**-derivation of the sequent $\Box\neg(P \to \Box P) \Rightarrow \bot$, we start with the sequent at the root of a tree and apply **G3s'**-rules in reverse. From the root-node until the leaves of the tree above, there is only one rule applicable at each step. Since the left leaf is no axiom-instance, this is no **G3s'**-proof for the sequent $\Box\neg(P \to \Box P) \Rightarrow \bot$, and since this was the only possibility to derive the sequent in **G3s'**, we have
$\mathbf{G3s'} \nvdash \Box\neg(P \to \Box P) \Rightarrow \bot$.
So not for every sequent derivable in **G3s**, there is a proof in **G3s'**: Let $\mathcal{D}$ be a **G3s**-derivation of $\Box A, \Gamma \Rightarrow \Delta$, the last rule applied in $\mathcal{D}$ is L$\Box$3:

$$\dfrac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\,L\Box 3$$

The standard proof of the equivalence of two Gentzen systems proceeds by an induction on the depth of the deductions, see for example the proof of Theorem 3.41. Let $\mathcal{D}$ be a **G3s**-deduction of $\Box A, \Gamma \Rightarrow \Delta$, with depth $n + 1$. By induction hypothesis, we have $\mathbf{G3s'} \vdash A, \Box A, \Gamma \Rightarrow \Delta$. In **G3s'**, we proceed by an application of L$\Box$1:

$$\dfrac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Box A, \Gamma \Rightarrow \Delta}\,L\Box 1$$

and then we would need dp-admissibility of contraction to find a proof of $\Box A, \Gamma \Rightarrow \Delta$. $\Box$

Remark 3.49. As a conclusion of the fact that contraction is not dp-admissible for **G3s'** is that cut is not admissible for the system. Contraction-admissibility is used in the proof for cut-admissibility.

The conclusion of the previous two proofs is, that there are sequents derivable in **G3s**, but not derivable in **G3s'**, and we know the reason for those sequents not being derivable in the modified system. But what can we say about the other direction? Is there a **G3s**-proof for any sequent, derivable in **G3s'**? The following theorem gives an answer to this question:

Theorem 3.50. *The system* **G3s′** *is sound with respect to* **S4**.

Proof. What we have to prove is: if **G3s′** $\vdash \Gamma \Rightarrow \Delta$ then $\bigwedge \Gamma \to \bigvee \Delta$ is valid in the class of reflexive, transitive frames. Since **G3s** is a sound and complete system for **S4**, it's enough to show that whenever **G3s′** $\vdash \Gamma \Rightarrow \Delta$ then **G3s** $\vdash \Gamma \Rightarrow \Delta$. This is provable by an induction on the depth n of the **G3s′**-proof $\mathcal{D}$, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. Assume the statement to hold for derivations of smaller depth.

Case 1. If $\Gamma \Rightarrow \Delta$ is an instance of the axioms, it is a **G3s**-axiom-instance too.

Case 2. If the last rule of the deduction $\mathcal{D}$ is one of the **G3s′**-rules different from L□1, we apply the induction hypothesis to the premise and use the same rule in **G3s** to get a **G3s**-proof of $\Gamma \Rightarrow \Delta$.

Case 3. If the last rule of the deduction $\mathcal{D}$ is L□1:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 1$$

By induction hypothesis we get **G3s** $\vdash_{n-1} \Gamma, A \Rightarrow \Delta$ and since weakening is dp-admissible in **G3s** we have **G3s** $\vdash_{n-1} \Gamma, A, \Box A \Rightarrow \Delta$. Now we can apply the L□3-rule and have a **G3s**-derivation of $\vdash_n \Gamma, \Box A \Rightarrow \Delta$:

$$\frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 3.$$

□

Corollary 3.51. *If* **G3s′** $\vdash_n \Gamma \Rightarrow \Delta$ *then* **G3s** $\vdash_n \Gamma \Rightarrow \Delta$.

Proof. The statement follows from the proof of the previous theorem. □

4 A G3-style sequent calculus for the Logic of Proofs

The aim of this section is to define a **G3**-style system for the Logic of Proofs **LP** and to prove that this system enjoys the already well-known properties like dp-admissibility of contraction, weakening and cut as well as dp-invertibility of the rules.

4.1 A Hilbert system for the Logic of Proofs

LP introduced by S. Artemov in [Art95] was the first of those logics which are in summary called *justification logics* today. Artemov developed **LP** to solve the problem of a provability semantics for **S4**. Compared to the language of modal logic, the language of justification logic is richer, what allows us to do a finer analysis of formulas and proofs. There is a new construct occurring in justification logic, it is a formula of the type *term:formula*, for example $t : A$, with the intended semantics *term t is a proof of formula A*.

Definition 4.1. The language $\mathcal{L}_{LP}$ of **LP** contains the language of classical propositional logic, proof constants $a_0, a_1, ..., a_n, ...$ and proof variables $x_1, ..., x_n, ...$, one monadic function symbol $!$ such as two binary function symbols $\cdot$ and $+$ and a symbol of the type *term:formula*. *Justification terms* are defined by the grammar

$$t ::= \quad x \quad | \quad a \quad | \quad !t \quad | \quad (t_1 \cdot t_2) \quad | \quad (t_1 + t_2).$$

These terms are called *proof polynomials* and we denote them by p, r, s, t, etc. Constants correspond to proofs of a finite fixed set of axioms. We assume that $p \cdot r \cdot s...$ should be read as $(...((p \cdot r) \cdot s)...)$, and $p + r + s...$ as $(...((p + r) + s)...)$. Using t stand for any term.
Justification formulas are given by the grammar

$$A ::= \quad P \quad | \quad \bot \quad | \quad (A_1 \wedge A_2) \quad | \quad (A_1 \vee A_2) \quad | \quad (A_1 \to A_2) \quad | \quad t : A.$$

The formulas $\neg A$ and $A_1 \leftrightarrow A_2$ are defined as in Definition 2.1. In writing formulas we save on parentheses by assuming the following precedence from highest to lowest: $!, \cdot, +, :, \neg, \wedge, \vee, \to$.

Definition 4.2. We continue Definition 2.3 to define the *depth* or *complexity* of justification formulas by

$$|t : A| = |A| + 1, \text{ for all formulas } A.$$

Definition 4.3. [Art01, Definition 5.2] The Hilbert-style formalization for **LP** is obtained from **Hcp** by adding the following axioms:

$$\begin{array}{ll} A1. & t : F \to F \\ A2. & t : (F \to G) \to (s : F \to (t \cdot s) : G) \\ A3. & t : F \to !t : (t : F) \\ A4. & s : F \to (s + t) : F, t : F \to (s + t) : F \end{array}$$

and the *axiom necessitation* rule AN

$$\frac{A}{c : A}\,AN$$

where A is an axiom A0-A4, and c a proof constant. By A0 we denote the finite set of axioms from **Hcp** (cp. Definition 2.7).
The system $\mathbf{LP}_0$ is **LP** without the axiom necessitation rule.

Definition 4.4. Given a justification formula A, its *forgetful projection* is defined as:

1. $P^\circ = P$, for atomic propositions P;
2. $\perp^\circ = \perp$;
3. $(A_0 \circ A_1)^\circ = A_0^\circ \circ A_1^\circ$, for any formulas A_0, A_1 and $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. $(t : B)^\circ = \Box B^\circ$ for all formulas B.

The forgetful projection Γ° of a set of justification formulas $\Gamma = \{A_0, A_1, ..., A_n\}$ is defined as $\Gamma^\circ = \{A_0^\circ, A_1^\circ, ..., A_n^\circ\}$.

Finding the Logic of Proofs, Artemov shows that **S4** is nothing but the forgetful projection of **LP**. This result is stated in the Realization Theorem[8] in [Art01, section 9]:

- The forgetful projection of an **LP**-theorem is an **S4**-theorem (cp. [Art01, Lemma 9.1]).
- It is possible to *realize* all occurrences of $\Box$ in an **S4**-theorem by justification terms, such that the resulting justification formula is an **LP**-theorem (cp. [Art01, Theorem 9.4]). This process is called *realization*.

In other words, $\mathbf{LP}^\circ = \mathbf{S4}$, the forgetful projection of **LP** is exactly **S4**.

Working with **LP**, we have to know what a Constant Specification is:

Definition 4.5. A *Constant Specification (CS)* is a set of **LP**-formulas $c_1 : A_1, c_2 : A_2, ...$ where c_i's are constants and A_i's are instances of the axioms A0-A4. CS is *injective* if for each constant c there is at most one formula $c : A \in CS$. Each derivation in **LP** generates the CS consisting of all formulas introduced in this derivation by the *axiom necessitation* rule AN. For a constant specification CS, $\mathbf{LP}(CS)$ is $\mathbf{LP}_0$ plus formulas from CS as additional axioms.

Artemov proves arithmetical soundness of **LP**, for the standard provability interpretation of **LP** we refer to [Art01, section 6], and to prove completeness he introduces a sequent formulation of **LP**. The system he defines in [Art01, section 7] is a weakening-free Gentzen system, which contains rules for contraction. Thus, it is a system of the family of **G2**-Gentzen calculi.

[8] Realization Theorems have been proved (e.g. in [BGK10]) for many further modal logics and their justification counterparts, like (modal logic/justification logic) K/J, D/JD, T/JT, K4/J4 and so on.

4.2 The Gentzen systems LPG1 and LPG2

The sequent system **LPG** introduced by Artemov is a **G2**-style system for **LP**. Since we want to proceed the same way we did in the previous chapters about propositional logic and **S4**, respectively, we first introduce a **G1**-style system for **LP** and then define a weakening- and contraction-free sequent system.

Definition 4.6. The Gentzen system **LPG1** for **LP** is defined by the axioms and rules listed in Figure 4.

The system $\mathbf{LPG1}_0$ is **LPG1** without (Rc).

Remark 4.7. The system **LPG1** is an extension of the **G1**-system **G1c** for propositional logic by the rules (L:1), (R!1), (R+1), (R·1), and (Rc), introducing **LP**-formulas.

In the following definition, we already introduce the weakening-free system for **LP** and prove in a next step, that weakening is depth-preserving admissible for **LPG2**.

Definition 4.8. [Art01, Section 7][9] The system **LPG2** is obtained from **LPG1** by leaving out the weakening rules (LW), (RW), and replacing the axioms by the following two general ones

$$\Gamma, A \Rightarrow A, \Delta \quad (\text{Ax2}) \quad \text{and} \quad \bot, \Gamma \Rightarrow \Delta \quad (\text{L}\bot 2).$$

The system $\mathbf{LPG2}_0$ is **LPG2** without (Rc).

All the rules from the system **LPG1** with a number '1' in their rule-labeling (L∧1, R∨2, L:1, R!1, R+1, R·1) have the number '2' in **LPG2**, although the rules remain the same.

Lemma 4.9. *Weakening is depth-preserving admissible for* **LPG2***, that is*

$$\textit{if} \quad \mathbf{LPG2} \vdash_n \Gamma \Rightarrow \Delta \quad \textit{then} \quad \mathbf{LPG2} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta'.$$

Proof. We prove the lemma by an induction on the depth n of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. Since we already proved the statement for **G2c**, we refer to the corresponding Lemma 2.23 and restrict ourselves to the cases, where the last rule of the deduction $\mathcal{D}$ is one of the justification rules. Let $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$:

Case 1. If the last rule of $\mathcal{D}$ is (L:2)

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, t : A \Rightarrow \Delta}\, L{:}2$$

we apply the induction hypothesis to the premise and have $\mathbf{LPG2} \vdash_{n-1} \Gamma, \Gamma', A \Rightarrow \Delta, \Delta'$. Now we use L:2 to obtain the desired proof $\mathbf{LPG2} \vdash_n \Gamma, \Gamma', t : A \Rightarrow \Delta, \Delta'$.

[9] The system **LPG2** we define here is the system **LPG** Artemov introduces in his work, adapted to our notation.

$$A \Rightarrow A \ (\mathrm{Ax1}) \qquad \bot\Rightarrow \quad (\mathrm{L}\bot 1)$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\, RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\, RC$$

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\, L{\wedge}1\ , (\mathrm{i{=}0,1}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, R{\wedge}$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\, L{\vee} \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\, R{\vee}1\ , (\mathrm{i{=}0,1})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\, L \rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\, R \rightarrow$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, t : A \Rightarrow \Delta}\, L{:}1 \qquad \frac{\Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, !t : t : A}\, R!1$$

$$\frac{\Gamma \Rightarrow \Delta, t_i : A}{\Gamma \Rightarrow \Delta, (t_0 + t_1) : A}\, R{+}1, (\mathrm{i{=}0,1}) \qquad \frac{\Gamma \Rightarrow \Delta, s : (A \rightarrow B) \quad \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, (s \cdot t) : B}\, R{\cdot}1$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, c : A}\, Rc$$

where in (Rc), A is an axiom A0-A4, and c a proof constant.

Figure 4: Gentzen system **LPG1** for **LP**

Case 2. If the last rule of $\mathcal{D}$ is (R!2)

$$\frac{\Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, !t : t : A} R!2$$

we apply the induction hypothesis to the premise and use R!2 to obtain

$$\mathbf{LPG2} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, !t : t : A, \Delta'.$$

Case 3. If the last rule of $\mathcal{D}$ is (R+2)

$$\frac{\Gamma \Rightarrow \Delta, t_i : A}{\Gamma \Rightarrow \Delta, (t_0 + t_1) : A} R{+}2, \text{(i=0,1)}$$

we apply the induction hypothesis to the premise and have

$$\mathbf{LPG2} \vdash_{n-1} \Gamma, \Gamma' \Rightarrow \Delta, t_i : A, \Delta' \text{ for } i \in \{0, 1\}.$$

However, we use (R+2) to obtain

$$\mathbf{LPG2} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, (t_0 + t_1) : A, \Delta'.$$

Case 4. If the last rule of $\mathcal{D}$ is (R·2)

$$\frac{\Gamma \Rightarrow \Delta, s : (A \to B) \quad \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, (s \cdot t) : B} R{\cdot}2$$

we apply the induction hypothesis to the premises, which have deductions of depth $\leq n-1$, to get $\mathbf{LPG2} \vdash_{n-1} \Gamma, \Gamma' \Rightarrow \Delta, s : (A \to B), \Delta'$ and $\mathbf{LPG2} \vdash_{n-1} \Gamma, \Gamma' \Rightarrow \Delta, t : A, \Delta'$. Using (R·2) leads to $\mathbf{LPG2} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, (s \cdot t) : B, \Delta'$.

Case 5. If the last rule of $\mathcal{D}$ is (Rc)

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, c : A} Rc$$

where A is an instance of an axiom A0-A4, and c is a proof constant. We apply the induction hypothesis to the premise, to get $\mathbf{LPG2} \vdash_{n-1} \Gamma, \Gamma' \Rightarrow \Delta, A, \Delta'$ and use (Rc) to obtain $\mathbf{LPG2} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, c : A, \Delta'$.

□

Corollary 4.10. *Weakening is depth-preserving admissible for* $\mathbf{LPG2}_0$.

Proof. It follows from the proof of the lemma for **LPG2** that the statement holds for $\mathbf{LPG2}_0$ too, since we can just omit the case where the last rule of the deduction is (Rc). □

The dp-admissibility of weakening for **LPG2** implies that the systems **LPG1** and **LPG2** are equivalent:

Theorem 4.11.

$$\textbf{LPG1} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \textbf{LPG2} \vdash \Gamma \Rightarrow \Delta.$$

Proof. The justification rules of the two systems are identical, and for the remaining rules we refer to the corresponding proof for **G1c** and **G2c** in section 1 (Theorem 2.24). □

Corollary 4.12.

$$\textbf{LPG1}_\mathbf{0} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \textbf{LPG2}_\mathbf{0} \vdash \Gamma \Rightarrow \Delta.$$

Proof. By definition, the systems $\textbf{LPG1}_0$ and $\textbf{LPG2}_0$, are the systems **LPG1** and **LPG2** without the rule (Rc). In the proof of the equivalence of the systems **LPG1** and **LPG2**, the rule (Rc) is not needed to transfer a deduction from one system into the other, only if the last rule of the deduction is (Rc). But this can not happen, if we consider proofs in the reduced systems. □

Corollary 4.13.

$$\textit{If} \quad \textbf{LPG1} \vdash_n \Gamma \Rightarrow \Delta \quad \textit{then} \quad \textbf{LPG2} \vdash_n \Gamma \Rightarrow \Delta.$$

The contrary does not hold. The statement follows from the corresponding corollary for **G1c** and **G2c** (Corollary 2.25), and the fact that the justification rules of the two systems are the same.
Artemov proved for his systems **LPG2**, $\textbf{LPG2}_0$ that cut is admissible and that the Hilbert system **LP** is equivalent to the Gentzen system **LPG2**.

Theorem 4.14. *Cut is admissible for the systems* **LPG2***,* $\textbf{LPG2}_0$.

Proof. The statement for $\textbf{LPG2}_0$ follows from the arithmetical completeness theorem for $\textbf{LP}_\mathbf{0}$ [TS00, Theorem 8.1], and the proof for **LPG2** is given in [Art01, Corollary 8.13]. □

Theorem 4.15. *The Hilbert system* **LP** *and the sequent system* **LPG2** *are equivalent. The same can be stated for* $\textbf{LP}_0$ *and* $\textbf{LPG2}_0$.

Proof. Again, the proof for $\textbf{LP}_0$ follows from [Art01, Theorem 8.1] and in the proof of [Art01, Corollary 8.13] Artemov states that an analog of his Theorem 8.1 can be proved for **LP**. From this analogue, the equivalence of **LP** and **LPG2** follows. □

From the equivalence of the systems $\textbf{LPG1}_{(0)}$ and $\textbf{LPG2}_{(0)}$ and the previous two theorems we can follow the same properties for $\textbf{LPG1}_{(0)}$ as for $\textbf{LPG2}_{(0)}$:

Corollary 4.16. *Cut is admissible for the systems* **LPG1***,* $\textbf{LPG1}_0$.

Corollary 4.17. *The Hilbert system* **LP** *and the sequent system* **LPG1***, as well as the systems* $\textbf{LP}_0$ *and* $\textbf{LPG1}_0$ *are equivalent.*

Unlike the propositional and the **S4** case, our starting-point was the weakening-free system for **LP** defined in [Art01] and then we found the system **LPG1** containing rules for weakening by reverse engineering. To keep the order of the definitions of the systems (1. system containing structural rules, 2. system not containing weakening, 3. system not containing weakening and contraction), we first introduced the system containing rules for weakening and contraction, although we obtained it from the system **LPG2**. In the next subsection, we make a short side trip to the Kripke semantics for **LP**, and then, we will be ready to define a weakening-, contraction- and cut-free sequent system for **LP**.

4.3 F-models for LP

In this subsection, we introduce the semantics of F-models for the Logic of Proofs, due to Melvin Fitting [Fit05]. However, for F-models, admissible evidence functions and so on, we will use the notation from [Kuz09]. The Kripke semantics for **LP** will be used in the following subsection, to prove the non-invertibility of some rules of the system **LPG2**.

Definition 4.18. [Fit05, Definitions 3.1, 3.2, 3.3] An *F-Model* for $\mathbf{LP}(CS)$ is a quadruple
$\mathcal{M} = (W, R, \mathcal{A}, V)$, where (W, R, V) is a Kripke model with

- a set of worlds $W \neq \emptyset$,
- an accessibility relation $R \subseteq W \times W$, R is reflexive and transitive, and
- a valuation function $V : \Phi \to V(P)$, that assigns to an atomic proposition P the set $V(P) \subseteq W$ of all worlds where this proposition is deemed true.

Finally, an *admissible evidence function* $\mathcal{A} : Tm \times Fm \to 2^W$ assigns to a pair of a term t and a formula F a set $\mathcal{A}(t, F) \subseteq W$ of all worlds where t is deemed admissible evidence for F. $\mathcal{A}$ satisfies the following closure conditions:

$$\begin{array}{rl} C2. & \mathcal{A}(t, F \to G) \cap \mathcal{A}(s, F) \subseteq \mathcal{A}(t \times s, G); \\ C3. & \mathcal{A}(t, F) \subseteq \mathcal{A}(!t : t : F); \\ C4. & \mathcal{A}(t, F) \cup \mathcal{A}(s, F) \subseteq \mathcal{A}(t + s, F); \\ CS. & \mathcal{A}(c, A) = W, \text{ where } c : A \in CS; \\ \text{Monotonicity.} & wRu \text{ and } w \in \mathcal{A}(t, F) \text{ imply } u \in \mathcal{A}(t, F). \end{array}$$

Closure conditions C2, C3 and C4 are required to validate axioms A2, A3, A4 respectively, which is reflected in their numbering.
The forcing relation $\Vdash$ is defined as follows:

- $\mathcal{M}, w \Vdash P$ iff $w \in V(P)$, where $P \in \Phi$;
- $\mathcal{M}, w \Vdash \perp$, never;
- Boolean cases are standard (cp. Definition 3.8);

- $\mathcal{M}, w \Vdash t : F$ iff (1) $\mathcal{M}, u \Vdash F$ for all wRu and (2) $w \in \mathcal{A}(t, F)$.

For the proof of the following theorem, we refer to [Kuz09, Theorem 12]:

Theorem 4.19. *[Fit05, Theorem 8.4]* **LP**(CS) *is sound an complete with respect to its F-models.*

Definition 4.20. Let $\mathcal{F} = (W, R)$ be a Kripke frame. A *possible evidence function on* $\mathcal{F}$ is any function $\mathcal{B} : Tm \times Fm \to 2^W$.

It has to be noted that an admissible evidence function on $\mathcal{F}$ is, by definition, also a possible evidence function on $\mathcal{F}$.

Definition 4.21. [Kuz09, Definition 14] For a given Kripke frame $\mathcal{F} = (W, R)$, we say that a possible evidence function $\mathcal{B}_2$ on $\mathcal{F}$ is *based* on a possible evidence function $\mathcal{B}_1$, also on $\mathcal{F}$, and write $\mathcal{B}_1 \subseteq \mathcal{B}_2$ if $\mathcal{B}_1(t, F) \subseteq \mathcal{B}_2(t, F)$ for any term t and any formula F.

Intuitively, $\mathcal{B} \subseteq \mathcal{A}$ means that admissible evidence function $\mathcal{A}$ satisfies the positive conditions set forth in $\mathcal{B}$. The goal is typically to construct the minimal admissible evidence function based on the given possible evidence function $\mathcal{B}$:

Definition 4.22. [Kuz09, Definition 15] Let $\mathcal{B}$ be a possible evidence function on a Kripke frame $\mathcal{F} = (W, R)$. The *minimal* admissible evidence function $\mathcal{A}$ based on $\mathcal{B}$ must satisfy the following two conditions:

1. it is based on $\mathcal{B}$, i.e. $\mathcal{B} \subseteq \mathcal{A}$;
2. it is the smallest one, i.e., $\mathcal{B} \subseteq \mathcal{A}'$ implies that $\mathcal{A} \subseteq \mathcal{A}'$ for any other admissible evidence function $\mathcal{A}'$ on the same Kripke frame.

The following calculus is due to Nikolai Krupski [Kru06], while the idea goes back to Alexey Mkrtychev [Mkr97].

Definition 4.23. Let CS be a constant specification for **LP**. The axioms and rules of the $*!_{CS}$-calculus for **LP**(CS) are defined as follows:

$*CS$. Axioms: $*\,(c, A)$, where $c : A \in CS$.

$*A2$. *Application Rule* $\dfrac{*(s, F \to G) \quad *\,(t, F)}{*(s \cdot t, F)}$.

$*A3$. *Positive Introspection Rule* $\dfrac{*(t, F)}{*(!t, t : F)}$.

$*A4$. *Sum Rule* $\dfrac{*(s, F)}{*(s + t, F)}$, $\quad \dfrac{*(t, F)}{*(s + t, F)}$.

Definition 4.24. [Kuz09, Definition 22] For a possible evidence function $\mathcal{B}$ on a Kripke frame $\mathcal{F} = (W, R)$ and a world $w \in W$,

$$\mathcal{B}_w^* = \{*(t, F) : w \in \mathcal{B}(t, F)\}.$$

So B_w^* contains $*(t, F)$ iff $w \in \mathcal{B}(t, F)$. In this sense $*$ can be seen as an abbreviation for $w \in \mathcal{B}$.

Theorem 4.25. *[Kuz09, Theorem 23]*[10] *Let $\mathcal{B}$ be a possible evidence function on a Kripke frame $\mathcal{F} = (W, R)$ for* **LP***(CS). Define possible evidence function $\mathcal{A}$ as follows: let*

$$*(t, F) \in \mathcal{A}_w^* \qquad \Longleftrightarrow \qquad \mathcal{B}_w^* \cup \bigcup_{uRw} \mathcal{B}_u^* \vdash_{*!CS} *(t, F).$$

$\mathcal{A}$ so defined is the minimal evidence function based on $\mathcal{B}$.

4.4 A contraction-free sequent system for LP

The standard strategy to obtain a weakening- and contraction-free Gentzen system out of a weakening-free system is - roughly speaking - to leave out the contraction rules, restrict the axiom (Ax2) to atomic formulas, and to modify the rules to make them invertible, as we have shown for propositional logic and **S4**. The depth-preserving invertibility of the rules is the reason for restricting the axiom (Ax2), and the invertibility is necessary to verify depth-preserving admissibility of contraction for the **G3**-system. The procedure for constructing a **G3**-style system for **LP** is obvious for the propositional part: we just take **G3c** as a basis for our system. But to enlarge the system to a full system for **LP** is not that evident. We now continue by the definition of the system **LPG3**, and then the explanation of our approach will follow:

Definition 4.26. The weakening- and contraction-free Gentzen system **LPG3** for **LP** is specified by the axioms and rules listed in Figure 5.

The system $\mathbf{LPG3}_0$ is **LPG3** without (Axc).
The weakening- and contraction-free system **LPG3** is an extension of the system **G3c** for propositional logic, by the justification part, which consists of two axioms: (Axc), (Axt), and four rules: (L:3), (R!3), (R+3), (R·3).

The system **LPG3** has been obtained from **LPG2** by

1. taking the axiom (Ax3) instead of (Ax2);
2. leaving out the contraction rules (LC), (RC);
3. taking the invertible rules (L∧3), (R∨3) instead of (L∧2), (R∨2);
4. embedding contraction into the rules (L:2), (R+2), (R·2) leads to (L:3), (R+3), (R·3);

[10] We restrict ourselves to state the theorem only for **LP**(CS).

$$P, \Gamma \Rightarrow \Delta, P \qquad \text{(Ax3), P atomic} \qquad \bot, \Gamma \Rightarrow \Delta \qquad \text{(L}\bot\text{3)}$$

$$t : A, \Gamma \Rightarrow \Delta, t : A \qquad \text{(Axt)}$$

$$\Gamma \Rightarrow \Delta, c : A \qquad \text{(Axc), c proof constant, A axiom A0-A4}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\, L\wedge 3 \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\, R\vee 3$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\, L\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\, R\rightarrow$$

$$\frac{\Gamma, A, t : A \Rightarrow \Delta}{\Gamma, t : A \Rightarrow \Delta}\, L{:}3 \qquad \frac{\Gamma \Rightarrow \Delta, t : A, !t : t : A}{\Gamma \Rightarrow \Delta, !t : t : A}\, R!3$$

$$\frac{\Gamma \Rightarrow \Delta, t_0 : A, t_1 : A, (t_0 + t_1) : A}{\Gamma \Rightarrow \Delta, (t_0 + t_1) : A}\, R{+}3$$

$$\frac{\Gamma \Rightarrow \Delta, s : (A \rightarrow B), (s \cdot t) : B \quad \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B}{\Gamma \Rightarrow \Delta, (s \cdot t) : B}\, R{\cdot}3$$

Figure 5: Gentzen system **LPG3** for **LP**

5. adding the axiom (Axt);

6. embedding contraction into the rule (R!2) leads to (R!3);

7. replacing the rule (Rc) by an axiom (Axc).

Here is a try to explain, why this are the "right" modifications to obtain **LPG3** from **LPG2**. As we already mentioned, it was clear to take **G3c** as a basis for **LPG3**, since the Logic of Proofs is an extension of propositional logic. Thus the reasons for doing 1., 2. and 3. are clear. We continue with the justification rules. The rules (L:2), (R+2), (R·2) from the system **LPG2** are not invertible. To prove this claim, we have to resort to the semantics of F-Models for the Logic of Proofs, introduced in the previous subsection. We start with the rule (L:2):

- (L:2) is not invertible:

  To prove that (L:2) is not invertible, we construct an F-model $\mathcal{M}'$, such that $\mathcal{M}' \nVdash P \to x : P$. Let $\mathcal{M}' = (W, R, \mathcal{A}, V)$, with $W = \{u, v\}$ and $\mathcal{A}(t, F) = W$, for all terms t and all formulas F. Thus $\mathcal{A}$ is the total admissible evidence function. In addition, we set $V(P) = u$, $R = \{(u, v), (u, u), (v, v)\}$. In this model, we have

  $$\begin{aligned} \mathcal{M}', u \Vdash P \text{ and } \mathcal{M}', v \nVdash P \quad &\Rightarrow \quad \mathcal{M}', u \nVdash x : P \\ &\Rightarrow \quad \mathcal{M}', u \nVdash P \to x : P \\ &\Rightarrow \quad \mathcal{M}' \nVdash P \to x : P. \end{aligned}$$

  With the completeness Theorem 4.19 and with the equivalence of **LP** and **LPG2** (Theorem 4.15) we obtain **LPG2** $\nvdash P \Rightarrow x : P$.

  On the other hand, we have

  $$\textbf{LPG2} \vdash x : P \Rightarrow x : P$$

  since the sequent is an instance of (Ax2).

- (R+2) is not invertible:

  We construct a countermodel $\mathcal{M}'$, such that $\mathcal{M}' \nVdash x : P \to y : P$. Let $\mathcal{M}' = (W, R, \mathcal{A}^*, V)$, with $W = \{u\}$, $V(P) = \{u\} = W$, $R = \{(u, u)\}$, and $\mathcal{A}^*$ is the minimal evidence function based on $\mathcal{B}$, where $\mathcal{B}$ is defined as follows: $\mathcal{B}(x, P) = \{u\}$ and $\mathcal{B}(y, P) = \emptyset$. From Theorem 4.25 we have that

  $$\begin{aligned} *(y, P) \in \mathcal{A}^*_u \quad &\Longleftrightarrow \quad \mathcal{B}^*_u \cup \bigcup_{wRu} \mathcal{B}^*_w \vdash_{*!_{CS}} *(y, P), \\ &\Longleftrightarrow \quad \mathcal{B}^*_u \vdash_{*!_{CS}} *(y, P), \\ &\Longleftrightarrow \quad *(x, P) \vdash_{*!_{CS}} *(y, P). \end{aligned}$$

  From the definition of the $*!_{CS}$-calculus it is clear that $*(x, P) \nvdash_{*!_{CS}} *(y, P)$, and therefore $*(y, P) \notin \mathcal{A}^*_u$. This means nothing but $u \notin \mathcal{A}^*(y, P)$. Thus we have $\mathcal{M}', u \nVdash x : P \to y : P$, and therewith $\mathcal{M}' \nVdash x : P \to y : P$. With the completeness Theorem 4.19 and the equivalence of the systems **LP** and **LPG2** (Theorem 4.15) we obtain:

$$\mathbf{LPG2} \nvdash x : P \Rightarrow y : P.$$

On the other hand, the following **LPG2**-proof

$$\frac{x : P \Rightarrow x : P}{x : P \Rightarrow (x + y) : P} R{+}2$$

shows that $\mathbf{LPG2} \vdash x : P \Rightarrow (x + y) : P$, which proves the non-invertibility of R+2.

- (R·2) is not invertible:

  We construct a countermodel $\mathcal{M}'$, such that $\mathcal{M}' \nVdash (x \cdot y) : P \rightarrow y : Q$. Let $\mathcal{M}' = (W, R, \mathcal{A}, V)$, with $W = \{u\}$, $V(P) = \{u\}$, $V(Q) = \emptyset$, $R = \{(u, u)\}$, and $\mathcal{A}$ is the total evidence function $\mathcal{A}(t, F) = W$ for all terms t and all formulas F. Then we have:

  $$\begin{array}{lcl} \mathcal{M}', u \Vdash P & \Rightarrow & \mathcal{M}', u \Vdash (x \cdot y) : P, \text{ since } u \text{ is the only accessible world for } u, \\ & & \text{and } u \in \mathcal{A}(x \cdot y, P) \text{ by definition of } \mathcal{A}. \\ \mathcal{M}', u \nVdash Q & \Rightarrow & \mathcal{M}', u \nVdash y : Q, \text{ since } u \text{ is the only accessible world for } u. \end{array}$$

  Thus we have

  $$\mathcal{M}', u \nVdash (x \cdot y) : P \rightarrow y : Q \quad \Rightarrow \quad \mathcal{M}' \nVdash (x \cdot y) : P \rightarrow y : Q.$$

  Again, with the completeness Theorem 4.19 and the equivalence of the systems **LP** and **LPG2** (Theorem 4.15) we obtain:

  $$\mathbf{LPG2} \nvdash (x \cdot y) : P \Rightarrow y : Q.$$

  On the other hand, we have that $\mathbf{LPG2} \vdash (x \cdot y) : P \Rightarrow (x \cdot y) : P$, since the sequent is an instance of (Ax2).

It is clear from the three proofs above that we have to take the invertible versions of the rules (L:2), (R+2) and (R·2), that is, the rules with embedded contraction (L:3), (R+3) and (R·3). The dp-invertibility of those rules will just follow from the dp-admissibility of weakening for **LPG3**. Thus, 4. from our list above is also clear.
The next step is to add an additional axiom, namely (Axt) to the system. Why do we have to do that? It turned out to be impossible to derive the sequent $t : A \Rightarrow t : A$ for arbitrary formulas of the form $t : A$ within our temporary system. One reason therefore is that the axiom (Ax3) is restricted to atomic propositions, thus $t : A \Rightarrow t : A$ is obviously no instance of (Ax3). The other reason will become clear with the following example:

$$\cfrac{\cfrac{(x + y) : P, P \Rightarrow x : P, y : P, (x + y) : P}{(x + y) : P, P \Rightarrow (x + y) : P} R + 3}{(x + y) : P \Rightarrow (x + y) : P} L : 3$$

There is no rule we could apply to the top-most sequent of the tree above, to get rid of the terms x and y of the formulas $x : P, y : P$ in the top-most node, to obtain a single P to have a derivation of the desired sequent. The same problem occurs, if we want to derive the sequent $(x \cdot y) : P \Rightarrow (x \cdot y) : P$. These are the reasons for doing 5.
But with (Axt) as an additional axiom in our new temporary system, let us call it **LPG3'**, another problem occurs: the rule (R!2) is not depth-preserving invertible. Counterexample: Since $!t : t : P \Rightarrow !t : t : P$ is an instance of (Axt), we have that $\mathbf{LPG3'} \vdash_0 !t : t : P \Rightarrow !t : t : P$. But obviously $!t : t : P \Rightarrow t : P$ is no instance of (Axt) and no instance of one of the other axioms, too, thus $\mathbf{LPG3'} \nvdash_0 !t : t : P \Rightarrow t : P$. This is the reason for taking the invertible version of (R!2), in other words, the reason for embedding contraction in the rule, thus taking (R!3).
It remains to explain 7.: Instead of having a (Rc)-rule with embedded contraction, we decided to replace the rule by the axiom (Axc). We prefer the axiom, since with this additional axiom in our system, we do not need another rule with contraction embedded, which is absolutely not beneficial if we do proof search. In the rules (L:3), (R!3), (R+3) and (R·3), where contraction has been built in, the premise is even more complicated than the conclusion of the rule, which gives more space to continue, while doing proof search.
This is how the system has been constructed. In the following we prove that **LPG3** has all the properties we want it to have, namely dp-admissibility of weakening, contraction and cut, and dp-invertibility of the rules.

Lemma 4.27. *Weakening is depth-preserving admissible in* **LPG3***, that is*

$$\text{if} \quad \mathbf{LPG3} \vdash_n \Gamma \Rightarrow \Delta \quad \text{then} \quad \mathbf{LPG3} \vdash_n \Gamma, \Gamma' \Rightarrow \Delta, \Delta'.$$

Proof. By induction on n. It remains to consider the cases, where $\Gamma \Rightarrow \Delta$ is an axiom instance of (Axc) or (Axt), and the cases where the last rule of the proof $\mathbf{LPG3} \vdash_n \Gamma \Rightarrow \Delta$ is one of the justification rules. Let $\mathcal{D}$ be a deduction of depth n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$.

Case 1. If the sequent $\Gamma \Rightarrow \Delta$ is an instance of (Axc) or (Axt), then so is $\Gamma', \Gamma \Rightarrow \Delta, \Delta'$.

Case 2. If the last rule of $\mathcal{D}$ is (L:3)

$$\frac{\Gamma, A, t : A \Rightarrow \Delta}{\Gamma, t : A \Rightarrow \Delta} L:3$$

we apply the induction hypothesis to the premise and use (L:3) to obtain
$\mathbf{LPG3} \vdash_n \Gamma', \Gamma, t : A \Rightarrow \Delta, \Delta'$.

Case 3. If the last rule of $\mathcal{D}$ is (R·3)

$$\frac{\Gamma \Rightarrow \Delta, s : (A \rightarrow B), (s \cdot t) : B \quad \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B}{\Gamma \Rightarrow \Delta, (s \cdot t) : B} R\cdot$$

the two premises have derivations of smaller depth, thus we apply the induction hypothesis to obtain

$\mathbf{LPG3} \vdash_{n-1} \Gamma', \Gamma \Rightarrow \Delta, s : (A \rightarrow B), (s{\cdot}t) : B, \Delta'$ and $\mathbf{LPG3} \vdash_{n-1} \Gamma', \Gamma \Rightarrow \Delta, t : A, (s{\cdot}t) : B, \Delta'$.

We use (R·3) to find a proof of $\mathbf{LPG3} \vdash_n \Gamma', \Gamma \Rightarrow \Delta, (s \cdot t) : B, \Delta'$.

The remaining cases are proved similar. □

Corollary 4.28. *Weakening is depth-preserving admissible for* $\mathbf{LPG3}_0$.

Proof. Axiom (Axc) is only used in the proof of the previous lemma, if the sequent $\Gamma \Rightarrow \Delta$ is an instance of (Axc), thus we take the same proof for $\mathbf{LPG3}_0$ and just omit the mentioned case. □

In the following lemma we prove dp-invertibility of the rules in **LPG3**:

Lemma 4.29 (Inversion Lemma)**.** *Let* $\vdash$ *be deducibility in* **LPG3**.

1. *If* $\vdash_n A \wedge B, \Gamma \Rightarrow \Delta$*, then* $\vdash_n A, B, \Gamma \Rightarrow \Delta$.
2. *If* $\vdash_n \Gamma \Rightarrow \Delta, A \vee B$*, then* $\vdash_n \Gamma \Rightarrow \Delta, A, B$.
3. *If* $\vdash_n A \vee B, \Gamma \Rightarrow \Delta$*, then* $\vdash_n A, \Gamma \Rightarrow \Delta$ *and* $\vdash_n B, \Gamma \Rightarrow \Delta$.
4. *If* $\vdash_n \Gamma \Rightarrow \Delta, A \wedge B$*, then* $\vdash_n \Gamma \Rightarrow \Delta, A$ *and* $\vdash_n \Gamma \Rightarrow \Delta, B$.
5. *If* $\vdash_n \Gamma \Rightarrow A \rightarrow B, \Delta$*, then* $\vdash_n \Gamma, A \Rightarrow \Delta, B$.
6. *If* $\vdash_n \Gamma, A \rightarrow B \Rightarrow \Delta$*, then* $\vdash_n \Gamma \Rightarrow \Delta, A$ *and* $\vdash_n \Gamma, B \Rightarrow \Delta$.
7. *If* $\vdash_n \Gamma, t : A \Rightarrow \Delta$*, then* $\vdash_n \Gamma, A, t : A \Rightarrow \Delta$.
8. *If* $\vdash_n \Gamma \Rightarrow \Delta, !t : t : A$*, then* $\vdash_n \Gamma \Rightarrow \Delta, t : A, !t : t : A$.
9. *If* $\vdash_n \Gamma \Rightarrow \Delta, (t + s) : A$*, then* $\vdash_n \Gamma \Rightarrow \Delta, t : A, s : A, (t + s) : A$.
10. *If* $\vdash_n \Gamma \Rightarrow \Delta, (s \cdot t) : B$*, then* $\vdash_n \Gamma \Rightarrow \Delta, s : (A \rightarrow B), (s \cdot t) : B$ *and* $\vdash_n \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B$*, for a formula* A.

Proof. The statements 1.-6. have been proved for **G3c** in the corresponding lemma of section 1 (Lemma 2.31). The proof for **LPG3** works exactly the same, the only case we have to look at, is, if the conclusion of the rule (which we want to be invertible) is an axiom-instance of (Axc) or (Axt): We show the case for the first statement: If $A \wedge B, \Gamma \Rightarrow \Delta$ is an instance of (Axc) or (Axt), then $A \wedge B$ is not principal, thus $A, B, \Gamma \Rightarrow \Delta$ is an axiom-instance too. In addition, it has to be mentioned that no new rule (compared to **G3c**) can have the formula $A \wedge B$ as a principal formula. The same argumentation works for the statements 2.-6.
The statements 7.-10. follow from the dp-admissibility of weakening for **LPG3**. □

Corollary 4.30. *Depth-preserving invertibility of the rules also holds for* $\mathbf{LPG3}_0$.

Proof. (Axc) is only needed in the proof above, when the conclusion of the rule (we want to be invertible) is an instance of (Axc), but this can not be the case in $\mathbf{LPG3}_0$. dp-admissibility of weakening for $\mathbf{LPG3}_0$ has been proved in Corollary 4.28. □

Lemma 4.31. *Contraction is depth-preserving admissible for* **LPG3***, that is*

1. *If* $\quad \mathbf{LPG3} \vdash_n A, A, \Gamma \Rightarrow \Delta \quad$ *then* $\quad \mathbf{LPG3} \vdash_n A, \Gamma \Rightarrow \Delta$*, and*

2. *if* $\quad \mathbf{LPG3} \vdash_n \Gamma \Rightarrow \Delta, A, A \quad$ *then* $\quad \mathbf{LPG3} \vdash_n \Gamma \Rightarrow \Delta, A$

Proof. We prove the lemma by a simultaneous induction on n of both statements. Assume the two statements to have been proved for n.
Let $\mathbf{LPG3} \vdash_{n+1} A, A, \Gamma \Rightarrow \Delta$ by a deduction $\mathcal{D}$.

Case 1. If the sequent $A, A, \Gamma \Rightarrow \Delta$ is an instance of one of the axioms, then so is $A, \Gamma \Rightarrow \Delta$.

Case 2. If the sequent $A, A, \Gamma \Rightarrow \Delta$ is no axiom-instance, and none of the formulas A is principal, we apply the induction hypothesis to the premise(s), which has (have) deduction(s) of smaller depth, and then use the same rule to obtain the deduction of $A, \Gamma \Rightarrow \Delta$:

$$\frac{\vdash_n A, A, \Gamma' \Rightarrow \Delta'}{\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta} R \qquad \Rightarrow (IH) \qquad \frac{\vdash_n A, \Gamma' \Rightarrow \Delta'}{\vdash_{n+1} A, \Gamma \Rightarrow \Delta} R.$$

Case 3. If the sequent $A, A, \Gamma \Rightarrow \Delta$ is no axiom-instance, and one the formulas A is principal, we have to distinct between $A \equiv A_0 \wedge A_1$, $A \equiv A_0 \vee A_1$, $A \equiv A_0 \rightarrow A_1$ and $A \equiv t : B$.

1. If $A \equiv A_0 \wedge A_1$

$$\frac{\vdash_n \Gamma, A_0, A_1, A_0 \wedge A_1 \Rightarrow \Delta}{\vdash_{n+1} \Gamma, A_0 \wedge A_1, A_0 \wedge A_1 \Rightarrow \Delta} L\wedge 3$$

we use the dp-invertibility of the rule L$\wedge$3 to find a proof of $\vdash_n \Gamma, A_0, A_1, A_0, A_1 \Rightarrow \Delta$. Applying the induction hypothesis to this sequent twice, we get $\vdash_n \Gamma, A_0, A_1 \Rightarrow \Delta$ from what we derive $\vdash_{n+1} \Gamma, A_0 \wedge A_1 \Rightarrow \Delta$ by one application of L$\wedge$3.

2. If $A \equiv A_0 \vee A_1$

$$\frac{\vdash_n \Gamma, A_0, A_0 \vee A_1 \Rightarrow \Delta \qquad \vdash_n \Gamma, A_1, A_0 \vee A_1 \Rightarrow \Delta}{\vdash_{n+1} \Gamma, A_0 \vee A_1, A_0 \vee A_1 \Rightarrow \Delta} L\vee$$

we use dp-invertibility of the rule L$\vee$ to obtain $\vdash_n \Gamma, A_0, A_0 \Rightarrow \Delta$ from the left premise, and $\vdash_n \Gamma, A_1, A_1 \Rightarrow \Delta$ from the right premise. Now we apply the induction hypothesis to the two derivations of depth n to contract the two occurrences of A_0, A_1 respectively and apply L$\vee$ to get the desired deduction.

3. If A is of the form $A_0 \rightarrow A_1$,

$$\frac{\vdash_n \Gamma, A_0 \rightarrow A_1 \Rightarrow \Delta, A_0 \qquad \vdash_n A_1, \Gamma, A_0 \rightarrow A_1 \Rightarrow \Delta}{\vdash_{n+1} \Gamma, A_0 \rightarrow A_1, A_0 \rightarrow A_1 \Rightarrow \Delta} L\rightarrow$$

we use dp-invertibility of the rule L$\rightarrow$ to obtain $\vdash_n \Gamma \Rightarrow \Delta, A_0, A_0$ (from the left premise) and $\vdash_n A_1, A_1, \Gamma \Rightarrow \Delta$ (from the right premise). Applying induction hypothesis to those two derivations of depth n leads to $\vdash_n \Gamma \Rightarrow \Delta, A_0$ and $\vdash_n A_1, \Gamma \Rightarrow \Delta$, and we apply one instance of L$\rightarrow$ to obtain a proof of $\vdash_{n+1} \Gamma, A_0 \rightarrow A_1 \Rightarrow \Delta$.

4. If A is of the form $t : B$,

$$\frac{\vdash_n \Gamma, t : B, B, t : B \Rightarrow \Delta}{\vdash_{n+1} \Gamma, t : B, t : B \Rightarrow \Delta} L:3$$

applying the induction hypothesis to the premise we get

$$\vdash_n \Gamma, t : B, B \Rightarrow \Delta$$

and using the rule (L:3) delivers the proof of $\vdash_{n+1} \Gamma, t : B \Rightarrow \Delta$.

Let $\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A$ by a deduction $\mathcal{D}$.

Case 4. If the sequent $\Gamma \Rightarrow \Delta, A, A$ is an instance of one of the axioms, then so is $\Gamma \Rightarrow \Delta, A$.

Case 5. If the sequent $\Gamma \Rightarrow \Delta, A, A$ is no axiom-instance, and none of the formulas A is principal, we apply the IH to the premise(s), which has (have) deduction(s) of smaller depth, and then use the same rule to obtain the deduction of $\Gamma \Rightarrow \Delta, A$:

$$\frac{\vdash_n \Gamma' \Rightarrow \Delta', A, A}{\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A} R \quad \Rightarrow (IH) \quad \frac{\vdash_n \Gamma' \Rightarrow \Delta', A}{\vdash_{n+1} \Gamma \Rightarrow \Delta, A} R.$$

Case 6. If the sequent $\Gamma \Rightarrow \Delta, A, A$ is no axiom-instance, and one the formulas A is principal, we have to distinct between $A \equiv A_0 \wedge A_1$, $A \equiv A_0 \vee A_1$, $A \equiv A_0 \rightarrow A_1$, $A \equiv !t : t : B$, $A \equiv (t + s) : B$, and $A \equiv (s \cdot t) : B$. The propositional cases work similar to the shown cases above. We just consider the justification cases:

1. If A is of the form $!t : t : B$,

$$\frac{\vdash_n \Gamma \Rightarrow \Delta, !t : t : B, t : B, !t : t : B}{\vdash_{n+1} \Gamma \Rightarrow \Delta, !t : t : B, !t : t : B} R!3$$

we apply the induction hypothesis to the premise and find a proof of

$$\vdash_n \Gamma \Rightarrow \Delta, !t : t : B, t : B$$

and applying the rule $R!3$ delivers the proof of $\vdash_{n+1} \Gamma \Rightarrow \Delta, !t : t : B$.

2. If $A \equiv (t + s) : B$

$$\frac{\vdash_n \Gamma \Rightarrow \Delta, (t + s) : B, t : B, s : B, (t + s) : B}{\vdash_{n+1} \Gamma \Rightarrow \Delta, (t + s) : B, (t + s) : B} R{+}3$$

we apply the induction hypothesis to the premise to obtain

$$\vdash_n \Gamma \Rightarrow \Delta, (t + s) : B, t : B, s : B$$

and then we use one application of R+3 to have a derivation of

$$\vdash_{n+1} \Gamma \Rightarrow \Delta, (t + s) : B.$$

3. If $A \equiv (s \cdot t) : B$

$$\frac{\vdash_n \Gamma \Rightarrow \Delta, s : (C \rightarrow B), (s \cdot t) : B, (s \cdot t) : B \qquad \vdash_n \Gamma \Rightarrow \Delta, t : C, (s \cdot t) : B, (s \cdot t) : B}{\vdash_{n+1} \Gamma \Rightarrow \Delta, (s \cdot t) : B, (s \cdot t) : B} R{\cdot}3$$

we apply the induction hypothesis to the premises, to obtain

$$\vdash_n \Gamma \Rightarrow \Delta, s : (C \to B), (s \cdot t) : B \text{ and } \vdash_n \Gamma \Rightarrow \Delta, t : C, (s \cdot t) : B$$

and then use R·3 to find a proof of $\vdash_{n+1} \Gamma \Rightarrow \Delta, (s \cdot t) : B$.

□

Corollary 4.32. *Contraction is dp-admissible for* $\mathbf{LPG3}_0$.

Proof. We replace the dp-invertibility of the rules for **LPG3** in the previous proof by the dp-invertibility of the rules for $\mathbf{LPG3}_0$ (Corollary 4.30). □

Lemma 4.33. *The sequent $A \Rightarrow A$ is derivable in* **LPG3** *for all A.*

Proof. Induction on the complexity of the formula A. The proof for propositional formulas A has been done in the corresponding lemma for the system **G3c** (Lemma 2.34). If A is a formula of the form $t : B$, the sequent $A \Rightarrow A$ is an instance of (Axt). □

Corollary 4.34. *The sequent $A \Rightarrow A$ is derivable in* $\mathbf{LPG3}_0$ *for all A.*

Proof. To prove that $A \Rightarrow A$ is derivable in **LPG3** for all formulas A the axiom (Axc) is not necessary, thus the statement holds for the reduced system $\mathbf{LPG3}_0$ too. □

Now, after we have shown that weakening and contraction are still depth-preserving admissible for **LPG3** ($\mathbf{LPG3}_0$), in other words, that weakening and contraction are still present in the system not containing the structural rules, we are ready to prove the equivalence of **LPG1** and **LPG3**.

Theorem 4.35. *The sequent systems* **LPG1** *and* **LPG3** *are equivalent, that is*

$$\mathbf{LPG1} \vdash \Gamma \Rightarrow \Delta \qquad iff \qquad \mathbf{LPG3} \vdash \Gamma \Rightarrow \Delta.$$

Proof. Both directions are provable by an induction on the depth n of the deduction $\mathcal{D}$ of the sequent $\Gamma \Rightarrow \Delta$.

"$\Rightarrow$": Let $\mathcal{D}$ be a deduction of depth n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$ in **LPG1**.

Case 1. If $\Gamma \Rightarrow \Delta$ is an instance of (Ax1), the sequent is of the form $A \Rightarrow A$, and we know by the previous lemma, that $A \Rightarrow A$ is derivable in **LPG3** for all formulas A. If $\Gamma \Rightarrow \Delta$ is an instance of (L⊥1), it is an instance of (L⊥3) too.

Case 2. If the last rule of $\mathcal{D}$ is (RW)

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\, RW$$

we apply the induction hypothesis to the premise and have $\mathbf{LPG3} \vdash \Gamma \Rightarrow \Delta$, and then by dp-admissibility of weakening we obtain $\mathbf{LPG3} \vdash \Gamma \Rightarrow \Delta, A$.

The case where the last rule of $\mathcal{D}$ is (LW) can be treated symmetrically.

Case 3. If the last rule of the deduction $\mathcal{D}$ is (LC):

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LC$$

By induction hypothesis, **LPG3** $\vdash A, A, \Gamma \Rightarrow \Delta$. Since contraction is admissible in **LPG3**, we have that **LPG3** $\vdash A, \Gamma \Rightarrow \Delta$.

The case where the last rule of $\mathcal{D}$ is (RC) is treated symmetrically.

Case 4. The last rule of the deduction $\mathcal{D}$ is (L$\wedge$1):

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\, L\wedge 1$$

By induction hypothesis, **LPG3** $\vdash A_i, \Gamma \Rightarrow \Delta$. Since weakening is dp-admissible in **LPG3** we can add A_0 or A_1 (it depends on which of the formulas is already there) to the antecedent and get **LPG3** $\vdash A_0, A_1, \Gamma \Rightarrow \Delta$. Now we apply (L$\wedge$3), to obtain a proof of the desired sequent:

$$\frac{A_0, A_1, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\, L\wedge 3.$$

Case 5. The last rule of the deduction $\mathcal{D}$ is (R$\vee$1):

$$\frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\, R\vee 1$$

By induction hypothesis, **LPG3** $\vdash \Gamma \Rightarrow \Delta, A_i$. Since weakening is dp-admissible in **LPG3** we are able to add one of the formulas A_0 or A_1 (it depends on which of the formulas is already there) to the succedent and get **LPG3** $\vdash \Gamma \Rightarrow \Delta, A_0, A_1$. Now we apply (R$\vee$3), to obtain a **LPG3**-proof of the desired sequent:

$$\frac{\Gamma \Rightarrow \Delta, A_0, A_1}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\, R\vee 3.$$

In the cases where the last rule of $\mathcal{D}$ is (L$\vee$), (R$\wedge$), (L$\rightarrow$) and (R$\rightarrow$), we just apply the induction hypothesis to the premise of the rule and apply the same rule in **LPG3**.

Case 6. If the last rule of the deduction $\mathcal{D}$ is (L:1),

$$\frac{A, \Gamma \Rightarrow \Delta}{t : A, \Gamma \Rightarrow \Delta}\, L{:}1$$

we apply the induction hypothesis to the premise and obtain **LPG3** $\vdash A, \Gamma \Rightarrow \Delta$. Since weakening is dp-admissible in **LPG3** we have that **LPG3** $\vdash A, t : A, \Gamma \Rightarrow \Delta$ and we are able to apply the rule (L:3) in **LPG3**:

$$\frac{A, t : A, \Gamma \Rightarrow \Delta}{t : A, \Gamma \Rightarrow \Delta}\, L{:}3$$

The case where the last rule of $\mathcal{D}$ is (R!1) is treated similarly.

Case 7. If the last rule of $\mathcal{D}$ is (R+1)

$$\frac{\Gamma \Rightarrow \Delta, t_i : A}{\Gamma \Rightarrow \Delta, (t_0 + t_1) : A} R{+}1, i \in \{0, 1\}$$

we apply the induction hypothesis to the premise and get **LPG3** $\vdash \Gamma \Rightarrow \Delta, t_i : A$, $i \in \{0, 1\}$. From dp-admissibility of weakening we obtain

$$\textbf{LPG3} \vdash \Gamma \Rightarrow \Delta, t_0 : A, t_1 : A, (t_0 + t_1) : A,$$

and we are able to use (R+3) in **LPG3** to find a desired **LPG3**-derivation of $\Gamma \Rightarrow \Delta, (t_0 + t_1) : A$.

Case 8. If the last rule of $\mathcal{D}$ is (R·1)

$$\frac{\Gamma \Rightarrow \Delta, s : (A \to B) \quad \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, (s \cdot t) : B} R{\cdot}1$$

we apply the induction hypothesis to the premises, which have deductions of smaller depth, and have

$$\textbf{LPG3} \vdash \Gamma \Rightarrow \Delta, s : (A \to B) \text{ and } \textbf{LPG3} \vdash \Gamma \Rightarrow \Delta, t : A.$$

From dp-admissibility of weakening we obtain

$$\textbf{LPG3} \vdash \Gamma \Rightarrow \Delta, s : (A \to B), (s \cdot t) : B \text{ and } \textbf{LPG3} \vdash \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B.$$

Applying (R·3) in **LPG3** leads to a derivation of $\Gamma \Rightarrow \Delta, (s \cdot t) : B$.

Case 9. If the last rule of $\mathcal{D}$ is (Rc)

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, c : A} Rc$$

we do not have a corresponding rule in **LPG3**, but we have an axiom, namely

$$\Gamma \Rightarrow \Delta, c : A \qquad \text{(Axc)}.$$

If **LPG1** $\vdash \Gamma \Rightarrow \Delta, c : A$ then **LPG3** $\vdash \Gamma \Rightarrow \Delta, c : A$, since this sequent will always be an instance of (Axc).

"$\Leftarrow$": Let $\mathcal{D}$ be a deduction of depth n, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$ in **LPG3**.

Case 1. 1. If the sequent $\Gamma \Rightarrow \Delta$ is an instance of (Ax3) or (L$\bot$3), we start by the corresponding axiom instances in **LPG1**, namely $P \Rightarrow P$ (Ax1) and $\bot \Rightarrow$ (L$\bot$1), and derive the desired sequent with a finite number of right- and left weakening-applications.

2. If the sequent $\Gamma \Rightarrow \Delta$ is an instance of (Axt), we start by the axiom instance $t : A \Rightarrow t : A$ of (Ax1) in **LPG1**, and derive the desired sequent with applications of (LW) and (RW).

3. If the sequent $\Gamma \Rightarrow \Delta$ is an instance of (Axc), the sequent is of the form $\Gamma \Rightarrow \Delta, c : A$, where c is a proof constant, and A one of the Hilbert-system axioms A0-A4. Since **LPG1** is equivalent to the Hilbert system **LP** (cp. Corollary 4.17), there is a **LPG1**-derivation of the Hilbert-system axiom A, thus **LPG1** $\vdash \Rightarrow A$. Applying one instance of (Rc) implies that **LPG1** $\vdash \Rightarrow c : A$. To obtain a derivation of the desired sequent, we use (RW) and (LW) until we have **LPG1** $\vdash \Gamma \Rightarrow \Delta, c : A$.

Case 2. If the last rule of the deduction $\mathcal{D}$ is (L∧3):

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\, L{\wedge}3$$

By induction hypothesis we have that **LPG1** $\vdash A, B, \Gamma \Rightarrow \Delta$ and we get a **LPG1**-proof of the desired sequent as follows:

$$\cfrac{\cfrac{\cfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, B, \Gamma \Rightarrow \Delta}\, L{\wedge}1}{A \wedge B, A \wedge B, \Gamma \Rightarrow \Delta}\, L{\wedge}1}{A \wedge B, \Gamma \Rightarrow \Delta}\, LC$$

Case 3. If the last rule of the deduction $\mathcal{D}$ is (R∨3):

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\, R{\vee}3$$

by induction hypothesis we have that **LPG1** $\vdash \Gamma \Rightarrow \Delta, A, B$ and we get a **LPG1**-proof of the desired sequent as follows:

$$\cfrac{\cfrac{\cfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B, B}\, R{\vee}1}{\Gamma \Rightarrow \Delta, A \vee B, A \vee B}\, R{\vee}1}{\Gamma \Rightarrow \Delta, A \vee B}\, RC$$

If the last rule of $\mathcal{D}$ is $L\vee$, $R\wedge$, $L \rightarrow$ or $R \rightarrow$ we apply the induction hypothesis to the premise and apply the same rule in **LPG1**.

Case 4. If the last rule of the deduction $\mathcal{D}$ is (R!3),

$$\frac{\Gamma \Rightarrow \Delta, t : A, !t : t : A}{\Gamma \Rightarrow \Delta, t : t : A}\, R!3$$

by induction hypothesis we have that **LPG1** $\vdash \Gamma \Rightarrow \Delta, t : A, !t : t : A$ and we get a **LPG1**-proof of the desired sequent as follows:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta, t : A, !t : t : A}{\Gamma \Rightarrow \Delta, !t : t : A, !t : t : A}\, R!1}{\Gamma \Rightarrow \Delta, !t : t : A}\, RC$$

The case where the last rule of $\mathcal{D}$ is (L:3) can be treated similarly.

Case 5. If the last rule of the deduction $\mathcal{D}$ is (R+3),

$$\dfrac{\Gamma \Rightarrow \Delta, t : A, s : A, (t+s) : A}{\Gamma \Rightarrow \Delta, (t+s) : A}\, R{+}3$$

by induction hypothesis we have that **LPG1** $\vdash \Gamma \Rightarrow \Delta, t : A, s : A, (t+s) : A$ and we get a **LPG1**-proof of the desired sequent as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, t : A, s : A, (t+s) : A}{\Gamma \Rightarrow \Delta, t : A, (t+s) : A, (t+s) : A}\, R{+}1}{\Gamma \Rightarrow \Delta, t : A, (t+s) : A}\, RC}{\Gamma \Rightarrow \Delta, (t+s) : A, (t+s) : A}\, R{+}1}{\Gamma \Rightarrow \Delta, (t+s) : A}\, RC$$

Case 6. If the last rule of the deduction $\mathcal{D}$ is (R·3),

$$\dfrac{\Gamma \Rightarrow \Delta, s : (A \to B), (s \cdot t) : B \quad \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B}{\Gamma \Rightarrow \Delta, (s \cdot t) : B}\, R{\cdot}3$$

by induction hypothesis we have that **LPG1** $\vdash \Gamma \Rightarrow \Delta, s : (A \to B), (s \cdot t) : B$ and **LPG1** $\vdash \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B$. We get a **LPG1**-proof of the desired sequent as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, s : (A \to B), (s \cdot t) : B \quad \Gamma \Rightarrow \Delta, t : A, (s \cdot t) : B}{\Gamma \Rightarrow \Delta, (s \cdot t) : B, (s \cdot t) : B}\, R{\cdot}1}{\Gamma \Rightarrow \Delta, (s \cdot t) : B}\, RC$$

□

Corollary 4.36.

$$\mathbf{LPG1_0} \vdash \Gamma \Rightarrow \Delta \qquad iff \qquad \mathbf{LPG3_0} \vdash \Gamma \Rightarrow \Delta.$$

Proof. By definition, **LPG1**$_0$ and **LPG3**$_0$ are the corresponding systems without the rule (Rc) and the axiom (Axc), respectively. In the proof of the equivalence of the two systems, the axiom (Axc) in **LPG3** is only needed, if the last rule of the **LPG1**-deduction is (Rc), and (Rc) is only necessary, if the sequent $\Gamma \Rightarrow \Delta$ is an instance of the **LPG3**-axiom (Axc). Thus, the systems **LPG**$_0$ and **LPG3**$_0$ are equivalent. □

Corollary 4.37. *Cut is admissible for the systems* **LPG3** *and* **LPG3**$_0$.

Proof. The statement for **LPG3** follows from the equivalence of **LPG1** and **LPG3** (Theorem 4.35) and cut-admissibility for **LPG1** (Corollary 4.16). The statement for **LPG3**$_0$ follows from the previous corollary and cut-admissibility for **LPG1**$_0$ (Corollary 4.16). □

The following corollary summarizes the equivalences between the different systems for the Logic of Proofs:

Corollary 4.38.

(1) *The systems* **LP**, **LPG1**, **LPG2** *and* **LPG3** *are equivalent.*

(2) *The systems* $\mathbf{LP}_0$, $\mathbf{LPG1}_0$, $\mathbf{LPG2}_0$ *and* $\mathbf{LPG3}_0$ *are equivalent.*

Proof. (1) follows from Theorem 4.15, Theorem 4.11 and Theorem 4.35, statement (2) from Theorem 4.15, Corollary 4.12 and Corollary 4.36. ◻

5 Self-referentiality in G3-systems for S4

In this section we will analyze self-referentiality in contraction-free fragments of **S4**. The Logic of Proofs is the justification counterpart of **S4**, by verifying the realization theorem. In the justification language, it is easy to see, when self-referentiality occurs. Self-referential proofs are valid proofs that prove statements about these same proofs, for instance, $\vdash t : F(t)$. But this *direct* self-referentiality is not the only type of self-referentiality. It could happen that $\vdash t_1 : F(t_2)$ and $\vdash t_2 : F(t_1)$, in other words, one proof refers to the other and vice versa. On the other hand, it is not always clear in the modal language whether this knowledge and that knowledge of the same statement are, in fact, related, and thus self-referentiality occurs. There is a clear connection between the modal language and the language with explicit justifications, so the role of self-referentiality in different fragments of **S4**, can be investigated through its justification counterpart, the Logic of Proofs.

In [Kuz09], Roman Kuznets scrutinized self-referentiality on the logic-level: he proved that for the logics **K**, **D**, **T**, **S4**, **D4** and **K4**, either direct self-referentiality is required already on the level of atomic justifications, or self-referentiality can be avoided. From our perspective, one of his results [Kuz09, Theorem 26] is of particular interest: *the realization of* **S4** *in* **LP** *requires directly self-referential constants and, hence, self-referentiality*. In other words, there is at least one **S4**-theorem ϕ which cannot be realized by an **LP**-theorem ϕ^r such that ϕ^r can be derived in **LP** without using any self-referential statements $t : F(t)$. Since **G3s** is an appropriate sequent system for **S4**, we can follow that there are realizations of proofs in **G3s** with self-referential constants. But how can we find those **G3s**-proofs, whose realizations are calling for self-referentiality in **LP**? This is exactly, what Junhua Yu's paper [Yu09] is about. Compared to R. Kuznets, Yu works on the theorem-level: he considers self-referentiality in realizations of specified theorems. Yu is asking for criteria to find out whether an **S4**-theorem has to call for a self-referential constant specification to prove its realized form in **LP**. Yu introduces *prehistoric phenomena* in **G3s**-derivations and then a specific prehistoric phenomenon, *the left prehistoric loop*, is shown to be necessary for self-referentiality.

We proceed on the theorem-level-track and show that different contraction-free sequent systems represent different fragments of **S4**. First, we introduce the notion of prehistoric phenomena in **G3s**, which we will keep on using for the systems **G3s'** and **G3s***. Then we summarize Yu's work about **G3s**-systems and self-referentiality, and apply his machinery of prehistoric phenomena to **G3s'**-proofs, to check whether realizations of such proofs still call for self-referential constant specifications. In the last subsection, we present the sequent system **G3s***, representing a non-self-referential fragment of **S4**.

5.1 Notations

The notations presented in the following are more or less adopted from [Yu09, chapters 2-5]. Since we will restrict our observations to **G3**-style systems, we omit the '3' added in the rule-names of L$\wedge$3, R$\vee$3, L$\Box$3 and R$\Box$3, and write L$\wedge$, R$\vee$, L$\Box$ and R$\Box$ in the following.

Definition 5.1. In a **G3s**-prooftree $\mathcal{T}$ we define negative, positive and principal-positive families of boxes.

- *Positive* and *negative occurrences of boxes* in a formula, sequent respectively are defined as follows:

  1. The indicated occurrence of $\Box$ in $\Box F$ is positive;
  2. any occurrence of $\Box$ from F in $G \to F$, $G \wedge F$, $F \wedge G$, $G \vee F$, $F \vee G$, $\Box F$ and $\Gamma \Rightarrow \Delta, F$ has the same polarity as the corresponding occurrence of $\Box$ in F;
  3. any occurrence of $\Box$ from F in $\neg F$, $F \to G$ and $F, \Gamma \Rightarrow \Delta$ has a polarity opposite to that of the corresponding occurrence of $\Box$ in F.

- The *direct relation* between occurrences of boxes in a **G3s**-rule application is defined as follows:

  1. Each occurrence of $\Box$ in a side formula A in a premise is directly related only to the corresponding occurrence of $\Box$ in A in the conclusion;
  2. Each occurrence of $\Box$ in an active formula in a premise is directly related only to the corresponding occurrence of $\Box$ in the principal formula of the conclusion.

  A *family* of boxes is an equivalence class with respect to the reflexive transitive closure of the direct relation defined above. We denote families of boxes by $f_0, f_1, ...$ Since cut is not contained in **G3s**, all rules of **G3s** respect the polarity of formulas, hence, each family consists of boxes of the same polarity.

- A family of boxes is positive (negative) if it consists of positive (negative) boxes.

- We say that *a rule introduces a box* from a family f_i, if the box is present in the principal formula of the rule application but not in the active one(s). From this definition it follows that there is only one single rule from **G3s**, which introduces boxes, the R$\Box$-rule. The L$\Box$-rule does not introduce boxes, since the principal formula is always one of the active formulas by definition of the rule.

- The box introduced in an instance of an R$\Box$-rule is called *principal*. A principal box is always positive, and will be denoted by $\blacksquare$:

  $$\frac{\Box\Gamma \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \blacksquare A, \Delta'} R\Box.$$

  The box introduced in an instance of an R$\Box$-rule is the only principal box in the conclusion of this R$\Box$-application.

- A positive family of boxes is principal-positive (essential), if at least one of its boxes is related to a principal box. In other words, we call a family principal-positive, if at least one member of the family has been introduced by an instance of R$\Box$. Also the boxes which belong to principal-positive families, will be denoted by $\blacksquare$. In $\mathcal{T}$, we have only finitely many principal-positive families of boxes, say, $f_1, ..., f_m$. An occurrence of a member of the principal-positive family f_i is denoted by $\blacksquare_i$. For each principal-positive family f_i, there are only finitely many R$\Box$-rules introducing members of this family. This R$\Box$-applications are denoted by $(R\Box)_{i.1}$,...,$(R\Box)_{i.m_i}$, they introduce finitely many boxes, denoted by $\blacksquare_{i.1}, ..., \blacksquare_{i.m_i}$, of the principal positive family f_i.

- In $\mathcal{T}$, the sequent in the premise (conclusion) of $(R\Box)_{i.j}$ is denoted by $P_{i.j}$ $(C_{i.j})$.

Remark 5.2. The L$\Box$-rule is the only rule which can relate two occurrences of boxes in one sequent together. We give two examples for such a rule-application, which relates two occurrences of boxes to one occurrence, one for principal positive boxes and the second one for negative boxes:

1. $$\frac{\Box\neg\blacksquare P, \neg\blacksquare P, \Gamma \Rightarrow \Delta}{\Box\neg\blacksquare P, \Gamma \Rightarrow \Delta} L\Box,$$

2. $$\frac{\Gamma, A \to \Box B, \Box(A \to \Box B) \Rightarrow \Delta}{\Box(A \to \Box B), \Gamma \Rightarrow \Delta} L\Box.$$

In the first example, the two occurrences of the principal positive box (boxing P) in the premise are related to the single occurrence of the same box in the conclusion. In the second example, the two occurrences of the box boxing the formula B are related to the single occurrence of the same box in the conclusion.

Now, we will present some results about properties of families of boxes in **G3s**-proofs, due to Yu. For the proofs of the following lemmas and theorems we refer to the corresponding lemma, theorem in [Yu09].

Lemma 5.3. *[Yu09, Lemma 15] In a* **G3s**-*proof* $\mathcal{T}$*, each family of boxes has exactly one occurrence in the root of the prooftree.*

Theorem 5.4. *[Yu09, Theorem 16] In any sequent in a* **G3s**-*proof, any pair of nested boxes belongs to different families.*

Proof. By induction on the depth of the prooftree $\mathcal{T}$. For the induction step we need the fact that no **G3s**-rule relates two nested boxes in a premise to a same box in the conclusion. $\Box$

Theorem 5.5. *[Yu09, Theorem 17] In a* **G3s**-*proof* $\mathcal{T}$*, if a* $\blacksquare_j$ *occurs in the scope of a* $\blacksquare_i$ *in a sequent, then for any* $\blacksquare_i$ *in any sequent of* $\mathcal{T}$*, there is a* $\blacksquare_j$ *occurring in the scope of this* $\blacksquare_i$*.*

5.2 Prehistoric loops in G3s-proofs and self-referentiality

Definition 5.6. In a given **G3s**-prooftree $\mathcal{T}$, we will identify nodes with their labels. Thus we will talk about occurrences of a sequent in the prooftree. Let T be a set of such occurrences, then $\mathcal{T} = (T, R)$, where $T := \{s_0, s_1, ..., s_z\}$ is the set of occurrences of sequents, and

$$R := \{(s_i, s_j) \in T \times T : s_i \text{ is the conclusion of a rule with the premise } s_j\}$$

is a binary relation. As usual, the reflexive transitive closure of R is denoted by R^*. By s_r, we denote the sequent at the root-node of the prooftree. A path in $\mathcal{T}$ is identified with its maximum node. In particular, each branch is identified with its leaf. Since each path in $\mathcal{T}$ from the root of the prooftree s_r is associated with an unique end-node, we can denote paths by their end-nodes. For any branch s_0 of the form $s_r R^* C_{i.j} R P_{i.j} R^* s_0$ in a **G3s**-proof $\mathcal{T}$, the path $C_{i.j}$ is called a *history of f_i in branch s_0*.

Definition 5.7. [Yu09, Definition 20] *Prehistoric relation* is defined with respect to branches at first, and then with respect to a prooftree. For any principal-positive families of boxes f_i, f_h, and any branch s of the form $s_r R^* C_{i.j} R P_{i.j} R^* s$:

- If $P_{i.j}$ has the form
$$\Box\alpha_1, ..., \Box\alpha_k(\blacksquare_h), ..., \Box\alpha_n \Rightarrow \beta,$$
where $\alpha_k(\blacksquare_h)$ is any formula with an occurrence of $\blacksquare_h$, then f_h is a *left prehistoric family in s of f_i*. Notation: $h \prec^s_L i$.

- If $P_{i.j}$ has the form
$$\Box\alpha_1, ..., \Box\alpha_n \Rightarrow \beta(\blacksquare_h),$$
then f_h is a *right prehistoric family in s of f_i*. Notation: $h \prec^s_R i$.

- The relation of *prehistoric family in s* is defined by: $\prec^s := \prec^s_L \cup \prec^s_R$.

- In a **G3s**-proof $\mathcal{T}$, binary relations of left prehistoric, right prehistoric and prehistoric is defined by: $\prec_L := \bigcup\{\prec^s_L : s \text{ is a leaf of } \mathcal{T}\}$, $\prec_R := \bigcup\{\prec^s_R : s \text{ is a leaf of } \mathcal{T}\}$, $\prec := \prec_L \cup \prec_R$.

- To denote one of $\prec$, $\prec_L$, $\prec_R$, $\prec^s$, $\prec^s_L$ or $\prec^s_R$, we write $\lhd$.

While the right prehistoric relation can be seen from the form of the succedent of the conclusion of an R$\Box$-application, the left prehistoric relation is not very obvious. This is why we give an example for this relation:

Example 5.8. We consider the following **G3s**-proof of the **S4**-formula

$$\Phi = \Box\neg\Box\Diamond\Box P \rightarrow \neg\Box P$$

$$
\begin{prooftree}
\AxiomC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, P, \Box\neg\blacksquare_1 P \Rightarrow P$}
\RightLabel{$L\Box$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow P$}
\RightLabel{$R\Box_1$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow \blacksquare_1 P, \bot$}
\AxiomC{$\bot, \ldots \Rightarrow \bot$}
\RightLabel{$L\rightarrow$}
\BinaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P, \neg\blacksquare_1 P \Rightarrow \bot$}
\RightLabel{$L\Box$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow \bot$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P \Rightarrow \neg\Box\neg\blacksquare_1 P$}
\RightLabel{$R\Box_2$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \bot$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \neg\Box P$}
\AxiomC{$\bot, \ldots \Rightarrow \neg\Box P$}
\RightLabel{$L\rightarrow$}
\BinaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P$}
\RightLabel{$L\Box$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Rightarrow \Box\neg\blacksquare_2\Diamond\blacksquare_1 P \rightarrow \neg\Box P$}
\end{prooftree}
$$

We just indicate the right premises of the two L→-applications, but it is clear that the sequents are instances of (L⊥3). The two R□-applications are denoted by R$\Box_1$ and R$\Box_2$, they introduce one principal positive family each, f_1 an f_2. Each occurrence of a principal positive box in the prooftree, is a member of either family f_1 or f_2, and denoted by $\blacksquare_1$, $\blacksquare_2$ respectively. So let us consider the premises of the two R□-rules to check whether there are such left (right) prehistoric families. The premise of R$\Box_1$ is

$$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow P.$$

From the succedent it is clear that none of the families f_1, f_2, is a right prehistoric family of the other one, since there is no principal positive box-occurrence on the right side of the premise. In the antecedent, we have to look for principal positive boxes in the scope of a negative box, and find the two formulas

$$\phi_1 = \Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \qquad\qquad \phi_2 = \Box\neg\blacksquare_1 P.$$

In ϕ_1, there are even two occurrences of principal positive boxes ($\blacksquare_1, \blacksquare_2$) in the scope of a negative box-occurrence. Since we are considering the rule-application introducing $\blacksquare_1$, we have that f_1 is a left prehistoric family of itself, notation: $1 \prec_L 1$, and f_2 is a left prehistoric family of f_1, $2 \prec_L 1$. From ϕ_2, it follows again that f_1 is a left prehistoric family of itself. To continue our observations, we consider the premise of the second R□-application, R$\Box_2$, introducing family f_2:

$$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P \Rightarrow \neg\Box\neg\blacksquare_1 P.$$

There is an occurrence of $\blacksquare_1$ in the formula present in the succedent of the sequent. From the conclusion of R$\Box_2$ it is even more evident that f_1 is a right prehistoric family of f_2: $\blacksquare_2\Diamond\blacksquare_1 P$. In the antecedent, there is an occurrence of ϕ_1 again, and therefore we have that f_2 is a left prehistoric family of itself, $2 \prec_L 2$, and f_1 is a left prehistoric family of f_2, $1 \prec_L 2$. In summary, we have $1 \prec_L 1, 1 \prec_L 2, 2 \prec_L 1, 2 \prec_L 2$ and $1 \prec_R 2$.

Lemma 5.9. *[Yu09, Lemma 21] The following are equivalent:*

1. *$h \prec^s i$;*

2. *In the branch s, there is a sequent s' with an occurrence of $\blacksquare_h$ in it. There is also a history of f_i in s, which does not include s'.*

Remark 5.10.

1. Each history in a branch s breaks the branch into two parts, the historic period (from the conclusion of the R□-rule to the root of the proof tree), and the prehistoric period (from the leaf of the branch to the premise of the R□-rule):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{s}{\dots}}{P_{i.j}}}{C_{i.j}}\, R\Box_{i.j}}{\dots}}{s_r}$$

2. By the previous lemma we know that $h \prec^s i$ iff $\blacksquare_h$ has an occurrence in a prehistoric period of f_i in s. This is the reason why Yu calls the $\prec$-relation prehistoric relation.

The following corollaries are left without proof:

Corollary 5.11. *[Yu09, Corollary 22] For any principal-positive family f_i, $i \nprec_R i$.*

Corollary 5.12. *[Yu09, Corollary 24] If $k \prec_R j$ and $j \lhd i$, then $k \lhd i$.*

Definition 5.13. [Yu09, Definition 25] (Prehistoric Loop) In a **G3s**-proof $\mathcal{T}$, the ordered sequence of principal-positive families $f_{i_1}, ..., f_{i_n}$ is called

- a *prehistoric loop*, if $i_1 \prec i_2 \prec ... \prec i_n \prec i_1$, and
- a *left prehistoric loop*, if $i_1 \prec_L i_2 \prec_L ... \prec_L i_n \prec_L i_1$.

Remark 5.14. The **G3s**-proof of the formula $\Phi = \Box\neg\Box\Diamond\Box p \to \neg\Box p$ presented in Example 5.8, has a left prehistoric loop: $1 \prec_L 2 \prec_L 1$.

The following theorem states that the $\prec_L$'s are the only essential steps in a prehistoric loop:

Theorem 5.15. *[Yu09, Theorem 26] $\mathcal{T}$ has a prehistoric loop iff $\mathcal{T}$ has a left prehistoric loop.*

Before we present the main result of Yu's paper [Yu09], which provides a connection between his notion of prehistoric loops and self-referentiality, we have to define (direct) self-referential modal reasoning:

Definition 5.16. [Kuz09, Definition 3] Modal reasoning in a modal logic ML, as represented by its justification counterpart JL, is *not directly self-referential* if each modal theorem ϕ of ML can be realized by an justification theorem ϕ^r that can be derived in JL without using any self-referential statements $t : F(t)$.
The reasoning of ML and JL is *not self-referential* if the realization of each modal theorem ϕ can be achieved without using any cycles of references, such as

$$t_2 : F_1(t_1), ..., t_n : F_{n-1}(t_{n-1}), t_1 : F_n(t_n).$$

Theorem 5.17. *[Kuz09, Theorem 26] Realization of* **S4** *in* **LP***, of* **D4** *in* **JD4***, and of* **T** *in* **JT** *requires directly self-referential constants and, hence, direct self-referentiality.*

Definition 5.18. [Yu09, Definition 6] The constant specification CS of a derivation in **LP** is *not direct self-referential* if CS does not contain any formulas of the form $t : F(t)$.
Notation: CS^*.
CS is *not self-referential*, if it does not contain cycles of references

$$t_2 : F_1(t_1), ..., t_n : F_{n-1}(t_{n-1}), t_1 : F_n(t_n).$$

Notation: $CS^{\otimes}$.

Theorem 5.19. *[Yu09, Theorem 30](Necessity of Left Prehistoric Loop for Self-ref.) If an* **S4***-theorem ϕ has a left-prehistoric-loop-free* **G3s***-proof, then there is an* **LP***-formula ψ such that $\psi^\circ = \phi$ and $\vdash_{\mathbf{LP}(CS^{\otimes})} \psi$, for some $CS^{\otimes}$.*

The previous theorem states that whenever we can derive an **S4**-theorem ϕ in the sequent system **G3s** without loop, there is a realization of ϕ which is derivable in **LP** with a non-self-referential constant specification. In this subsection, the left prehistoric relation turned out to be the determinant relation among the prehistoric phenomena. Considering the rules in **G3s**, the modal rules introducing boxes play an important rule concerning left prehistoric loops. The L□-rule, the only rule which can relate two box-occurrences in one sequent together, determines the family-wise-situation of a proof. On the other hand, the R□-rule defines the prehistoric relation between principal positive families of boxes. Roughly speaking, it is the L□-rule with its embedded contraction, which causes at least partially, the development of a left prehistoric loop. But what happens, if we replace the L□-rule with the built-in contraction (L□3), by the rule without contraction (L□1)? Are there still occurring loops in such derivations? The following subsection will give an answer to this question.

5.3 Prehistoric loops in G3s'-proofs and self-referentiality

In the previous subsection, the contraction embedded in the L□-rule turned out to be one reason, or maybe even *the* reason, for a loop occurring in a **G3s**-proof. In the current subsection, we dispel contraction from the L□-rule, which leads us to the modified system **G3s'**, and try to find out whether loops can still appear in such proofs.
In section 3.5 we introduced the sequent system **G3s'**, the system we get from **G3s** if we replace L□3 by

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} L\Box 1.$$

L$\Box$1 is not invertible, which transpired to be the reason for contraction not being admissible for the system. This fact in turn, breeds the incompleteness of **G3s'**. At first sight, it could seem strange to work with an incomplete system. But fortunately, the system is sound (Theorem 3.50) and therefore does not derive sequents, which are not valid in **S4**. The question we would like to answer now is, if all the proofs in **G3s'** can be realized without self-referentiality.

We use the same definition of families of boxes for the system **G3s'** as for **G3s** (cp. Definition 5.1).
A a first important property of **G3s'**-proofs is, that in **G3s**$'$ there is no rule, which relates two occurrences of boxes in one sequent together. The reason therefore is, of course, the missing contraction in the L$\Box$1-rule of this system. As a consequence of this important fact we are able to amplify Lemma 5.3 for the system **G3s**$'$. It is not only in the root sequent of a derivation, where each family of boxes has exactly one occurrence, it is the case for each sequent of the derivation:

Lemma 5.20. *In all the sequents of a* **G3s**$'$*-proof, more than one box of the same family can never occur.*

Proof. From Lemma 5.3 (which also holds for **G3s**$'$, since none of the rules relates two occurrences of boxes in the conclusion to a same occurrence in a premise) we know that each family of boxes has exactly one occurrence in the root-sequent of each prooftree. There is no rule in **G3s**$'$ which relates an occurrence of a box from the conclusion to two occurrences of boxes in the premise of the rule. The three propositional two-premise rules, namely L$\rightarrow$, L$\vee$ and R$\wedge$, relate an occurrence of a box in the conclusion to two occurrences of boxes, but only each in one premise. For example:

$$\frac{\Gamma \Rightarrow \Box A, B \quad \Gamma \Rightarrow \Box A, C}{\Gamma \Rightarrow \Box A, B \wedge C} R\wedge.$$

Thus, we will never find two occurrences of boxes of the same family in one sequent of a **G3s**$'$-derivation. □

Remark 5.21. In the following, we will use the same definitions of left and right prehistoric families and (left) prehistoric loops for **G3s**$'$-proofs as we defined it for **G3s**-derivations.

The next properties of the sequent system we introduce, are consequences of Lemma 5.20:

Corollary 5.22. *If f_i is a principal positive family in a* **G3s**$'$*-proof, then*

1. $i \not\prec_L i$,
2. $i \not\prec_R i$.

Proof. The first statement is a consequence of the previous lemma: Assume that there is a principal positive family f_k such that $k \prec_L k$. In the **G3s′**proof, there has to occur an R$\Box$ application of the form:

$$\frac{\Box\Gamma, \Box\alpha(\blacksquare_k) \Rightarrow A}{\Gamma', \Box\Gamma, \Box\alpha(\blacksquare_k) \Rightarrow \blacksquare_k A, \Delta'}\ R\Box_k.$$

In the conclusion of this rule occur two boxes of the same family f_k, which is a contradiction to Lemma 5.20.
The second statement can be proved by an induction on the **G3s′**-derivation. For the induction step, we need the fact, that no **G3s′**-rule relates two nested boxes in a premise to a same box in the conclusion. $\Box$

Remark 5.23. The fact, that any principal positive family in an arbitrary **G3s’**-proof can not be a left prehistoric family of itself ($i \nprec_L i$) constitutes a significant difference between the systems **G3s′** and **G3s**.

Lemma 5.24. *The system* **G3s’** *enjoys the subformula property, that is, every rule from* **G3s’** *has the subformula property.*

Proof. By Definition 2.17, a sequent system rule has the subformula property, if the active formulas of the rule are subformulas of the principal formula. An inspection of the rules from **G3s’** shows that there is no active formula, which is not a subformula of the principal formula of the rule. $\Box$

Corollary 5.25. *In a* **G3s’**-*proof $\mathcal{T}$, if the formula A occurs in a sequent q of any branch, A occurs as a subformula in each sequent below the node labeled with q, until s_r.*

Proof. Lets consider the node s_q of the prooftree labeled with the sequent q. From the previous lemma we know that no matter which **G3s’**-rule R we apply to the node s_q (if R is a two-premise rule, s_q is labeled with q and another sequent), the formula A occurs as a subformula in the sequent of the conclusion of R. Since this holds for any rule, we can be sure that A occurs as a subformula in every single node below s_q until the root-node s_r of the derivation $\mathcal{T}$. $\Box$

Corollary 5.26. *In* **G3s′**, *each principal positive family of boxes $\blacksquare_i$ can be introduced only once in each branch s of the proof. In other words, there is only one R$\Box$-application, R$\Box_i$, for each principal positive family f_i in each branch s of the proof.*

Proof. Consider the branch s in a **G3s′**-proof $\mathcal{T}$ and assume that there are two R$\Box$-rules, R$\Box_{i,j_1}$ and R$\Box_{i,j_2}$, introducing the same principal positive family f_i. The branch s

has the form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s}{\ldots}}{\Box\Gamma \Rightarrow A}}{\Gamma', \Box\Gamma \Rightarrow \blacksquare_{i.j_1} A, \Delta'}\, R\Box_{i.j_1}}{\ldots}}{\Box\psi(\blacksquare_{i.j_1} A), \Box\Psi \Rightarrow A}}{\Psi', \Box\psi(\blacksquare_{i.j_1} A), \Box\Psi \Rightarrow \blacksquare_{i.j_2} A, \Phi'}\, R\Box_{i.j_2}
$$

$$\ldots$$

From Corollary 5.25 we know that the formula $\blacksquare_i A$ occurs as a subformula in the conclusion of the second R$\Box$ application, which contradicts Lemma 5.20. $\Box$

Corollary 5.27. *In* **G3s'***-proofs, occurrences of principal positive boxes* $\blacksquare_i$ *appear only in the historic period of* $R\Box_i$*, that is, the sequents from the conclusion of the* $R\Box_i$*-rule to the root of the tree.*

Proof. Assume that $\blacksquare_i$ which is going to be introduced by R$\Box_i$, occurs already in the premise of the rule:

Case 1. If $\blacksquare_i$ occurs in the antecedent of the premise

$$\dfrac{\Box\Gamma, \Box\alpha(\blacksquare_i) \Rightarrow A}{\Gamma', \Box\Gamma, \Box\alpha(\blacksquare_i) \Rightarrow \blacksquare_i A, \Delta'}\, R\Box_i$$

then there are obviously two occurrences of boxes of the same family in the conclusion of the R$\Box$-rule, which is a contradiction to Lemma 5.20.

Case 2. If $\blacksquare_i$ occurs in the succedent of the premise

$$\dfrac{\Box\Gamma \Rightarrow \alpha(\blacksquare_i)}{\Gamma', \Box\Gamma \Rightarrow \blacksquare_i \alpha(\blacksquare_i), \Delta'}\, R\Box_i$$

then we have that $i \prec_R i$, which is a contradiction to Corollary 5.22.

$\Box$

Breaking down the definition of a (left) prehistoric loop from a whole derivation to the branches of a prooftree, we obtain an interesting property of **G3s'** from the previous two corollaries:

Definition 5.28. Let $\mathcal{T}$ be a **G3s'**-proof. The ordered sequence of principal-positive families $f_{i_1}, ..., f_{i_n}$ occurring in the branch s is called

- a *prehistoric loop in the branch* s, if $i_1 \prec^s i_2 \prec^s ... \prec^s i_n \prec^s i_1$, and
- a *left prehistoric loop in* s, if $i_1 \prec^s_L i_2 \prec^s_L ... \prec^s_L i_n \prec^s_L i_1$.

Corollary 5.29. *Let* $\mathcal{T}$ *be a* **G3s'***-proof. Any branch* s *in* $\mathcal{T}$ *cannot have a left prehistoric loop.*

Proof. Assume, for the sake of a contradiction, that there is a branch s in a **G3s'**-derivation with a left prehistoric loop:

$$i_1 \prec_L^s i_2 \prec_L^s \ldots \prec_L^s i_n \prec_L^s i_1.$$

W.l.o.g. the branch s is of the form:

$$\begin{array}{c} \vdots \\ \dfrac{\Box\Gamma_2, \Box\alpha_1(\blacksquare_1) \Rightarrow A_2}{\Gamma_2', \Box\Gamma_2, \Box\alpha_1(\blacksquare_1) \Rightarrow \blacksquare_2 A_2, \Delta_2'} R\Box_2 \\ \vdots \\ \dfrac{\Box\Gamma_n, \Box\alpha_{n-1}(\blacksquare_{n-1}) \Rightarrow A_n}{\Gamma_n', \Box\Gamma_n, \Box\alpha_{n-1}(\blacksquare_{n-1}) \Rightarrow \blacksquare_n A_n, \Delta_n'} R\Box_n \\ \vdots \\ \dfrac{\Box\Gamma_1, \Box\alpha_n(\blacksquare_n) \Rightarrow A_1}{\Gamma_1', \Box\Gamma_1, \Box\alpha_n(\blacksquare_n) \Rightarrow \blacksquare_1 A_1, \Delta_1'} R\Box_1 \\ \vdots \end{array}$$

Since $i_1 \prec_L^s i_2$, there has to be an occurrence of $\blacksquare_1$ in the scope of a negative box in the antecedent of the premise of R$\Box_2$. From Corollary 5.25 we know that there is an occurrence of $\blacksquare_1$ in each sequent of the derivation below the R$\Box_2$-application until the root sequent. So there is an occurrence of $\blacksquare_1$ in the conclusion of the R$\Box_1$-application too. If the $\blacksquare_1$ occurs in the succedent of the R$\Box_1$-rule, we have that $i_1 \prec_R^s i_1$ which contradicts Corollary 5.22. If the $\blacksquare_1$ occurs in the antecedent of the R$\Box_1$-rule, there are two occurrences of $\blacksquare_1$ in the same sequent (the second occurrence is the $\blacksquare_1$ introduced by R$\Box_1$). This is a contradiction to Lemma 5.20. □

The following theorem is a property of **G3s** which holds for **G3s**$'$ too:

Theorem 5.30. *In a* **G3s**$'$*-proof* $\mathcal{T}$*, if a* $\blacksquare_j$ *occurs in the scope of a* $\blacksquare_i$ *in a sequent s, then for any* $\blacksquare_i$ *in any sequent of* $\mathcal{T}$*, there is a* $\blacksquare_j$ *occurring in the scope of this* $\blacksquare_i$*.*

Proof. Consider a formula of the form $\phi(\blacksquare_i\psi(\blacksquare_j))$ in the node, labeled with the sequent s_l, of $\mathcal{T}$. From Corollary 5.25 we know that the formula ϕ occurs as a subformula in each sequent below, until the root s_r of $\mathcal{T}$.
Assume, for the sake of a contradiction, that there is an occurrence of $\blacksquare_i$ in the formula $\phi'(\blacksquare_i)$ in the node, labeled with the sequent s_k, of $\mathcal{T}$, such that there is no $\blacksquare_j$ in the scope of it. We have to consider the following cases:

Case 1. If $s_l = s_k$, this is a contradiction to Lemma 5.20.

Case 2. If $s_l \neq s_k$, but s_l and s_k are sequents of the same branch in $\mathcal{T}$. W.l.o.g. let s_k be closer to the root-node s_r than s_l. From Corollary 5.25 we know that there is an occurrence of $\blacksquare_i\psi(\blacksquare_j)$ in s_k. By assumption, there is another occurrence of $\blacksquare_i$ (without $\blacksquare_j$ in the scope of it) in s_k, which contradicts Lemma 5.20 again.

Case 3. If $s_l \neq s_k$, and s_l, s_k are sequents in different branches of $\mathcal{T}$. In one node of $\mathcal{T}$, the two branches get connected by one of the two-premise-rules. Since the formula Φ, where $\phi(\blacksquare_i\psi(\blacksquare_j))$ occurs as a subformula, is different from the formula Φ', where $\phi'(\blacksquare_i)$ occurs as a subformula, the formulas Φ, Φ' have to be the active formulas of the two-premise rule applied. Thus, there are two occurrences of $\blacksquare_i$ in the principal formula of the rule connecting the two branches, which is a contradiction to Lemma 5.20.

$\square$

The following property of **G3s** (Corollary 5.12) is also provable for the system **G3s′**:

Corollary 5.31. *If i, j and k are different principal positive families in a* **G3s′***-proof, then*

$$\text{if } k \prec_R j \text{ and } j \lhd i, \text{ then } k \lhd i.$$

Proof. From $k \prec_R j$ we know that there is an R$\square_j$-rule of the form

$$\frac{\square\Gamma \Rightarrow \alpha(\blacksquare_k)}{\Gamma', \square\Gamma \Rightarrow \blacksquare_j\alpha(\blacksquare_k), \Delta'} R\square_j.$$

With the previous theorem we can be sure, that wherever $\blacksquare_j$ occurs, there is a $\blacksquare_k$ occurring in the scope of it. Keeping this fact in mind, we prove the corollary for the different relations:

Case 1. $\lhd = \prec_L^s$. $j \prec_L^s i$ implies that branch s is of the form:

$$\cfrac{\cfrac{\cfrac{s}{\cdots}}{\square\Gamma, \gamma(\blacksquare_j) \Rightarrow \beta}}{\cfrac{\Gamma', \square\Gamma, \gamma(\blacksquare_j) \Rightarrow \blacksquare_i\beta, \Delta'}{\cfrac{\cdots}{s_r}}} R\square_i$$

Since we know that there is a $\blacksquare_k$ occurring in the scope of $\blacksquare_j$, we have the desired property $k \prec_L^s i$.

Case 2. $\lhd = \prec_R^s$. $j \prec_R^s i$ implies that branch s is of the form:

$$\cfrac{\cfrac{\cfrac{s}{\cdots}}{\square\Gamma \Rightarrow \beta(\blacksquare_j)}}{\cfrac{\Gamma', \square\Gamma \Rightarrow \blacksquare_i\beta(\blacksquare_j), \Delta'}{\cfrac{\cdots}{s_r}}} R\square_i$$

Again we can use the fact, that there is a $\blacksquare_k$ in the scope of $\blacksquare_j$ to get $k \prec_R^s i$.

Case 3. $\lhd = \prec^s$. This case follows from cases 1 and 2.

Case 4. $\lhd = \prec_L$. $j \prec_L i$ implies that there is a branch s with $j \prec_L^s i$. From case 1 it follows that $k \prec_L^s i$, that is $k \prec_L i$.

Case 5. $\lhd = \prec_R$. $j \prec_R i$ implies that there is a branch s with $j \prec_R^s i$. From case 2 it follows that $k \prec_R^s i$, that is $k \prec_R i$.

Case 6. $\lhd = \prec$. This case follows from cases 4 and 5.

□

Now we showed all the properties of the system **G3s′** we need, to prove the following theorem:

Theorem 5.32. *If $\mathcal{T}$ is a* **G3s′***-proof, the following are equivalent:*

1. *$\mathcal{T}$ has a prehistoric loop.*
2. *$\mathcal{T}$ has a left prehistoric loop.*

Proof. The direction from 2. to 1. is clear. Direction from 1. to 2.: Consider a prehistoric loop $i_1 \prec i_2 \prec \ldots \prec i_n \prec i_1$ of smallest length.

Case 1. If all relations $\prec$ in the loop we consider are of the form $\prec_R$, the loop is $i_1 \prec_R i_2 \prec_R \ldots \prec_R i_n \prec_R i_1$ and from Corollary 5.31 it follows that $i_1 \prec_R i_1$ which is a contradiction to Lemma 5.22.

Case 2. If there are both, occurrences of $\prec_L$ and $\prec_R$ in the loop, w.l.o.g. we assume the loop to be of the form $i_1 \prec_R i_2 \prec_L i_3 \prec \ldots \prec i_1$. By Corollary 5.31 we obtain that $i_1 \prec_L i_3 \prec \ldots \prec i_1$ is a shorter prehistoric loop. This is a contradiction to our assumption that the loop we consider is of smallest length.

Case 3. If all relations $\prec$ in the loop we consider are of the form $\prec_L$, the loop is a left prehistoric loop. So the shortest prehistoric loop is always a left one.

□

In Corollary 5.29 we proved that in any branch of a **G3s’**-derivation, there is no left prehistoric loop. To state that there are no prehistoric loops at all, we need the following corollary, an analog of Corollary 5.31.

Corollary 5.33. *If i, j and k are different principal positive families in a* **G3s′***-proof, then*

1. *If $k \prec_R^s j$ and $j \prec_R^s i$, then $k \prec_R^s i$.*
2. *If $k \prec_R^s j$ and $j \prec_L^s i$, then $k \prec_L^s i$.*

Proof. From $k \prec_R^s j$ we know that the R$\Box_j$-application in the branch s is of the form

$$\frac{\Box\Gamma \Rightarrow \alpha(\blacksquare_k)}{\Gamma', \Box\Gamma \Rightarrow \blacksquare_j \alpha(\blacksquare_k), \Delta'}\, R\Box_j.$$

With Theorem 5.30 we can be sure, that wherever $\blacksquare_j$ occurs in this **G3s’**-derivation, there is a $\blacksquare_k$ occurring in the scope of it.

1.: $j \prec_L^s i$ implies that the R$\Box_i$-application in branch s is of the form:

$$\frac{\Box\Gamma, \gamma(\blacksquare_j) \Rightarrow \beta}{\Gamma', \Box\Gamma, \gamma(\blacksquare_j) \Rightarrow \blacksquare_i\beta, \Delta'} R\Box_i$$

Since we know that there is a $\blacksquare_k$ occurring in the scope of $\blacksquare_j$, we have the desired property $k \prec_L^s i$.
2.: $j \prec_R^s i$ implies that the R$\Box_i$-application in branch s is of the form:

$$\frac{\Box\Gamma \Rightarrow \beta(\blacksquare_j)}{\Gamma', \Box\Gamma \Rightarrow \blacksquare_i\beta(\blacksquare_j), \Delta'} R\Box_i$$

Again we can use the fact, that there is a $\blacksquare_k$ in the scope of $\blacksquare_j$ to get $k \prec_R^s i$.
From 1. and 2. it follows that if $k \prec_R^s j$ and $j \prec^s i$ then $k \prec^s i$. $\Box$

Corollary 5.34. *Let $\mathcal{T}$ be a* **G3s'***-proof. Any branch s in $\mathcal{T}$ cannot have a prehistoric loop.*

Proof. Assume the branch s in $\mathcal{T}$ to have a prehistoric loop, $i_1 \prec^s i_2 \prec^s ... \prec^s i_n \prec^s i_1$.

Case 1. If all occurrences of $\prec^s$ in the loop are of the form $\prec_R^s$, we have that $i_1 \prec_R^s i_2 \prec_R^s ... \prec_R^s i_n \prec_R^s i_1$. With the previous corollary it follows that $i_1 \prec_R^s i_1$, which contradicts Lemma 5.22.

Case 2. If there are both, occurrences of $\prec_L^s$ and $\prec_R^s$ in the prehistoric loop we consider, w.l.o.g. the loop is of the form $i_1 \prec_R^s i_2 \prec_L^s i_3 \prec^s ... \prec^s i_n \prec^s i_1$. The previous corollary implies that $i_1 \prec_L^s i_3 \prec^s ... \prec^s i_n \prec^s i_1$ is a prehistoric loop too, while having less occurrences of $\prec_R^s$. Since there are only finitely many occurrences of $\prec_R^s$ in the original loop, we eventually gain a prehistoric loop with occurrences of $\prec_L^s$ only, which is a contradiction to Corollary 5.29.

Case 3. If there are only occurrences of $\prec_L^s$ in the prehistoric loop, the loop is a left prehistoric loop, which is a contradiction to Corollary 5.29.

$\Box$

Definition 5.35. If $\mathcal{D} := \Gamma \Rightarrow \Delta$ and $\mathcal{D}' := \Gamma' \Rightarrow \Delta'$ are any sequents of a **G3s′** (**G3s**) derivation, then we say that the sequent $\mathcal{D}$ is a *subsequent* of the sequent $\mathcal{D}'$, if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Notation: $\mathcal{D} \subseteq \mathcal{D}'$.

Theorem 5.36. *If $\mathcal{T}$ is a* **G3s′***-proof of the sequent $\Gamma \Rightarrow \Delta$, we gain a* **G3s***-proof $\mathcal{T}'$ of $\Gamma \Rightarrow \Delta$, such that for every sequent $\mathcal{D}'$ in $\mathcal{T}'$ we have that $\mathcal{D} \subseteq \mathcal{D}'$, where $\mathcal{D}$ is the corresponding* **G3s′***-sequent in $\mathcal{T}$.*

Proof. Since **G3s′** is sound with respect to **S4** (Theorem 3.50), we know that whenever **G3s′** $\vdash \Gamma \Rightarrow \Delta$, then **G3s** $\vdash \Gamma \Rightarrow \Delta$.
If $\mathcal{T}$ has no application of L$\Box$1, then $\mathcal{T}$=$\mathcal{T}'$. If there are L$\Box$1-applications in $\mathcal{T}$, we replace all instances of L$\Box$1 (there are only finitely many, L$\Box_1$1, L$\Box_2$1,..., L$\Box_k$1) by instances of L$\Box$:

$$\frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, \Box A_i \Rightarrow \Delta} L\Box_i 1 \qquad \rightsquigarrow \qquad \frac{\Gamma, A_i, \Box A_i \Rightarrow \Delta}{\Gamma, \Box A_i \Rightarrow \Delta} L\Box_i.$$

We have to write an additional copy of $\Box A_i$ in the premise of each L$\Box$1-rule, accordingly we have to do that in all the sequents above the considered instance of L$\Box_i$1 until the top-nodes of the proof. Thus each sequent $\mathcal{S}$ in $\mathcal{T}$ is a subsequent ($\mathcal{S} \subseteq \mathcal{S}'$) of the corresponding $\mathcal{S}'$ in $\mathcal{T}'$. All this additional copies of $\Box A_1, ..., \Box A_k$ are, with the exception of the premise of the corresponding L$\Box$-rule, only side formulas in the derivation. No matter which rule R is applied above a L$\Box$1-rule in the **G3s**$'$-proof $\mathcal{T}$, the corresponding rule of **G3s** is applicable in $\mathcal{T}'$: We consider the premise $\Gamma' \Rightarrow \Delta'$ of the i-th L$\Box$1-rule in $\mathcal{T}$ and call it $\mathcal{D}$. The corresponding sequent $\mathcal{D}'$ in $\mathcal{T}'$ is of the form $\Gamma', \Box\Pi \Rightarrow \Delta'$, where $\Box\Pi$ is the set of boxed formulas $\Box A_1, \Box A_2, ..., \Box A_i$ which were added in the L$\Box$1-applications L$\Box_1$1,..,L$\Box_i$1. Actually, $\Box\Pi$ is the set of principal formulas of the L$\Box$1-applications that occur on the branch below $\Gamma' \Rightarrow \Delta'$. Thus we have $\mathcal{D} \subseteq \mathcal{D}'$. The principal formula of the rule-application above L$\Box_i$1, has to be in Γ' or Δ'. Since the principal formula of the rule-application R above L$\Box_i$1 is contained in Γ' or Δ', it is also contained in $\mathcal{D}'$ and there is nothing which precludes us from applying the same rule R in $\mathcal{T}'$. Even R$\Box$ is applicable, since $\Box\Pi$ contains only boxed formulas and Γ' too, if R$\Box$ was the rule above L$\Box_i$ in the **G3s**$'$-derivation $\mathcal{T}$.
If $\dfrac{\Gamma' \Rightarrow \Delta'}{\Gamma'' \Rightarrow \Delta''}\,R$ is any rule-application of **G3s'** in $\mathcal{T}$, then $\dfrac{\Box\Pi, \Gamma' \Rightarrow \Delta'}{\Box\Pi, \Gamma'' \Rightarrow \Delta''}\,R$ is the corresponding application in **G3s**. The set of additional boxed formulas $\Box\Pi$ in the premise and in the conclusion of R in $\mathcal{T}'$ is the same, unless the rule R is L$\Box$. When R is L$\Box$1 in $\mathcal{T}$, then

$$\frac{A, \Gamma' \Rightarrow \Delta'}{\Box A, \Gamma' \Rightarrow \Delta'}\,L\Box 1 \qquad \text{becomes} \qquad \frac{\Box\Pi, \Box A, A, \Gamma' \Rightarrow \Delta'}{\Box\Pi, \Box A, \Gamma' \Rightarrow \Delta'}\,L\Box.$$

The formulas added in the premise of L$\Box$ in $\mathcal{T}'$ contain one additional formula ($\Box A$) compared to the conclusion. $\Box$

Corollary 5.37. *If $\mathcal{T}$ is a left-prehistoric-loop-free* **G3s**$'$*-proof of the sequent* $\Gamma \Rightarrow \Delta$, *there is a left-prehistoric-loop-free* **G3s**-*proof* $\mathcal{T}'$ *of* $\Gamma \Rightarrow \Delta$.

Proof. From Theorem 5.36 we know that it is possible to construct a **G3s**-proof from a **G3s**$'$-derivation. It remains to show that whenever $\mathcal{T}$ is left-prehistoric-loop-free, then $\mathcal{T}'$ is it too. Assume $\mathcal{T}$ to be left-prehistoric-loop-free and construct the corresponding **G3s**-proof $\mathcal{T}'$. If there are no R$\Box$-applications in $\mathcal{T}$, $\mathcal{T}'$ is left-prehistoric-loop-free, too and we are done. If there are R$\Box$-applications R$\Box_1$,...,R$\Box_j$ in $\mathcal{T}$, the corresponding R$\Box$-application in $\mathcal{T}'$ is of the form:

$$\frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \blacksquare_i A, \Delta'}\,R\Box_i \qquad \rightsquigarrow \qquad \frac{\Box\Gamma, \Box\Pi \Rightarrow A}{\Gamma', \Box\Gamma, \Box\Pi \Rightarrow \blacksquare_i A, \Delta'}\,R\Box_i,$$

where $\Box\Pi$ denotes the additional boxed formulas $\Box A_1, ..., \Box A_l$ which were added in the L$\Box$-rules (L$\Box_1$,..., L$\Box_l$) below R$\Box_i$. To be sure that there is no left prehistoric loop in $\mathcal{T}'$, we have to modify the premises of the R$\Box$-rules as follows:

$$\frac{\Box\Gamma, \Box\Pi \Rightarrow A}{\Gamma', \Box\Gamma, \Box\Pi \Rightarrow \blacksquare_i A, \Delta'}\,R\Box_i, \qquad \rightsquigarrow \qquad \frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma, \Box\Pi' \Rightarrow \blacksquare_i A, \Delta'}\,R\Box_i.$$

$\Box\Pi'$ in the conclusion of the R$\Box_i$-application in $\mathcal{T}'$ is the set of principal formulas of the L$\Box$-rules that occur on the branch below R$\Box_i$, but above the closest R$\Box$-application. Since $\Box\Pi$ is the set of *all* principal formulas of the L$\Box$-rules occurring below R$\Box_i$, we have that $\Box\Pi' \subseteq \Box\Pi$. We make use of the built in weakening in the R$\Box$-rule and omit the additional copies of the elements from $\Box\Pi'$ in the premise of the R$\Box$-rules in $\mathcal{T}'$ and in all the sequents above. We get exactly the same sequents in the R$\Box$-premises as we have in the **G3s'**-derivation $\mathcal{T}$. The resulting prooftree, let us call it $\mathcal{T}''$, is still a derivation in **G3s**: In $\mathcal{T}'$, the formulas $\Box A_1, ..., \Box A_l$ in $\Box\Pi$ are side formulas above L$\Box_1$,...,L$\Box_l$. But L$\Box_1$,...,L$\Box_l$ are applied below R$\Box_i$, thus in $\Box\Pi$ are only contained boxed formulas, which are side formulas until the top nodes of the prooftree $\mathcal{T}'$. Since $\Box\Pi' \subseteq \Box\Pi$, we can be sure that in $\Box\Pi'$ too, are only contained boxed formulas that are side formulas until the leaves of the prooftree $\mathcal{T}''$.
If $\dfrac{\Box\Pi, \Gamma' \Rightarrow \Delta'}{\Box\Pi, \Gamma'' \Rightarrow \Delta''} R$ is a rule-application different from R$\Box$ and L$\Box$ in $\mathcal{T}'$, where $\Box\Pi$ is the set of principal formulas of L$\Box$-applications on the same branch below R, then the corresponding rule-application in $\mathcal{T}''$ is of the form

$$\frac{\Box\Pi', \Gamma' \Rightarrow \Delta'}{\Box\Pi', \Gamma'' \Rightarrow \Delta''} R.$$

$\Box\Pi'$ is the set of principal formulas of L$\Box$-applications on the same branch below R, but above the closest R$\Box$-application. Thus $\Box\Pi' \subseteq \Box\Pi$.
If R is a L$\Box$-application in $\mathcal{T}'$

$$\frac{\Box\Pi, \Box A, A, \Gamma' \Rightarrow \Delta'}{\Box\Pi, \Box A, \Gamma' \Rightarrow \Delta'} L\Box, \quad \text{the corresponding rule in } \mathcal{T}'' \text{ is } \quad \frac{\Box\Pi', \Box A, A, \Gamma' \Rightarrow \Delta'}{\Box\Pi', \Box A, \Gamma' \Rightarrow \Delta'} L\Box.$$

Again, $\Box\Pi' \subseteq \Box\Pi$ is the set of principal formulas of L$\Box$-applications on the same branch below R, but above the closest R$\Box$-rule.
Since $\mathcal{T}$ is left-prehistoric-loop-free and $\mathcal{T}''$ has the same sequents in the R$\Box$-premises, we can be sure that $\mathcal{T}''$ is left-prehistoric-loop-free, too. $\Box$

We give an example for such a transformation of a derivation from **G3s′** to a loop-free derivation in **G3s**:

Example 5.38. The formula $\phi = \Box\neg\Box\Diamond\Box P \rightarrow \neg\Box P$ is an **S4**-theorem, derivable in **G3s'** by the derivation $\mathcal{T}$ in Figure 6.
$\mathcal{T}$ has no left prehistoric loop. This can be checked by considering the premises of the two R$\Box$-rules.
If we replace all instances of L$\Box'$ in $\mathcal{T}$ by instances of L$\Box$, we get the **G3s**-derivation $\mathcal{T}'$, presented in Figure 7.
If we consider the premises of the two R$\Box$-rules in $\mathcal{T}'$, we can see from R$\Box_1$ that f_2 is a left prehistoric family of f_1 and from R$\Box_2$ that f_1 is a left prehistoric family of f_2, thus $1 \prec_L 2 \prec_L 1$, the proof has a left prehistoric loop. The additional boxed formulas, the formulas we had to add because of the original L$\Box$-rule, are the reason for the loop. But since the formulas are only side-formulas above the premise of the L$\Box$ application, we can run the following step: We take the derivation $\mathcal{T}'$ and proceed from bottom to top:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \Rightarrow P}{\Box P \Rightarrow P}\,L\Box'}{\Box P \Rightarrow \blacksquare_1 P, \bot}\,R\Box_1 \qquad \bot, \Box P \Rightarrow \bot}{\Box P, \neg\blacksquare_1 P \Rightarrow \bot}\,L\to}{\Box P, \Box\neg\blacksquare_1 P \Rightarrow \bot}\,L\Box'}{\Box P \Rightarrow \neg\Box\neg\blacksquare_1 P}\,R\to}{\Box P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \bot}\,R\Box_2}{\Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \neg\Box P}\,R\to \qquad \bot \Rightarrow \neg\Box P}{\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P}\,L\to}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P}\,L\Box'}{\Rightarrow \Box\neg\blacksquare_2\Diamond\blacksquare_1 P \to \neg\Box P}\,R\to
$$

Figure 6: **G3s'**-derivation of $\Box\neg\Box\Diamond\Box P \to \neg\Box P$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, P, \Box\neg\blacksquare_1 P \Rightarrow P}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow P}\,L\Box}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow \blacksquare_1 P, \bot}\,R\Box_1 \qquad \bot, \Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow \bot}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P, \neg\blacksquare_1 P \Rightarrow \bot}\,L\to}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P, \Box\neg\blacksquare_1 P \Rightarrow \bot}\,L\Box}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P \Rightarrow \neg\Box\neg\blacksquare_1 P}\,R\to}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \bot}\,R\Box_2}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \neg\Box P}\,R\to \qquad \bot, \Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P}\,L\to}{\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P}\,L\Box}{\Rightarrow \Box\neg\blacksquare_2\Diamond\blacksquare_1 P \to \neg\Box P}\,R\to
$$

(We already considered the same **G3s**-derivation of the same formula in Example 5.8.)

Figure 7: **G3s**-derivation of $\Box\neg\Box\Diamond\Box P \to \neg\Box P$ with a left prehistoric loop.

consider the undermost L□-rule and mark the copy of the boxed formula ($\Box\neg\blacksquare_2\Diamond\blacksquare_1 P$) in the premise. If we come to the next R□-rule, we drop the superfluous copy of the marked formula in the premise of the R□-rule. If there is another L□-rule application (above the first one) below the first R□-rule, we mark the copy of this boxed formula too and drop both boxed formulas in the premise of the next R□-rule. Like that, we proceed from bottom to top and get another derivation in **G3s**, $\mathcal{T}''$:

$$
\begin{prooftree}
\AxiomC{$\Box P, P \Rightarrow P$}
\RightLabel{$L\Box$}
\UnaryInfC{$\Box P \Rightarrow P$}
\RightLabel{$R\Box_1$}
\UnaryInfC{$\Box P, \Box\neg\blacksquare_1 P \Rightarrow \blacksquare_1 P, \bot$}
\AxiomC{$\bot, \Box P, \Box\neg\blacksquare_1 P \Rightarrow \bot$}
\RightLabel{$L\rightarrow$}
\BinaryInfC{$\Box P, \Box\neg\blacksquare_1 P, \neg\blacksquare_1 P \Rightarrow \bot$}
\RightLabel{$L\Box$}
\UnaryInfC{$\Box P, \Box\neg\blacksquare_1 P \Rightarrow \bot$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Box P \Rightarrow \neg\Box\neg\blacksquare_1 P$}
\RightLabel{$R\Box_2$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \Box P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \bot$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \blacksquare_2\Diamond\blacksquare_1 P, \neg\Box P$}
\AxiomC{$\bot, \Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P$}
\RightLabel{$L\rightarrow$}
\BinaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P, \neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P$}
\RightLabel{$L\Box$}
\UnaryInfC{$\Box\neg\blacksquare_2\Diamond\blacksquare_1 P \Rightarrow \neg\Box P$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Rightarrow \Box\neg\blacksquare_2\Diamond\blacksquare_1 P \rightarrow \neg\Box P$}
\end{prooftree}
$$

Note, that the premises of R$\Box_1$ and R$\Box_2$ are exactly the same as in $\mathcal{T}$ and therefore $\mathcal{T}''$ cannot have a left prehistoric loop.

With the previous corollary we can state an analog of Yu's main theorem for the system **G3s**$'$:

Theorem 5.39. *If an* **S4***-theorem* ϕ *has a left-prehistoric-loop-free* **G3s'***-proof, then there is an* **LP***-formula* ψ *such that* $\psi^\circ = \phi$ *and* $\vdash_{\mathbf{LP}(CS^\circledast)} \psi$.

Proof. From the previous corollary we know that if there is a left-prehistoric-loop-free **G3s'**-proof of $\Rightarrow \phi$ we can gain a **G3s**-derivation of $\Rightarrow \phi$ with the same property. With Yu's main theorem we get that there is an **LP**-formula ψ such that $\psi^\circ = \phi$ and ψ is derivable with a non-self-referential constant specification in **LP**. □

To answer the question we posed at the very beginning of this subsection, we have to say that loops can still occur in **G3s'**-proofs, although contraction is not embedded in the L□1-rule. Nevertheless, we have found some interesting differences between the occurrences of prehistoric phenomena in the two systems **G3s'** and **G3s**, and we have shown how to transform **G3s'**-proofs into proofs in **G3s**. With the possibility to construct a **G3s**-proof out of a **G3s'**-derivation, we were able to state an analog of Yu's main theorem, relating prehistoric loops in sequent system proofs and self-referentiality, for the modified system **G3s'**.
In opposite to families of boxes in **G3s**, a principal positive family in a **G3s'**-derivation cannot be a left prehistoric family of itself. In addition, we can limit the left prehistoric

loops to occur in a whole **G3s'**-derivation, but not in single branches of a derivation. This leads us to the next question that has to be asked, namely, where the remaining left prehistoric loops come from. Since we already know that a left prehistoric loop can not appear within a single branch of a **G3s'**-derivation, the question is not that hard to answer: it is the point where two different branches of a prooftree meet, the two-premise rules (L∨, R∧, L→). If we consider the following example of an L∨-application, it becomes clear that there is still a form of contraction present in the system **G3s'**, even though contraction is not dp-admissible for **G3s'**:

$$\frac{A, \Box\neg\blacksquare P \Rightarrow \Delta \quad B, \Box\neg\blacksquare P \Rightarrow \Delta}{A \vee B, \Box\neg\blacksquare P \Rightarrow \Delta} L\vee.$$

The L∨-application relates the principal positive box-occurrence in the conclusion of the rule-application to two occurrences of the same box, one occurrence in each premise. Of course this is even a weaker form of contraction than we have in the L□-rule of the system **G3s**, but it is still an indication for contraction to be present in the system. So the next step will be to avoid the form of contraction we have observed in **G3s'**-derivations, and check, whether loops can still occur in this modified system. This is what we will do in the next subsection.

5.4 G3s*-proofs and self-referentiality

In this subsection, our goal is to avoid the contraction observed in **G3s'**-proofs by replacing the two-premise rules (L∨, R∧, L→) by their context-splitting formulation, and to analyze, whether prehistoric loops can still appear in this modified system.

Definition 5.40. The system we gain from **G3s′** by replacing the three two-premise rules by the context-splitting formulation, namely

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} R\wedge \qquad \rightsquigarrow \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma' \Rightarrow \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \wedge B} R\wedge_{cs}$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \rightsquigarrow \qquad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma' \Rightarrow \Delta'}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\vee_{cs}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} L\to \qquad \rightsquigarrow \qquad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma' \Rightarrow \Delta'}{A \to B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\to_{cs}$$

is called **G3s***.

Since **G3s*** differs from **G3s′** only by the formulation of the two-premise rules, **G3s*** is incomplete too. Let us consider the first proof of the incompleteness Theorem 3.48 for **G3s'**, and explain the changes of the proof for **G3s***. The aim is to show that **G3s**$^* \nvdash \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$. In the prooftree (5), the first occurrence of a two-premise rule is the following instance of L→. Actually, the only two-premise rules in a

G3s' or **G3s***-derivation of the formula of interest are instances of L→, since there are no occurrences of ∧, ∨ in the formula.

$$\frac{\Rightarrow \bot, \Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \qquad \bot \Rightarrow \bot}{\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot}\, L \to$$

The principal formula of L→ is $\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$, the only side formula is $\bot$. There are two possible context-splitting rule-applications:

$$\frac{\Rightarrow \bot, \Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \qquad \bot \Rightarrow}{\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot}\, L \to_{cs} \tag{10}$$

and

$$\frac{\Rightarrow \Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \qquad \bot \Rightarrow \bot}{\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P) \Rightarrow \bot}\, L \to_{cs} \tag{11}$$

If we consider the second rule-application (11), the right premise is an instance of (L⊥), and to the left premise there is only one rule applicable in reverse, namely R□. Doing this, we obtain the following tree

$$\frac{\dfrac{\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}{\Rightarrow \neg\Box\neg P \to \Box\neg\Box\neg P}\, R \to}{\Rightarrow \Box(\neg\Box\neg P \to \Box\neg\Box\neg P)}\, R\Box$$

with the same sequent at the "top-node" as (5).

If we take the first possibility (10), we proceed on the left branch like we did it in prooftree (5).
In (6), there is one L→-application at the top of the tree, but again, replacing it by a context-splitting instance ($L\to_{cs}$) would not change anything on the fact that the left leaf is no axiom-instance.
The same holds for (7). The following two possible $L\to_{cs}$ which are applicable to the sequent $\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$:

$$\frac{\Rightarrow \Box\neg\Box\neg P, \Box\neg P \qquad \bot \Rightarrow}{\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}\, L \to_{cs} \qquad \frac{\dfrac{\dfrac{P \Rightarrow \bot}{\Rightarrow \neg P}\, R \to}{\Rightarrow \Box\neg P}\, R\Box \qquad \bot \Rightarrow \Box\neg\Box\neg P}{\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P}\, L \to_{cs}$$

The prooftree on the left leads to the same left premise as (7), while the right premise is an instance of (L⊥). The tree on the right is obviously no derivation of $\neg\Box\neg P \Rightarrow \Box\neg\Box\neg P$.
There is one L→-application left in (8). But again, replacing it by a context-splitting application would not lead to an axiom-instance on the left top-node of the tree.
Thus, **G3s*** $\nvdash \neg\Box\neg\Box(\neg\Box\neg P \to \Box\neg\Box\neg P)$, in other words, not all valid (with respect to **S4**) formulas are derivable in **G3s***. But the soundness is still preserved:

Theorem 5.41. **G3s*** *is sound with respect to* **S4**.

Proof. From Theorem 3.50 we know that **G3s′** is sound with respect to **S4**. It suffices to show that whenever **G3s**$^* \vdash \Gamma \Rightarrow \Delta$ then **G3s′** $\vdash \Gamma \Rightarrow \Delta$. This is proved by an induction on the depth n of the **G3s***-proof $\mathcal{D}$, such that $\mathcal{D} \vdash_n \Gamma \Rightarrow \Delta$. We consider only the cases, where the last rule of the deduction $\mathcal{D}$ is one of the two-premise rules, since this is the only point, where the two systems differs from each other. Assume the statement to be true for n, and let $\mathcal{D}$ be a deduction of depth $n+1$, such that $\mathcal{D} \vdash_{n+1} \Gamma \Rightarrow \Delta$

Case 1. The last rule of $\mathcal{D}$ is R$\wedge_{cs}$:

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma' \Rightarrow \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \wedge B} R\wedge_{cs}$$

By induction hypothesis we get **G3s′** $\vdash \Gamma \Rightarrow \Delta, A$ and **G3s′** $\vdash \Gamma' \Rightarrow \Delta', B$. Since weakening is dp-admissible in **G3s′**, we have that **G3s′** $\vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta', A$ and **G3s′** $\vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta', B$. Applying the R$\wedge$-rule in **G3s′**, we get a proof of the desired sequent:

$$\frac{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \quad \Gamma, \Gamma' \Rightarrow \Delta, \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \wedge B} R\wedge.$$

Case 2. The last rule of $\mathcal{D}$ is L$\vee_{cs}$:

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma' \Rightarrow \Delta'}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\vee_{cs}$$

Again by induction hypothesis and dp-admissibility of weakening in **G3s′** it follows that **G3s′** $\vdash A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ and **G3s′** $\vdash B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. Applying the L$\vee$-rule in **G3s′**, we get a proof of the desired sequent:

$$\frac{A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\vee.$$

Case 3. If the last rule of $\mathcal{D}$ is L$\rightarrow_{cs}$:

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma' \Rightarrow \Delta'}{A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L \rightarrow_{cs}$$

By the same argumentation we gain a **G3s′**-derivation of the sequent $A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Thus, we can be sure that the system **G3s*** does not prove formulas, which are not valid. □

The following properties of derivations in **G3s*** can be proved the same way it is done for **G3s′**.

Lemma 5.42. *If $\mathcal{T}$ is a* **G3s****-proof, then*

1. *in all the sequents of $\mathcal{T}$, there occurs at most one box of each family.*
2. *for principal positive families f_i in $\mathcal{T}$, we have that $i \nprec_L i$ and $i \nprec_R i$.*

3. $\mathcal{T}$ *has a left prehistoric loop iff* $\mathcal{T}$ *has a prehistoric loop.*

For derivations in **G3s′** we proved that there is at most one R$\Box$-application for each family f_i in each branch s_j of the prooftree. For **G3s***-proofs, we can even state more:

Lemma 5.43. *In a* **G3s****-derivation, each principal positive family of boxes* $\blacksquare_i$ *can be introduced only once in the prooftree. In other words, there is only one R*$\Box$*-application, R*$\Box_i$*, for each principal positive family* f_i *in the proof.*

Proof. Consider a **G3s***-derivation $\mathcal{T}$. If there are no applications of two-premise rules in $\mathcal{T}$, the proof consists of one branch only and by Lemma 5.42 we can be sure that there is only one R$\Box$-rule for each family of boxes occurring in the proof.
If there are applications of two-premise rules in $\mathcal{T}$, we assume that there are two R$\Box$-applications, R$\Box_{i,j}$ and R$\Box_{i,k}$, which introduce boxes of the family f_i in two different branches s_j and s_k. At some point of the prooftree, the two branches s_j and s_k are connected by one of the three two-premise rules R$\wedge_{cs}$, L$\vee_{cs}$, L$\rightarrow_{cs}$. Since the two-premise rules are formulated in a context-splitting way in **G3s***, there is no rule in the system which relates the two occurrences of $\blacksquare_i$ in the premises to one occurrence in the conclusion. Thus there are two occurrences of $\blacksquare_i$ in the conclusion of the applied two-premise rule, which is a contradiction to Lemma 5.42. $\Box$

Corollary 5.44. *A* **G3s****-proof cannot have a left prehistoric loop.*

Proof. This property follows from the fact, that there is only one R$\Box$-application for each principal positive family of boxes in each **G3s***-proof. To get a loop of the form

$$i_1 \prec_L i_2 \prec_L \ldots \prec_L i_n \prec_L i_1,$$

there have to be at least two R$\Box_{i_k}$-applications for one principal positive family f_{i_k} occurring in the left prehistoric loop, which is a contradiction to the previous lemma. $\Box$

Theorem 5.45. *If* $\mathcal{T}$ *is a* **G3s****-proof of the sequent* $\Gamma \Rightarrow \Delta$*, we gain a left-prehistoric-loop-free* **G3s′***-proof* $\mathcal{T}'$ *of* $\Gamma \Rightarrow \Delta$*, such that for every sequent* $\mathcal{D}'$ *in* $\mathcal{T}'$ *we have that* $\mathcal{D} \subseteq \mathcal{D}'$*, where* $\mathcal{D}$ *is the corresponding* **G3s****-sequent in* $\mathcal{T}$*.*

Proof. First, we show how to construct a **G3s'**-derivation $\mathcal{T}'$ out of a **G3s***-proof, and then we can follow that this constructed **G3s'**-derivation is left-prehistoric-loop-free.
If $\mathcal{T}$ has no application of R$\wedge_{cs}$, L$\vee_{cs}$ and L$\rightarrow_{cs}$, then $\mathcal{T} = \mathcal{T}'$. $\mathcal{T}'$ is left-prehistoric-loop-free, since $\mathcal{T}' = \mathcal{T}$, and $\mathcal{T}$ is loop-free, since **G3s***-derivations cannot have a left prehistoric loop.
Otherwise, we replace all instances of the context-splitting formulated two-premise rules of **G3s*** by the context-sharing two-premise rules of **G3s′**. Replace instances of

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma' \Rightarrow \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \wedge B} R\wedge_{cs} \quad \text{by} \quad \frac{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \quad B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \wedge B} R\wedge,$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma' \Rightarrow \Delta'}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\vee_{cs} \quad \text{by} \quad \frac{A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\vee,$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma' \Rightarrow \Delta'}{A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\rightarrow_{cs} \quad \text{by} \quad \frac{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \quad B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\rightarrow.$$

We add an additional copy of each formula in Γ' and Δ' to the left premise of L$\vee_{cs}$ (R$\wedge_{cs}$, L$\rightarrow_{cs}$) and an additional copy of the formulas in Γ and Δ to the right premise of L$\vee_{cs}$ (R$\wedge_{cs}$, L$\rightarrow_{cs}$). But since the resulting tree should still be a derivation, we add the additional formulas in every sequent between the premise of the considered L$\vee_{cs}$-, (R$\wedge_{cs}$-, L$\rightarrow_{cs}$-) rule and the conclusion of the closest R$\Box$-rule on each branch. If there is no R$\Box$-rule above the considered L$\vee_{cs}$-, (R$\wedge_{cs}$-, L$\rightarrow_{cs}$-) rule application, the additional formulas have to be added up to the top nodes of the prooftree. Thus, each sequent $\mathcal{S}$ in $\mathcal{T}$ is a subsequent of the corresponding $\mathcal{S}'$ in $\mathcal{T}'$. All this additional formulas are side formulas in the premise of the corresponding two-premise rule and in all sequents above. It remains to show that no matter which rule R is applied above a L$\vee_{cs}$ (R$\wedge_{cs}$, L$\rightarrow_{cs}$) application in $\mathcal{T}$, the corresponding rule of **G3s**$'$ is applicable in $\mathcal{T}'$: We consider the sequents in the premises of the i-th L$\vee$-rule in $\mathcal{T}$, $A, \Gamma \Rightarrow \Delta$ and $B, \Gamma' \Rightarrow \Delta'$, and call them $\mathcal{D}_L$, $\mathcal{D}_R$ respectively. The corresponding sequents $\mathcal{D}'_L$ and $\mathcal{D}'_R$ in the **G3s'**-proof $\mathcal{T}'$ are of the form $A, \Gamma, \Gamma', \Phi \Rightarrow \Delta, \Delta', \Psi$ and $B, \Gamma, \Gamma', \Phi \Rightarrow \Delta, \Delta', \Psi$, where Φ and Ψ are the sets of formulas which were added in the L$\vee_{cs}$-, R$\wedge_{cs}$- and L$\rightarrow_{cs}$-applications below the considered rule-application. We have $\mathcal{D}_L \subseteq \mathcal{D}'_L$ and $\mathcal{D}_R \subseteq \mathcal{D}'_R$.

$$\textbf{G3s}^*: \quad \frac{\dfrac{\cdots}{A, \Gamma \Rightarrow \Delta} R_L \qquad \dfrac{\cdots}{B, \Gamma' \Rightarrow \Delta'} R_R}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L\vee_{cs}$$

$$\textbf{G3s}': \quad \frac{\dfrac{\cdots}{A, \Gamma, \Gamma', \Phi \Rightarrow \Delta, \Delta', \Psi} R_L \qquad \dfrac{\cdots}{B, \Gamma, \Gamma', \Phi \Rightarrow \Delta, \Delta', \Psi} R_R}{A \vee B, \Gamma, \Gamma', \Phi \Rightarrow \Delta, \Delta', \Psi} L\vee$$

Consider the **G3s***-derivation: the principal formulas of the rule applications R_L and R_R immediately above L$\vee$, have to be in A, Γ, Δ for R_L and in B, Γ', Δ' for R_R. From this observations, and the fact that we have $\mathcal{D}_L \subseteq \mathcal{D}'_L$ and $\mathcal{D}_R \subseteq \mathcal{D}'_R$, it follows, that the principal formulas of R_L and R_R in the **G3s'**-proof are contained in $\mathcal{D}'_L$, $\mathcal{D}'_R$ respectively. So there is nothing which precludes us from applying the same rule R in $\mathcal{T}'$, and the resulting prooftree $\mathcal{T}'$ is a **G3s'**-derivation of the sequent $\Gamma \Rightarrow \Delta$. The argumentation for the cases where we consider the *i*-th R$\wedge$-rule and the *i*-th L$\rightarrow$-rule works similarly. But why can we be sure, that $\mathcal{T}'$ is a loop-free derivation of $\Gamma \Rightarrow \Delta$? First it has to be noted, that we construct $\mathcal{T}'$ out of a left-prehistoric-loop-free derivation $\mathcal{T}$ (**G3s**$*$-proofs can not have a left prehistoric loop). Secondly, we add the additional formulas, in the construction described, from the premises of the context-sharing two-premise rule up to the conclusion of the closest R$\Box$-rule on each branch. This is no problem, because of the built-in weakening in R$\Box$. Thus, the *premise* of the *k*-th R$\Box$-rule, R$\Box_k$, in $\mathcal{T}'$, let us call it $\mathcal{B}'_k$, is exactly the same as the corresponding sequent $\mathcal{B}_k$ in $\mathcal{T}$, in short form $\mathcal{B}'_k = \mathcal{B}_k$, for any $k = 1, ..., n$, where n is the number of R$\Box$-applications in $\mathcal{T}$, $\mathcal{T}'$. Since the premise of the R$\Box$-rules decide, whether or not the derivation has a loop, we can be sure that $\mathcal{T}'$ has no left prehistoric loop. $\Box$

Corollary 5.46. *If $\mathcal{T}$ is a* **G3s****-proof of the sequent* $\Gamma \Rightarrow \Delta$*, we gain a left-prehistoric-loop-free* **G3s***-proof $\mathcal{T}'$ of* $\Gamma \Rightarrow \Delta$*, such that for every sequent $\mathcal{D}'$ in $\mathcal{T}'$ we have that $\mathcal{D} \subseteq \mathcal{D}'$, where $\mathcal{D}$ is the corresponding* **G3s***-sequent in $\mathcal{T}$.*

Proof. By Theorem 5.45 we are able to construct a left-prehistoric-loop-free **G3s**$'$-derivation out of a **G3s***-proof. From Corollary 5.37 we know, that the left-prehistoric-loop-free **G3s**$'$-proof itself, can be transformed into a left-prehistoric-loop-free **G3s**-derivation. $\Box$

Theorem 5.47. *If an* **S4***-theorem ϕ is derivable in* **G3s****, then there is an* **LP***-formula ψ, such that $\psi^\circ = \phi$ and $\vdash_{\mathbf{LP}(CS^\circledast)} \psi$.*

Proof. By the previous corollary we know that if $\Rightarrow \phi$ is derivable in **G3s***, we can gain a loop-free **G3s**-proof of $\Rightarrow \phi$. With Theorem 5.19 we obtain, that there is an **LP**-formula ψ, such that ϕ is the forgetful projection of ψ, and ψ is derivable with a non-self-referential constant specification in **LP**. $\Box$

Finally, we achieved to define a **G3**-system, the system **G3s***, where left prehistoric loops do not occur. This was possible, since we eliminated every form of contraction we observed in derivations from the systems **G3s** and **G3s'**. With the possibility to construct a **G3s**-proof out of a derivation from **G3s***, we can indirectly apply Yu's main theorem to the class of **S4**-theorems, deducible in the system **G3s***. Since the incomplete system **G3s*** is still sound with respect to **S4**, the system can be used to represent a non-self-referential fragment of modal logic **S4**. If an **S4**-theorem ϕ is derivable in the system **G3s***, there is a realization of ϕ, which can be derived in **LP** with a non-self-referential constant specification. Thus, the criterion for an **S4**-theorem calling for a self-referential constant specification, which Yu obtained from studying prehistoric phenomena in **G3s**-proofs, can be complemented by a second one. In addition, we found a source of the appearance of prehistoric loops: it is the embedded contraction in the L$\Box$-rule of the system **G3s**, and the hidden contraction in the context-sharing formulation of the two-premise rules of the systems **G3s** and **G3s'**.

6 Conclusion

In this thesis, we analyze the strategy pursued by Troelstra and Schwichtenberg in [TS00] to define a weakening-, contraction- and cut-free sequent system, a so called **G3**-system, for classical (propositional) logic and **S4**. In the section about **S4**, we introduce a modified version of the **G3**-system for **S4**, the system **G3s'**. For this system, contraction is not dp-admissible and therefore the system is incomplete. The example of **G3s'** represents the importance of the dp-invertibility of the rules, for the dp-admissibility of contraction.
In chapter 3, we apply the results from the previous analysis to construct a **G3**-system for the justification counterpart of **S4**, the Logic of Proofs **LP**. We define the system **LPG3**, which does not contain structural rules, but weakening and contraction are depth-preserving admissible for this system. The sequent system **LPG** for **LP** introduced by Artemov in [Art01] to prove completeness of **LP**, is a system of the **G2**-family of Gentzen calculi. We can show that our system **LPG3** is equivalent to **LPG**.
In the last section of this thesis, we introduce and continue the work of Junhua Yu in [Yu09], and study prehistoric phenomena in different contraction-free sequent systems for **S4**. R. Kuznets' approach to study self-referentiality in [Kuz09] takes place on a logical level that is to decide, whether or not a modal logic can be realized non-self-referentially. In opposite to this approach, Yu considers the topic at a theorem-level. He defines prehistoric phenomena in **G3s**-proofs and shows that left prehistoric loops are necessary for self-referentiality. This is the initial point for our considerations. We retrieve the source of left prehistoric loops in **G3s**-derivations to lie in the embedded contraction of the L$\Box$-rule, and therefore we apply Yu's machinery of prehistoric phenomena to the system **G3s'**, where contraction is no more implemented in the left box-rule. It turns out that this modification is not enough to avoid prehistoric loops in **G3s'**-proofs. Although we can prove that within a single branch of a **G3s'**-derivation loops cannot occur, they still appear in the whole derivation. The reason therefore is that there is still a form of contraction present in **G3s'**, namely in the two-premise rules. Avoiding this form of contraction by replacing the two-premise rules (L$\vee$, R$\wedge$, L$\rightarrow$) by their context-splitting formulation leads us to the system **G3s***, where prehistoric loops actually do not occur. By finding a way to transform **G3s***-proofs into **G3s'**-proofs, and **G3s'**-proofs into **G3s**-proofs, we can prove that **G3s*** represents a non-self-referential fragment of **S4**.
There is a very interesting question concerning the system **G3s'**, which is not answered in this thesis: The examples of **G3s'**-derivations with a left prehistoric loop occurring, are very constructed and it seems to be possible that the loops occurring in **G3s'**-proofs are not necessary. So is it possible to find a loop-free **G3s'**-derivation for all theorems of **G3s'**? If this question cannot be answered positively, there is an even more interesting question to pose: Does **G3s'** represents the fragment of **S4**, whose realization is calling for self-referentiality, but not for direct self-referentiality? There is one behavior of prehistoric families in **G3s'**-derivations, which argues for the second claim to be true. In **G3s'**-proofs, any principal positive family f_i of boxes cannot be a left principal family of itself, thus $i \not\prec_L i$. Intuitively, $i \prec_L i$ is how we imagine ourselves direct self-reference represented by prehistoric phenomena, and $i_1 \prec_L i_2 \prec_L ... \prec_L i_n \prec_L i_1$ would be the corresponding visualization of non-direct self-reference. In **G3s**-derivations,

there are both types of loops occurring, in **G3'**-derivations, the ”direct” form of visualized self-reference within prehistoric phenomena is proved not to occur.
Another conjecture, which is already mentioned in the conclusions of [Yu09], has not been proved yet. There is supposed that if all **G3s**-proofs of an **S4**-theorem ϕ have left prehistoric loops, then any realizations of ϕ will necessarily call for self-referential constant specifications, thus it is unknown whether a left prehistoric loop is sufficient for self-referentiality.

7 Bibliography

[Art95] Sergei N. Artemov, *Operational modal logic*. Technical Report MSI 95-29, Cornell University, December 1995.

[Art01] Sergei N. Artemov, *Explicit provability and constructive semantics*. The Bulletin of Symbolic Logic, vol. 7 (2001), no. 1, pages 1-36.

[Art08] Sergei N. Artemov, *The logic of justification*. The Review of Symbolic Logic 1 (2008), pages 477-513.

[BGK10] Kai Brünnler, Remo Goetschi and Roman Kuznets, *A Syntactic Realization Theorem for Justification Logics.* In Lev Beklemishev, Valentin Goranko and Valentin Shehtman, editors, *Advances in Modal Logic*, volume 8, pages 38-56. College Publications, 2010.

[BRV01] Patrick Blackburn, Maarten de Rijke and Yde Venema, *Modal Logic*. Volume 53 of Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[Fit05] Melvin Fitting, *The logic of proofs, semantically.* Annals of Pure and Applied Logic, 132(1), pages 1-25. February 2005.

[Gen35] Gerhard Gentzen, *Untersuchungen über das logische Schliessen I, II*. Mathematische Zeitschrift, 39, pages 176-210, 405-431. Translation in [Gen69].

[Gen69] *The Collected Papers of Gerhard Gentzen*. English translation of Gentzen's papers, edited and introduced by M. E. Szabo, North-Holland Publishing Company, 1969.

[Goe33] Kurt Gödel, *Eine Interpretation des intuitionistischen Aussagenkalküls*. Ergebnisse eines mathematischen Kolloquiums 4, pages 34-40, 1933.

[Gor92] Rajeev P. Goré, *Cut-free Sequent and Tableau Systems for Propositional Normal Modal Logics*. PhD thesis, University of Cambridge, 1992.

[Ind10] Andrzej Indrzejczak, *Natural Deduction, Hybrid Systems and Modal Logics*. Volume 30 of Trends in Logic. Springer, 2010.

[Kan57] Stig Kanger, *Provability in Logic*. In: G. Holmström-Hintikka, S. Linstrom and R. Sliwinski (eds), *Collected papers of Stig Kanger with essays on his life and work, Vol. 1*. Volume 303 of the Synthese Library, 2001.

[Kru06] Nikolai V. Krupski. *On the complexity of the reflected logic of proofs.* Theoretical Computer Science, 357(1-3), pages 136-142. August 2006.

[Kuz09] Roman Kuznets, *Self-referential justifications in epistemic logic*. Theory of Computing Systems, 46(4), pages 636-661, Mai 2010. Published online April 2009.

[LL32] C.I. Lewis and C. H. Langford, *Symbolic Logic*. Dover, 1932.

[Men97] Elliott Mendelson, *Introduction to Mathematical Logic*. Fourth edition, Chapman & Hall, 1997.

[Mkr97] Alexey Mkrtychev, *Models for the logic of proofs.* In Sergei Adian and Anil Nerode, editors, *Logical Foundations of Computer Science*, 4th International Symposium, LFCS'97, Yaroslavl, Russia, July 6-12, 1997, Proceedings, volume 1234 of Lecture Notes in Computer Science, pages 266-275. Springer, 1997.

[TS00] A. S. Troelstra and Helmut Schwichtenberg, *Basic Proof Theory*. Volume 43 of Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, second edition, 2000.

[Yu09] Junhua Yu, *Prehistoric Phenomena and Self-referentiality in Realization Procedure*. Technical Report PP-2009-45, Institute for Logic, Language and Computation, University of Amsterdam, 2009. Later version published as [Yu10].

[Yu10] Junhua Yu, *Prehistoric phenomena and self-referentiality*. In Farid Ablayev and Ernst W. Mayr, editors, *Computer Science - Theory and Applications*, 5th International Computer Science Symposium in Russia, CSR 2010, Kazan, Russia, June 2010, Proceedings, volume 6072 of Lecture Notes in Computer Science, pages 384-396. Springer, 2010.