Applicative theories on tree ordinal numbers

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Contents

1 Introduction

In [8] the so-called *basic theory of operations and numbers,* which represents an applicative basis of explicit mathematics, is introduced. BON is a theory defined in the language of partial terms; it has notions of definedness, application, combinatory algebra, complete induction and typedness as well as some additional axioms. In particular, because of the partial combinatory algebra, a term for λ-abstraction can be defined, details can be read upon in [13]. The λ-abstraction also yields the existence of a term `fix`, which acts as a fixed-point combinator for functions. BON does not have a syntactic notion of types, but rather has typedness as a formal statement and, using the induction scheme, totality of many functions can be proven. The axioms of BON contain the basis of types, namely they contain the atomic type N and axioms about it.

In this master thesis, the ultimate goal is to introduce an untyped theory similar to BON. This newly introduced theory bears the name BON_Ω and is an extension of BON that is obtained by adding tree ordinals as a second atomic type. Tree ordinals being the least set that contains $\mathsf{0}_\Omega$ and is closed under the sup operation. The sup operation always takes a *function* from natural numbers to ordinals as input and yields an ordinal number. Another way of thinking of this is by imagining a *tree*, hence the name.

The strength of BON_Ω is equal to the one of ID_1; to prove this, we reduce the theory QT_Ω from [1] to BON_Ω in the lower bounds and $\mathsf{BON}_\Omega + \mu$ to PA_Ω from [11] in the upper bounds, whereas $\mathsf{BON}_\Omega + \mu$ the theory BON_Ω extended by the non-constructive μ-operator. The non-constructive μ-operator yields a zero of a function if it exists, and $\mathsf{0}_\mathsf{N}$ otherwise. The main motivation behind defining an untyped theory lies therein, that less information about the terms, i.e. the whole typedness, needs to be coded into the language itself. We achieve this by a rather strong form of induction in addition to assuming the typedness – as a formal statement – of the defining constants. Then the typedness of terms follows logically, rather than syntactically.

For the lower bounds, we essentially take the theory QT_Ω—which is proven to be equivalent to ID_1. Then we show, that all provable statements of QT_Ω can be proven in BON_Ω, assuming the typedness of the free variables.

For the upper bounds, we embed BON_Ω in PA_Ω from [11]—which is proven to be equivalent to ID_1. This is done by defining an inductive operator and using the theorems about fixed points and inductive operators as shown in the very same paper. The translation is done by defining a valuation function and then interpreting the application from BON_Ω as fulfilling the inductive operator.

Firstly, we show lower bounds for the proof theoretical strength of BON_Ω. We start off, with defining the set of all type symbols, defined inductively by applying the $\longrightarrow$ operation to the base types N and Ω. Then we formally introduce the theory BON_Ω and show some basic properties about BON_Ω, in particular, that in BON_Ω it is always possible for any given type to construct a term of the given type and that typedness of a term for some type implies its definedness.

After the introduction of BON_Ω, we define the theory QT_Ω from [2]. QT_Ω is a quantified version of Gödel's theory T over ordinal numbers. Most of the axioms of QT_Ω can be translated into BON_Ω in a straight-forward manner. There are two issues that need to be addressed though: QT_Ω has two recursors built into the theory that have no correspondence in BON_Ω, those are translated by specifically-crafted terms that behave in an equivalent manner, those terms are explicitly written down as part of the proof. The other issue that we need to deal with in the embedding is, that in QT_Ω, due to its typed nature, all terms that can be applied to each other from a syntactical point of view automatically are total, because of the typedness requirements for any term in QT_Ω. In BON_Ω, however, typedness is not part of the language, but rather just a formula like any other. We therefore require the typedness of certain constants by axioms, and we prove the typedness – and therefore definedness – of terms built by the application function using the strong induction principles of BON_Ω. Using those techniques we manage to embed QT_Ω into BON_Ω and so we indirectly embed ID_1 into BON_Ω.

Secondly, we show upper bounds for the proof theoretical strength of $\mathsf{BON}_\Omega + \mu$. As mentioned before, $\mathsf{BON}_\Omega + \mu$ is an extension of BON_Ω that has the non-constructive μ-operator scheme added. The μ-operator gives the least zero of the (coding of the) function it is applied to, if the function has any zero at all and it gives 0_N, otherwise. In this setting we have a typedness axiom that requires μ to be a total function of functions (of natural numbers to natural numbers) to natural numbers. The typedness is such a strong property that this operator cannot be constructively built up. As it turn out, though, the extension of BON_Ω by the μ-operator is harmless, i.e. BON_Ω and $\mathsf{BON}_\Omega + \mu$ are proof-theoretically equivalent.

We give a formal definition of PA_Ω from [11] and define an inductive operator form thereon. This inductive operator form is used to simulate the behaviour of $\mathsf{BON}_\Omega + \mu$ in PA_Ω. It is a parallel inductive definition; on one hand the axioms of $\mathsf{BON}_\Omega + \mu$ are coded into the operator on the other hand the set of tree ordinal numbers is defined. The induction needs to be simultaneous, because each part needs the lower layers of the other. A vital property of this inductive operator form is the functionality of the inductive operator. This and the fixed-point theorem due to [11] are two main tool used to embed $\mathsf{BON}_\Omega + \mu$ into PA_Ω.

Once the theory PA_Ω and those properties are introduced, we define a valuation of terms of $\mathsf{BON}_\Omega + \mu$ that has the intended meaning „the term t has the value x“. This valuation is used to define a translation of formulae of $\mathsf{BON}_\Omega + \mu$ to PA_Ω. We then can use the translation to embed $\mathsf{BON}_\Omega + \mu$ into PA_Ω by proving the translation of every axiom of $\mathsf{BON}_\Omega + \mu$. In the embedding proof, one challenge is the translation of the transfinite induction scheme, this part is proved by an induction along the layers of the inductive definition of the inductive operator form.

In the end we wrap up the results in a formal proof-theoretical equivalence theorem between ID_1, BON_Ω and $\mathsf{BON}_\Omega + \mu$.

2 The Theory BON_Ω

Our main theory BON_Ω is an untyped theory.

2.1 Definitions of the language, terms and formulae for BON_Ω

Technically we could skip a definition of what a type is at this spot an only talk of abbreviations of formulae, it turns out, however, that having a notion of types is useful here already, because we do have axioms that involve „types“ (or abbreviations of formulae). And later in the setting of QT_Ω we will need the types in the formal definition of the language. Therefore, we define what we consider types. In particular, there are are no product types in our setting, we define those using currying.

Definition 1. The set of all type symbols $\mathbb{T}$ is defined inductively:

1. $\mathsf{N} \in \mathbb{T}$
2. $\Omega \in \mathbb{T}$
3. $\sigma \in \mathbb{T} \wedge \tau \in \mathbb{T} \Rightarrow \sigma \longrightarrow \tau \in \mathbb{T}$

It is convenient to have product types to formulate certain properties, but it is also simpler to not have them as syntactical objects, therefore we introduce the following notation.

Notation 2. Let $\sigma_1, \ldots, \sigma_n, \tau \in \mathbb{T}$, then

$$(\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n) \longrightarrow \tau := \sigma_1 \longrightarrow (\sigma_2 \longrightarrow \cdots (\sigma_n \longrightarrow \tau) \cdots)$$

We try to define a theory BON_Ω based on BON from [13]. The following definition is basically an extension of the definition of BON_Ω, where we add additional constants for the ordinal part of BON_Ω. We also add a constant symbol μ to the language. This would not be necessary at this point, but it allows us to only use one language for BON_Ω and $\mathsf{BON}_\Omega + \mu$, which is defined later.

Definition 3. The language $\mathcal{L}(\mathsf{BON}_\Omega)$. There is a countably infinite supply of variables $v_1, v_{2,} \ldots$, the logical symbols $\neg, \vee, \exists$, an unary symbol $\downarrow$ for definedness and the binary symbol $=$ for equality. Furthermore, we have the following constants: k, s (combinators), $\mathsf{0_N}$, $\mathsf{0}_\Omega$ (numerical and ordinal zero), $\mathsf{s_N}$ (numerical successor), $\mathsf{p_N}$ (numerical predecessor), $\mathsf{d_N}$, d_Ω (definition by numerical and ordinal cases), sup, sup^{-1} (supremum on tree ordinals), μ (non-constructive minimum operator). We have a binary function symbol $\cdot$ and two unary relation symbols N, Ω.

The terms can be defined in the standard way. Also consider, that we only have one function symbol, namely $\cdot$.

Definition 4. $\mathcal{L}(\mathsf{BON}_\Omega)$ terms.

1. Every variable and every constant is a term

2. If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol with $n \geq 1$, then $f(t_{1,} \ldots, t_n)$ is a term.

If $s, t, t_1, \ldots, t_n$ are terms and R is an n-ary relation symbol, then the expressions $s\downarrow$, $s = t$ and $R(t_1 \ldots, t_n)$ are called *atomic formulae.*

As for formulae, the definition is standard; noteworthy is, that conjunction and universal quantification are abbreviations, rather than formal symbols. This makes induction on the formula build-up easier, for we do not need to consider that many cases in the induction step.

Definition 5. $\mathcal{L}(\mathsf{BON}_\Omega)$ formulae.

1. Every atomic formula is a formula

2. If A is a formula, then $\neg A$ is a formula.

3. If A and B are formulae, then $A \vee B$ is a formula.

4. If A is a formula and x a variable, then $\exists x A$ is a formula.

As seen in the following, we define the logical operators and quantifiers using abbreviations 1-4. The partial equality only states, that the two sides are equal if one of them is defined 5. The non-equality states, that both sides are defined *and* the terms are not equal 6, in particular, $s \neq t$ is a stronger statement than $\neg(s = t)$, because the definedness is not required in the latter.

Notation 6. We shall use the following conventions

$$
\begin{aligned}
A \wedge B &:= \neg(\neg A \vee \neg B) && (1)\\
A \rightarrow B &:= \neg A \vee B && (2)\\
A \leftrightarrow B &:= (A \rightarrow B) \wedge (B \rightarrow A) && (3)\\
\forall x A &:= \neg \exists x \neg A && (4)\\
s \simeq t &:= (s\downarrow \vee t\downarrow) \rightarrow (s = t) && (5)\\
s \neq t &:= s\downarrow \wedge t\downarrow \wedge \neg(s = t) && (6)\\
t \in \mathsf{N} &:= \mathsf{N}(t) && (7)\\
t \in \Omega &:= \Omega(t) && (8)\\
t : \mathsf{N} &:= t \in \mathsf{N} && (9)\\
t : \Omega &:= t \in \Omega && (10)\\
(\exists x : \sigma) A &:= \exists x (x : \sigma \wedge A) && (11)\\
(\forall x : \sigma) A &:= \forall x (x : \sigma \rightarrow A) && (12)\\
t : \sigma \longrightarrow \tau &:= (\forall x : \sigma)(tx : \tau) && (13)\\
a_x &:= \left(\left(\mathsf{sup}^{-1} a\right) x\right) && (14)
\end{aligned}
$$

2.2 Definition of BON_Ω and basic properties

The theory BON_Ω is based on BON, whereas the axioms, that are equivalent to the typedness axioms, e.g. $0_\mathsf{N} : \mathsf{N}$ and $\mathsf{suc} : \mathsf{N} \longrightarrow \mathsf{N}$, were rewritten to fit the current context. Newly in BON_Ω, we have the special axioms for ordinals. Their intended meaning is the following:

- $0_\Omega : \Omega$

  The ordinal zero naturally is an ordinal numbers.

- $\mathsf{sup} : (\mathsf{N} \longrightarrow \Omega) \longrightarrow \Omega$

  The sup operator yields an ordinal number if applied to a function of natural numbers to ordinals.

- $\mathsf{sup}^{-1} : \Omega \longrightarrow (\mathsf{N} \longrightarrow \Omega)$

  The sup^{-1} operator is supposed to be the inverse of the sup operator, so it's type has to be inverse to the one of sup.

- $(e : \mathsf{N} \longrightarrow \Omega) \rightarrow \mathsf{sup} e \neq 0_\Omega \wedge \mathsf{sup}^{-1} (\mathsf{sup} e) = e$

  If e is (the coding of) a function from natural numbers to ordinals, i.e. a sequence of ordinals, then the supremum will never be 0_Ω and the supremum inverse is really the inverse function to the supremum function.

- $a : \Omega \rightarrow (a \neq 0_\Omega \rightarrow \mathsf{sup} (\mathsf{sup}^{-1} a) = a)$

  If a is an ordinal, other than 0_Ω, the the supremum is the inverse function of the supremum inverse.

- $x : \mathsf{N} \rightarrow (0_\Omega)_x = 0_\Omega$

  The supremum inverse of 0_Ω is the constant 0_Ω function, i.e. the supremum inverse function yields 0_Ω at every position x.

- $a = 0_\Omega \rightarrow \mathsf{d}_\Omega e_1 e_2 a = e_1$

  The definition by ordinal cases yields the first argument e_1 when 0_Ω is given. Note that we use here the constant 0_Ω rather than comparing two ordinals. This is due to the fact, that we did not define what the equality of two ordinals should be. Defining such an equality is not that simple a task to do, because the ordinals, with the exception of 0_Ω are the results of the sup function, which takes a whole function as input, rather than then a single value. Then there would be the question when are the inputs equal? Would that be just that the values need to be equal, or maybe the term would have to be equal, etc. And most importantly, it suffices that we can distinguish between 0_Ω and not 0_Ω.

- $a \in \Omega \wedge a \neq 0_\Omega \rightarrow \mathsf{d}_\Omega e_1 e_2 a = e_2$

  The definition by ordinal cases yields the second argument e_2 when an ordinal number other than 0_Ω is given.

- $(e : \mathsf{N} \longrightarrow \mathsf{N}) \wedge (\exists x \in \mathsf{N})\,(ex = 0_{\mathsf{N}}) \rightarrow e\,(\mu e) = 0_{\mathsf{N}}$

  The non-constructive μ operator yields a zero of the function e if it has one. As given by the typedness, the μ operator yields a value if applied to any function of natural numbers to natural numbers.

- $A\,(0_\Omega) \wedge (\forall a : \Omega)\,(a \neq 0_\Omega \wedge (\forall x : \mathsf{N})\,A\,(a_x) \rightarrow A\,(a)) \rightarrow (\forall a : \Omega)\,A\,(a)$

  The transfinite induction scheme says, that if the statement A holds for 0_Ω and we can show, that from A holding at each component of an ordinal number a, we can show that it also holds at a itself, then we can conclude that the statement A holds for all the ordinal numbers.

Definition 7. The theory BON_Ω has the following axioms:

1. propositional axioms and rules: as usual.

2. quantifier axioms and rules: for all formulae A, B, all terms t and all variables x:

$$A[t/x] \wedge\ t{\downarrow} \rightarrow \exists x A$$

$$\frac{A \rightarrow B}{\exists x A \rightarrow B} \qquad x \notin FV(B)$$

3. definedness axioms: for all n-ary function symbols f and relation symbols R and all terms s, t and $t_1, \ldots, t_n$:

$$t{\downarrow} \qquad \text{if } t \text{ is a variable or a constant}$$

$$f\,(t_1, \ldots, t_n){\downarrow} \rightarrow t_1{\downarrow} \wedge \cdots \wedge\ t_n{\downarrow}$$

$$(s = t) \rightarrow s{\downarrow} \wedge t{\downarrow}$$

$$R\,(t_1, \ldots, t_n) \rightarrow t_1{\downarrow} \wedge \cdots \wedge\ t_n{\downarrow}$$

4. equality axioms: or all n-ary function symbols f and relation symbols R and all terms s, t, $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$:

$$t = t \qquad \text{if } t \text{ is a variable or a constant}$$

$$(s = t) \rightarrow (t = s)$$

$$(t_1 = t_2) \wedge (t_2 = t_3) \rightarrow (t_1 = t_3)$$

$$R\,(s_1, \ldots, s_n) \wedge (s_1 = t_1) \wedge \cdots \wedge (s_n = t_n) \rightarrow R\,(t_1, \ldots, t_n)$$

$$(s_1 = t_1) \wedge \cdots \wedge (s_n = t_n) \rightarrow f\,(s_1, \ldots, s_n) \simeq f\,(t_1, \ldots, t_n)$$

5. Typedness axioms:

$$0_{\mathsf{N}} : \mathsf{N}$$

$$\mathsf{suc} : \mathsf{N} \longrightarrow \mathsf{N}$$

$$0_\Omega : \Omega$$

$$\mathsf{sup} : (\mathsf{N} \longrightarrow \Omega) \longrightarrow \Omega$$

$$\mathsf{sup}^{-1} : \Omega \longrightarrow (\mathsf{N} \longrightarrow \Omega)$$

6. partial combinatory algebra, for all variables x, y, z:

$$\mathsf{k}xy = x$$

$$\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq (xz)(yz)$$

7. natural numbers, for all variables x, y

$$(\forall x \in \mathsf{N})\left(x' \neq 0_\mathsf{N} \wedge \mathsf{p}_\mathsf{N}(x') = x\right)$$

$$(\forall x \in \mathsf{N})\left(x \neq 0_\mathsf{N} \rightarrow \mathsf{p}_\mathsf{N}x \in \mathsf{N} \wedge (\mathsf{p}_\mathsf{N}x)' = x\right)$$

$$A(0_\mathsf{N}) \wedge (\forall x : \mathsf{N})\left(A(x) \rightarrow A(x')\right) \rightarrow (\forall y : \mathsf{N})\, A(y)$$

8. definition by numerical cases, for all variables x, y, u, v

$$u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u = v \rightarrow \mathsf{d}_\mathsf{N}xyuv = x$$

$$u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u \neq v \rightarrow \mathsf{d}_\mathsf{N}xyuv = y$$

9. Ordinal Numbers, for all variables e, a, x

$$(e : \mathsf{N} \longrightarrow \Omega) \rightarrow \mathsf{sup}e \neq 0_\Omega \wedge \mathsf{sup}^{-1}(\mathsf{sup}e) = e$$

$$a : \Omega \rightarrow \left(a \neq 0_\Omega \rightarrow \mathsf{sup}\left(\mathsf{sup}^{-1}a\right) = a\right)$$

$$x : \mathsf{N} \rightarrow (0_\Omega)_x = 0_\Omega$$

10. Definition by cases on ordinal numbers, for all variables e_1, e_2, a

$$a = 0_\Omega \rightarrow \mathsf{d}_\Omega e_1 e_2 a = e_1 \tag{15}$$

$$a \in \Omega \wedge a \neq 0_\Omega \rightarrow \mathsf{d}_\Omega e_1 e_2 a = e_2 \tag{16}$$

11. Transfinite induction scheme for any formula $A(x)$

$$\begin{array}{rcl} A(0_\Omega) & \wedge & \\ (\forall a : \Omega)\left(a \neq 0_\Omega \wedge (\forall x : \mathsf{N})\, A(a_x) \rightarrow A(a)\right) & \rightarrow & (\forall a : \Omega)\, A(a) \end{array}$$

Remark 8. In the previous definition of the axiom schemes 6-10, we can use just the definition for variables, because from

$$\mathsf{BON}_\Omega \vdash \forall x A \wedge\ t\downarrow \rightarrow A[t/x]$$

we get immediately, that each of those axioms works for all defined terms.

As shown in detail in [13], BON_Ω has a term $\mathtt{fix}$, that yields a fixed point theorem in BON_Ω. The proof is not difficult and can be read in detail in the named paper.

Theorem 9. *There is a term* $\mathtt{fix}$, *such that*

$$\mathsf{BON}_\Omega \vdash \mathtt{fix}x\!\downarrow \wedge \mathtt{fix}xy \simeq x\,(\mathtt{fix}x)\,y$$

In the following, we try to get used to the notion of typedness in BON_Ω.

Remark 10. Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \tau \in \mathbb{T}$ and let t be a term of BON_Ω, then

$$t : (\sigma_1 \times \cdots \times \sigma_n) \longrightarrow \tau \equiv (\forall x_1 : \sigma_1) \cdots (\forall x_n : \sigma_n)\,(tx_1 \cdots x_n : \tau)$$

Example 11. As an example what exactly the types correspond to in BON_Ω, consider the following

$$t : (\mathsf{N} \longrightarrow \mathsf{N}) \longrightarrow (\mathsf{N} \longrightarrow \mathsf{N})$$

$$(\forall x : \mathsf{N} \longrightarrow \mathsf{N})\,(tx : \mathsf{N} \longrightarrow \mathsf{N})$$

$$\forall x\,((\forall y : \mathsf{N})\,(xy : \mathsf{N}) \rightarrow tx : \mathsf{N} \longrightarrow \mathsf{N})$$

$$\forall x\,((\forall y : \mathsf{N})\,(xy : \mathsf{N}) \rightarrow (\forall z : \mathsf{N})\,(txz : \mathsf{N}))$$

$$\forall x\,(\forall y\,(y : \mathsf{N} \rightarrow xy : \mathsf{N}) \rightarrow \forall z\,(z : \mathsf{N} \rightarrow txz : \mathsf{N}))$$

Product types are only abbreviations for linear types. To see how the terms correspond, you can check the following example.

Example 12. As an example for product types, consider the following

$$t : (\mathsf{N} \times \mathsf{N}) \longrightarrow \mathsf{N}$$

$$t : \mathsf{N} \longrightarrow (\mathsf{N} \longrightarrow \mathsf{N})$$

$$(\forall x : \mathsf{N})\,(tx : \mathsf{N} \longrightarrow \mathsf{N})$$

$$(\forall x : \mathsf{N})\,(\forall y : \mathsf{N})\,(txy : \mathsf{N})$$

so we see, that t takes two arguments of type N and returns $txy : \mathsf{N}$.

For every type there actually is a defined term that has the correct type, so we always can find a term for every type.

Lemma 13. *For any type* σ *there is a term* t *such that*

$$\mathsf{BON}_\Omega \vdash t : \sigma \wedge t\!\downarrow$$

Proof. By induction on the build-up of σ.

1. $\sigma \equiv \mathfrak{R}$ for $\mathfrak{R} \in \{\mathsf{N}, \Omega\}$. Then set $t := \mathsf{0}_\mathfrak{R}$. $\mathsf{0}_\mathfrak{R} : \mathfrak{R}$ is an axiom in both cases, $\mathfrak{R}\,(\mathsf{0}_\mathfrak{R}) \rightarrow \mathsf{0}_\mathfrak{R}\!\downarrow$ is an axiom too.

2. $\sigma \equiv \nu \longrightarrow \tau$. We apply the induction hypothesis to τ to get t_τ. Set $t := \mathsf{k}t_\tau$; we have from the induction hypothesis $t_\tau\downarrow$. The axiom $\mathsf{k}t_\tau x = t_\tau$ gives us that $\mathsf{k}t_\tau x\downarrow$ and so $\mathsf{k}t_\tau\downarrow$. But this is the same as $t\downarrow$.
$$t_\tau : \tau$$
$$\Longrightarrow (\forall x : \nu)(t_\tau : \tau)$$
$$\Longrightarrow (\forall x : \nu)(tx : \tau) \equiv t : \nu \longrightarrow \tau \equiv t : \sigma$$
□

It turns out, that we will later often want to get the definedness of a term from its typedness. This general property can be shown uniformly for any type. The following lemma simply states: If a term is of a particular type, then it is automatically defined.

Lemma 14. *Typedness implies definedness. For any type σ and any term t*
$$\mathsf{BON}_\Omega \vdash t : \sigma \to t\downarrow$$

Proof. By induction on the build-up of σ.

1. $\sigma \equiv \mathfrak{R}$ for $\mathfrak{R} \in \{\mathsf{N}, \Omega\}$. Then
$$t : \sigma \equiv t : \mathfrak{R} \equiv \mathfrak{R}(t)$$
since $\mathfrak{R}$ is a relation symbol, we get the definedness of t directly from the axiom $\mathfrak{R}(t) \to t\downarrow$.

2. $\sigma \equiv \nu \longrightarrow \tau$. Then
$$(t : \nu \longrightarrow \tau) \equiv (\forall x : \nu)(tx : \tau) \equiv \forall x (x : \nu \to tx : \tau) \equiv \neg\exists x \neg (x : \nu \to tx : \tau)$$
take the following axiom of BON_Ω, where $A := \neg(x : \nu \to tx : \tau)$ and s is a term such that $s : \nu \wedge s\downarrow$
$$A[s/x] \wedge s\downarrow \to \exists x A$$
$$(\neg(x : \nu \to tx : \tau)[s/x] \wedge s\downarrow) \to \exists x \neg (x : \nu \to tx : \tau)$$
contra position immediately yields
$$\forall x (x : \nu \to tx : \tau) \to ((s : \nu \to ts : \tau) \vee \neg s\downarrow)$$
Because $s\downarrow$, we get that $t : \sigma \to (s : \nu \to ts : \tau)$. With some propositional reasoning together with $s : \nu \wedge s\downarrow$ we get that $t : \sigma \to ts : \tau$. We can apply the induction hypothesis to get $ts : \tau \to ts\downarrow$. The definedness axioms then give us that $ts\downarrow \to t\downarrow$. Putting all those together gives us $t : \sigma \to t\downarrow$.

□

So when we put together the lemmas and definitions from this section, we get, that BON_Ω is an extended version of BON, that is extended by the notion of typedness of terms—as a formal statement. Furthermore we have some basic properties about how the typedness behaves, i.e. what product types mean and that we always find a term, that has a particular type and that typedness implies definedness.

3 The Theory QT_Ω

The typed theory QT_Ω has the typedness statements essentially coded into the types of the variables and the application. A term can only be applied to another term if their types match. Therefore this theory is total, in the sense, that all wrongly typed terms cannot be applied to each other already on a syntactical level. This convenience on the one side, yields two inconveniences: The typedness of the defining constants is coded into the language rather than the theory's axioms. And there is a need for the special recursor terms r^σ and $R_{\Omega,\sigma}$ in the language. Those can be proven to be just normal terms, without the need to being added in the case of BON_Ω.

3.1 Definitions of the language, terms and formulae for QT_Ω

All the constants from BON_Ω we find here as well, though they have types and therefore some of them occur multiple times, e.g. $\mathsf{k}^{\sigma,\tau}$ and $\mathsf{s}^{\rho,\sigma,\tau}$. Additionally, we have the combinators; those, as we will prove later in the translation, can be constructed in BON_Ω and were therefore not necessary in the definition of BON_Ω.

Definition 15. The Language $\mathcal{L}(\mathsf{QT}_\Omega)$. For each $\sigma \in \mathbb{T}$ there is a countably infinite supply of variables of type σ; we shall use $x^\sigma, y^\sigma, z^\sigma, u^\sigma, v^\sigma, w^\sigma$ for such variables. For each $\sigma \in \mathbb{T}$ there is a binary predicate $=_\sigma$ for equality at type σ; and for all $\sigma, \tau \in \mathbb{T}$ there is an application operator $\mathsf{Ap}^{\sigma,\tau}$. Furthermore, the language contains the following constants, for all $\sigma, \tau, \rho \in \mathbb{T}$, with „$c$ a constant of type σ“ indicated by „$c \in \sigma$“.

$$0_\mathsf{N} \in \mathsf{N}$$

$$\mathsf{suc} \in \mathsf{N} \longrightarrow \mathsf{N}$$

$$\mathsf{k}^{\sigma,\tau} \in (\sigma \times \tau) \longrightarrow \sigma$$

$$\mathsf{s}^{\rho,\sigma,\tau} \in ((\rho \longrightarrow (\sigma \longrightarrow \tau)) \times (\rho \longrightarrow \sigma) \times \rho) \longrightarrow \tau$$

$$\mathsf{r}^\sigma \in (\sigma \times ((\sigma \times \mathsf{N}) \longrightarrow \sigma) \times \mathsf{N}) \longrightarrow \sigma$$

$$0_\Omega \in \Omega$$

$$\mathsf{sup} \in (\mathsf{N} \longrightarrow \Omega) \longrightarrow \Omega$$

$$\mathsf{sup}^{-1} \in \Omega \longrightarrow (\mathsf{N} \longrightarrow \Omega)$$

$$R_{\Omega,\sigma} \in (((\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma) \times \sigma \times \Omega) \longrightarrow \sigma$$

The terms are defined similarly to the definition of terms in BON_Ω, just that the typedness is defined by the typedness of the $\mathsf{Ap}^{\sigma,\tau}$ function.

Definition 16. $\mathcal{L}(\mathsf{QT}_\Omega)$ terms. The terms are defined recursively:

1. variables and constants of type σ are terms of type σ
2. if t is a term of type $\sigma \longrightarrow \tau$, t' a term of type σ, then $\mathsf{Ap}^{\sigma,\tau}(t, t')$ is a term of type τ.

The definition of formulae is as usual, with the exception, that the equals relation $=_\sigma$ is only applicable to terms of equal types, so therefore we have an equals relation for every type. We usually skip writing the type explicitly, when it is clear from the context.

Definition 17. $\mathcal{L}(\mathsf{QT}_\Omega)$ formulae.

1. prime formulae are expressions of the form $t =_\sigma s$, where t and s are terms of type σ
2. a prime formula is a formula; arbitrary formulae are built from prime formulae with the help of the logical operators $\neg, \vee, \exists x^\sigma$.

3.2 Definition of QT_Ω and basic properties

In the context of the typed theory, some of the more complicated axioms, like e.g. the induction scheme, are simpler to formulate, because the correct typedness is already guaranteed by the way, how formulae are build. So writing something, where a term would not be of the correct type, would not be a formula according to the definition of formulae in QT_Ω. The intended meaning of the special axioms is:

- $\mathsf{r}xy0_\mathsf{N} = x \qquad \mathsf{r}xy\,(z') = y\,(\mathsf{r}xyz)\,z$

  The recursor r simulates primitive recursion. At the recursion level 0_N, it returns the value x, and the value at the level z' is the value of the function y applied to the value of the previous level and the level itself. With other words, the value at a level depends of the previous level, as well, as the level itself.

- $R_\Omega e_1 e_2 0_\Omega = e_2 \qquad a \neq 0_\Omega \to R_\Omega e_1 e_2 a = e_1 a\,(\lambda x.R_\Omega e_1 e_2 a_x)$

  Similarly to the previous case, the recursor states, that a value on a level a depends of the value at the levels of the components a_x. So here the next level depends on the whole function $\lambda x.R_\Omega e_1, e_2, a_x$.

Definition 18. QT_Ω has the following axioms.

1. propositional axioms and rules: as usual.
2. quantifier axioms and rules: for all formulae A, B, all terms t of type σ and all variables x of type σ:

$$A[t/x] \to \exists x A$$

$$\frac{A \to B}{\exists x A \to B} \qquad x \notin FV(B)$$

3. equality axioms: for equality at all types, we assume

$$x = x$$

$$x = y \to y = x$$

$$x = y \wedge y = z \to x = z$$

$$y = z \to xy = xz$$

$$x = y \to xz = yz$$

4. defining equations for the constants:

$$\mathsf{k}xy = x \qquad \mathsf{s}xyz = (xz)\,(yz)$$

$$\mathsf{r}xy0_{\mathsf{N}} = x \qquad \mathsf{r}xy\,(z') = y\,(\mathsf{r}xyz)\,z$$

5. arithmetical axioms (where x, y are of type N):

$$x' = y' \to x = y$$

$$0_{\mathsf{N}} \neq x'$$

$$A\,(0_{\mathsf{N}}) \wedge \forall x\,(A\,(x) \to A\,(x')) \to \forall y A\,(y)$$

6. additional axioms for tree ordinals: When e is of type $\mathsf{N} \longrightarrow \Omega$

$$\mathsf{sup} e \neq 0_\Omega \wedge \mathsf{sup}^{-1}\,(\mathsf{sup} e) = e$$

$$a \neq 0_\Omega \to \mathsf{sup}\left(\mathsf{sup}^{-1} a\right) = a$$

$$(0_\Omega)_x = 0_\Omega$$

7. the axioms about the recursor: where e_1 is of type $(\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma$, e_2 of type σ and a an ordinal number

$$R_\Omega e_1 e_2 0_\Omega = e_2$$

$$a \neq 0_\Omega \to R_\Omega e_1 e_2 a = e_1 a\,(\lambda x. R_\Omega e_1 e_2 a_x)$$

8. Induction on ordinal numbers

$$A\,(0_\Omega) \wedge \forall a\,(a \neq 0_\Omega \wedge \forall x A\,(a_x) \to A\,(a)) \to \forall a A\,(a)$$

4 Embedding QT_Ω in BON_Ω

In order to embed QT_Ω in BON_Ω, we need to be able to simulate all the axioms of PA_Ω. For that we define a translation of QT_Ω formulae into BON_Ω formulae and then prove, that if QT_Ω proves a formula, then BON_Ω proves the translation. As it turns out, most of the axioms of QT_Ω have their obvious counterparts in BON_Ω with the exception of the recursors. Earlier, we mentioned, that due to the strength of induction that we have in BON_Ω, we are able to prove that there are terms in BON_Ω that behave like the recursors, without having to add the recursors to the definition of our theory. Due to the fact, that those recursor terms are a little complicated and that the simulation of them need to show two things, namely that the resulting terms are the same as in QT_Ω and that the recursor is of the correct type, we make the following lemmas.

4.1 The numerical recursor term $\mathtt{rec}_\mathsf{N}$ in BON_Ω

The term $\mathtt{rec}_\mathsf{N}$ is the simulation of the r^σ recursor. From the proof we will get, that the recursor term does not depend of the type of the input and output; it is the same term for every type σ. In the first part, we show that the terms yield the expected and in the second part, we show that for every σ the recursor has the necessary type.

Lemma 19. *Numerical recursor. There is a term* $\mathtt{rec}_\mathsf{N}$*, such that*

1. $\mathsf{BON}_\Omega \vdash (z : \mathsf{N}) \to (\mathtt{rec}_\mathsf{N} xy\mathsf{0}_\mathsf{N} \simeq x \wedge \mathtt{rec}_\mathsf{N} xy\,(z') \simeq y\,(\mathtt{rec}_\mathsf{N} xyz)\,z)$
2. $\mathsf{BON}_\Omega \vdash \mathtt{rec}_\mathsf{N} : (\sigma \times (\sigma \times \mathsf{N} \longrightarrow \sigma) \times \mathsf{N}) \longrightarrow \sigma$

Proof. Set

$$\mathtt{rec}_\mathsf{N} := (\lambda xy.\mathtt{fix}t)$$

$$t := (\lambda ez.\mathsf{d}_\mathsf{N} x\,(y\,(e\,(\mathsf{p}_\mathsf{N} z))\,(\mathsf{p}_\mathsf{N} z))\,\mathsf{0}_\mathsf{N} z)$$

$$\begin{aligned} \mathtt{rec}_\mathsf{N} xyz &\simeq (\mathtt{fix}t)\,z \simeq t\,(\mathtt{fix}t)\,z \simeq t\,(\mathtt{rec}_\mathsf{N} xy)\,z \\ &\simeq \mathsf{d}_\mathsf{N} x\,(y\,((\mathtt{rec}_\mathsf{N} xy)\,(\mathsf{p}_\mathsf{N} z))\,(\mathsf{p}_\mathsf{N} z))\,\mathsf{0}_\mathsf{N} z \end{aligned}$$

from $z : \mathsf{N} \to z' : \mathsf{N} \wedge z' \neq \mathsf{0}_\mathsf{N}$ and $z : \mathsf{N}$ we get

$$\begin{aligned} \mathtt{rec}_\mathsf{N} xy\mathsf{0}_\mathsf{N} &\simeq x \\ \mathtt{rec}_\mathsf{N} xy\,(z') &\simeq y\,((\mathtt{rec}_\mathsf{N} xy)\,(\mathsf{p}_\mathsf{N}\,(z')))\,(\mathsf{p}_\mathsf{N}\,(z')) \\ &\simeq y\,((\mathtt{rec}_\mathsf{N} xy)\,z)\,z \\ &\simeq y\,(\mathtt{rec}_\mathsf{N} xyz)\,z \end{aligned}$$

So far, we only applied terms to other terms. As such, we a priori do not know, if there exist values and if such potential values have the necessary type.
To show the typedness, we need to recall remark 10:

$$\begin{aligned} &\mathtt{rec}_\mathsf{N} : (\sigma \times ((\sigma \times \mathsf{N}) \longrightarrow \sigma) \times \mathsf{N}) \longrightarrow \sigma \\ \equiv\ &(\forall x : \sigma)\,(\forall y : (\sigma \times \mathsf{N}) \longrightarrow \sigma)\,(\forall z : \mathsf{N})\,(\mathtt{rec}_\mathsf{N} xyz : \sigma) \end{aligned}$$

Now we can look at our partial equality from before, given those types. To show this, we need to apply induction on natural numbers. Set

$$A(z) := (\forall x : \sigma)(\forall y : (\sigma \times \mathsf{N}) \longrightarrow \sigma)(\mathtt{rec}_{\mathsf{N}} xyz : \sigma)$$

First we show $A(0_{\mathsf{N}})$.

$$\mathtt{rec}_{\mathsf{N}} xy 0_{\mathsf{N}} \simeq \mathsf{d}_{\mathsf{N}} x (y ((\mathtt{rec}_{\mathsf{N}} xy)(\mathsf{p}_{\mathsf{N}} 0_{\mathsf{N}}))(\mathsf{p}_{\mathsf{N}} 0_{\mathsf{N}})) 0_{\mathsf{N}} z \simeq x$$

from $x : \sigma$, we get $\mathtt{rec}_{\mathsf{N}} xy 0_{\mathsf{N}} : \sigma$, and so $A(0_{\mathsf{N}})$.
For $A(z) \to A(z')$ consider the following

$$\mathtt{rec}_{\mathsf{N}} xy (z') \simeq \mathsf{d}_{\mathsf{N}} x (y ((\mathtt{rec}_{\mathsf{N}} xy)(\mathsf{p}_{\mathsf{N}} 0_{\mathsf{N}}))(\mathsf{p}_{\mathsf{N}} 0_{\mathsf{N}})) 0_{\mathsf{N}} \simeq y (\mathtt{rec}_{\mathsf{N}} xyz) z$$

Consider, what $y : (\sigma \times \mathsf{N}) \longrightarrow \sigma$ means:

$$(\forall v : \sigma)(\forall n : \mathsf{N})(yvn : \sigma)$$

with other words, if we put v of type σ and a natural number n into y, we get something of type σ. $A(z)$, together with the premises, gives us $\mathtt{rec}_{\mathsf{N}} xyz : \sigma$. We also have given $z : \mathsf{N}$, but that means, that $\mathtt{rec}_{\mathsf{N}} xyz$ is such a v and z is such an n, that we get

$$y (\mathtt{rec}_{\mathsf{N}} xyz) z : \sigma$$

but given the partial equality $\mathtt{rec}_{\mathsf{N}} xy (z') \simeq y (\mathtt{rec}_{\mathsf{N}} xyz) z$, we get our $A(z')$:

$$\mathtt{rec}_{\mathsf{N}} xy (z') : \sigma$$

Applying induction, we get:

$$A(0_{\mathsf{N}}) \wedge (\forall x : \mathsf{N})(A(x) \to A(x')) \to (\forall z : \mathsf{N}) A(z)$$

so we get

$$(\forall z : \mathsf{N})(\forall x : \sigma)(\forall y : (\sigma \times \mathsf{N}) \longrightarrow \sigma)(\mathtt{rec}_{\mathsf{N}} xyz : \sigma)$$

this is exactly what we need. □

4.2 The ordinal recursor term $\mathtt{rec}_\Omega$ in BON_Ω

Analogously to the previous lemma, here we do the same for the ordinal recursor. Also the ordinal recursor is just one term, not depending of σ.

Lemma 20. *Ordinal recursor. There is a term* $\mathtt{rec}_\Omega$*, such that*

1. $\mathsf{BON}_\Omega \vdash a \in \Omega \wedge a \neq 0_\Omega \to \left(\mathtt{rec}_\Omega e_1 e_2 0_\Omega \simeq e_1 \wedge \mathtt{rec}_\Omega e_1 e_2 a \simeq e_1 a (\lambda x.\mathtt{rec}_\Omega e_1 e_2 a_x)\right)$
2. $\mathsf{BON}_\Omega \vdash \mathtt{rec}_\Omega : (((\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma) \times \sigma \times \Omega) \longrightarrow \sigma$

Proof. Set

$$\texttt{rec}_\Omega := (\lambda fa.\texttt{fix}t)$$

$$t := (\lambda ha.\mathsf{d}_\Omega e_2 \, (e_1 a \, (\lambda x.ha_x)) \, a)$$

$$\begin{aligned} \texttt{rec}_\Omega fa\alpha &\simeq (\texttt{fix}t) \, a \simeq t \, (\texttt{fix}t) \, a \simeq t \, (\texttt{rec}_\Omega e_1 e_2) \, a \\ &\simeq \mathsf{d}_\Omega e_2 \, (e_1 a \, (\lambda x.\texttt{rec}_\Omega e_1 e_2 a_x)) \, a \end{aligned}$$

So far, we only applied terms to other terms. As such, we a priori do not know, if there exist values and if such potential values have the necessary type. Recall the definition of typedness in BON_Ω and remark 10:

$$\begin{aligned} &\texttt{rec}_\Omega : (((\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma) \times \sigma \times \Omega) \longrightarrow \sigma \\ \equiv \;& (\forall e_1 : (\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma) \, (\forall e_2 : \sigma) \, (\forall a : \Omega) \, (\texttt{rec}_\Omega e_1 e_2 a : \sigma) \end{aligned}$$

To show this, we need induction on ordinal numbers. Set

$$A\,(a) := (\forall e_1 : (\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma) \, (\forall e_2 : \sigma) \, (\texttt{rec}_\Omega e_1 e_2 a : \sigma)$$

First we show $A\,(\mathsf{0}_\Omega)$.

$$\texttt{rec}_\Omega e_1 e_2 \mathsf{0}_\Omega \simeq \mathsf{d}_\Omega e_2 \, (e_1 a \, (\lambda x.\texttt{rec}_\Omega e_1 e_2 \mathsf{0}_{\Omega x})) \, \mathsf{0}_\Omega \simeq e_2$$

from $e_2 : \sigma$, we get $\texttt{rec}_\Omega e_1 e_2 \mathsf{0}_\Omega : \sigma$ and thus $A\,(\mathsf{0}_\Omega)$.
For $A\,(a_x) \rightarrow A\,(a)$, consider the following:
From $a : \Omega$ and $a \neq \mathsf{0}_\Omega$, we get

$$\texttt{rec}_\Omega e_1 e_2 a \simeq \mathsf{d}_\Omega e_2 \, (e_1 a \, (\lambda x.\texttt{rec}_\Omega e_1 e_2 a_x)) \, a \simeq e_1 a \, (\lambda x.\texttt{rec}_\Omega e_1 e_2 a_x)$$

The condition $e : (\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma$ gives us

$$e : (\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma$$

$$(\forall b : \Omega) \, (\forall e : \mathsf{N} \longrightarrow \sigma) \, (e_1 be : \sigma)$$

With other words, if we insert an ordinal number b and a function e from natural numbers to σ into the function e_1, we get a value in σ.
Consider the following partial equality:

$$\texttt{rec}_\Omega e_1 e_2 a_x \simeq (\lambda x.\texttt{rec}_\Omega e_1 e_2 a_x) \, x$$

From $(\forall x : \mathsf{N}) \, A\,(a_x)$, we get:

$$(\forall x : \mathsf{N}) \, (\texttt{rec}_\Omega e_1 e_2 a_x : \sigma)$$

$$(\forall x : \mathsf{N})\,((\lambda x.\mathtt{rec}_\Omega e_1 e_2 a_x)\, x : \sigma)$$

$$(\lambda x.\mathtt{rec}_\Omega e_1 e_2 a_x) : \mathsf{N} \longrightarrow \sigma$$

We therefore see, that $\lambda x.\mathtt{rec}_\Omega e_1 e_2 a_x$ is a function e, such that $e_1 b e : \sigma$. This gives us

$$e_1 a\,(\lambda x.\mathtt{rec}_\Omega e_1 e_2 a_x) : \sigma$$

and so

$$\mathtt{rec}_\Omega e_1 e_2 a : \sigma$$

As such, we now can apply the transfinite induction and we get:

$$\begin{aligned} & (\forall a : \Omega)\, A\,(a) \\ \equiv \quad & (\forall a : \Omega)\,(\forall e_1 : (\Omega \times (\mathsf{N} \longrightarrow \sigma)) \longrightarrow \sigma)\,(\forall e_2 : \sigma)\,(\mathtt{rec}_\Omega e_1 e_2 a : \sigma) \end{aligned}$$

and this is the typedness of $\mathtt{rec}_\Omega$. □

4.3 Translation of QT_Ω formulae to BON_Ω formulae and basic properties thereof

Now, that we showed, that we can simulate the recursors, we can finally define a translation of QT_Ω terms to BON_Ω terms. This translation will be used to define the translation of formulae which in turn is going to be used to state the embedding theorem. The typed constants QT_Ω are translated into their counterparts in BON_Ω, the same goes for variables. The two recursors are translated by the recursors terms we showed to exist in the previous lemmas. And the application in QT_Ω is translated to the application in BON_Ω.

Definition 21. Define a translation

$$\triangle : \mathcal{L}\,(\mathsf{QT}_\Omega) \longrightarrow \mathcal{L}\,(\mathsf{BON}_\Omega)$$

for terms, set

1. $\mathsf{0_N} \longmapsto \mathsf{0_N}$
2. $\mathsf{suc} \longmapsto \mathsf{suc}$
3. $\mathsf{k}^{\sigma,\tau} \longmapsto \mathsf{k}$
4. $\mathsf{s}^{\rho,\sigma,\tau} \longmapsto \mathsf{s}$
5. $\mathsf{0}_\Omega \longmapsto \mathsf{0}_\Omega$
6. $\mathsf{sup} \longmapsto \mathsf{sup}$
7. $\mathsf{sup}^{-1} \longmapsto \mathsf{sup}^{-1}$
8. $\mathsf{r}^\sigma \longmapsto \mathtt{rec}_\mathsf{N}$

9. $R_{\Omega,\sigma} \longmapsto \mathtt{rec}_\Omega$

10. $x^\sigma \longmapsto x$

11. $\mathsf{Ap}^{\sigma,\tau}(s,t) \longmapsto s^\triangle \cdot t^\triangle$

for formulae; let t and s be terms of type σ

1. $t =_\sigma s \longmapsto t^\triangle = s^\triangle$

2. $\neg A \longmapsto \neg A^\triangle$

3. $A \vee B \longmapsto A^\triangle \vee B^\triangle$

4. $\exists x^\sigma A \longmapsto (\exists x : \sigma)\, A^\triangle$

4.3.1 Substitution lemma of the translation

In the embedding proof, we will often need the following substitution lemma to apply the induction hypothesis.

Lemma 22. *Substitution lemma.*

$$A\,[t/x]^\triangle \equiv A^\triangle \left[t^\triangle/x^\triangle\right]$$

Proof. Consider the following cases:

1. A is a constant
$$A\,[t/x]^\triangle \equiv \underbrace{A^\triangle}_{\text{constant}} \equiv A^\triangle \left[t^\triangle/x^\triangle\right]$$

2. $A \equiv x$
$$A\,[t/x]^\triangle \equiv x\,[t/x]^\triangle \equiv t^\triangle \equiv x^\triangle \left[t^\triangle/x^\triangle\right]$$

3. $A \equiv y \not\equiv x$
$$A\,[t/x]^\triangle \equiv y\,[t/x]^\triangle \equiv y^\triangle \equiv y^\triangle \left[t^\triangle/x^\triangle\right]$$

4. $A \equiv s_1 =_\sigma s_2$
$$\begin{aligned} A\,[t/x]^\triangle &\equiv (s_1 =_\sigma s_2)\,[t/x]^\triangle \equiv (s_1\,[t/x] =_\sigma s_2\,[t/x])^\triangle \\ &\equiv (s_1\,[t/x])^\triangle = (s_2\,[t/x])^\triangle \\ &\equiv s_1^\triangle \left[t^\triangle/x^\triangle\right] = s_2^\triangle \left[t^\triangle/x^\triangle\right] \\ &\equiv \left(s_1^\triangle = s_2^\triangle\right) \left[t^\triangle/x^\triangle\right] \\ &\equiv A^\triangle \left[t^\triangle/x^\triangle\right] \end{aligned}$$

5. $A \equiv \neg B$
$$A\,[t/x]^\triangle \equiv (\neg B)\,[t/x]^\triangle \equiv \neg B\,[t/x]^\triangle \equiv \neg B^\triangle \left[t^\triangle/x^\triangle\right] \equiv A^\triangle \left[t^\triangle/x^\triangle\right]$$

6. $A \equiv B \vee C$

$$\begin{aligned} A[t/x]^\triangle &\equiv (B \vee C)[t/x]^\triangle \equiv (B[t/x] \vee C[t/x])^\triangle \equiv B[t/x]^\triangle \vee C[t/x]^\triangle \\ &\equiv B^\triangle\left[t^\triangle/x^\triangle\right] \vee C^\triangle\left[t^\triangle/x^\triangle\right] \equiv \left(B^\triangle \vee C^\triangle\right)\left[t^\triangle/x^\triangle\right] = A^\triangle\left[t^\triangle/x^\triangle\right] \end{aligned}$$

7. $A \equiv \exists x^\sigma B$

$$A[t/x]^\triangle \equiv A^\triangle \equiv A^\triangle\left[t^\triangle/x^\triangle\right]$$

8. $A \equiv \exists y^\sigma B$ and $y \not\equiv x$

$$\begin{aligned} A[t/x]^\triangle &\equiv ((\exists y^\sigma B)[t/x])^\triangle \equiv (\exists y^\sigma B[t/x])^\triangle \equiv (\exists y : \sigma)(B[t/x])^\triangle \\ &\equiv (\exists y : \sigma)\left(B^\triangle\left[t^\triangle/x^\triangle\right]\right) \equiv \left((\exists y : \sigma) B^\triangle\right)\left[t^\triangle/x^\triangle\right] \equiv A^\triangle\left[t^\triangle/x^\triangle\right] \end{aligned}$$

□

4.3.2 Typedness theorem of the translation

In the embedding theorem, we will assume the typedness of the variables and we will typically need that some term has the correct type. What the following theorem states, is that the typedness of the free variables implies the correct typedness of terms build up from those free variables. Or more precisely, in order to prove that the translation of a $\mathcal{L}(\mathsf{QT}_\Omega)$ term has the same type in BON_Ω, we only need to assume the correct typedness of the free variables, but not of the whole term. The correct typedness of the term is a *logical consequence* of the typedness of the free variables, rather than a *syntactical requirement.*

Theorem 23. *Typedness in* BON_Ω*. Let* $t[\vec{x}]$ *be an* $\mathcal{L}(\mathsf{QT}_\Omega)$ *term of type* σ*, with all free variables exposed. Further, let* x_1 *be of type* σ_1*,* x_2 *of type* σ_2*, ... and* x_n *of type* σ_n*, respectively. Then*

$$\mathsf{BON}_\Omega \vdash x_1 : \sigma_1 \wedge x_2 : \sigma_2 \wedge \cdots \wedge x_n : \sigma_n \to t^\triangle[\vec{x}] : \sigma$$

Proof. The proof is by induction on the build-up of the $\mathcal{L}(\mathsf{QT}_\Omega)$ term t.

1. $t \equiv \mathsf{0_N}$, so t is of type N:

$$t^\triangle = \mathsf{0_N}^\triangle = \mathsf{0_N}$$

   we have an axiom

$$\mathsf{0_N} : \mathsf{N}$$

   and this is exactly what we need.

2. $t \equiv \mathsf{suc}$, so t is of type $\mathsf{N} \longrightarrow \mathsf{N}$

$$t^\triangle = \mathsf{suc}^\triangle = \mathsf{suc}$$

   the following is an axiom and exactly what we need

$$\mathsf{suc} : \mathsf{N} \longrightarrow \mathsf{N}$$

3. $t \equiv \mathsf{k}^{\sigma,\tau}$, so $t : \sigma \longrightarrow (\tau \longrightarrow \sigma)$

$$t^{\triangle} = \left(\mathsf{k}^{\sigma,\tau}\right)^{\triangle} = \mathsf{k}$$

we need to show, that

$$\mathsf{k} : (\sigma \times \tau) \longrightarrow \sigma$$

$$(\forall x : \sigma)\,(\forall y : \tau)\,(\mathsf{k}xy : \sigma)$$

from $\mathsf{k}xy = x$, we immediately get the needed.

4. $t \equiv \mathsf{s}^{\rho,\sigma,\tau}$, so $t : ((\rho \longrightarrow (\sigma \longrightarrow \tau)) \times (\rho \longrightarrow \sigma) \times \rho) \longrightarrow \tau$

$$t^{\triangle} = \left(\mathsf{s}^{\rho,\sigma,\tau}\right)^{\triangle} = \mathsf{s}$$

we need to show the following

$$\mathsf{s} : ((\rho \longrightarrow (\sigma \longrightarrow \tau)) \times (\rho \longrightarrow \sigma) \times \rho) \longrightarrow \tau$$

$$(\forall x : \rho \longrightarrow (\sigma \longrightarrow \tau))\,(\forall y : \rho \longrightarrow \sigma)\,(\forall z : \rho)\,(\mathsf{s}xyz : \tau)$$

$$(\forall x : \rho \longrightarrow (\sigma \longrightarrow \tau))\,(\forall y : \rho \longrightarrow \sigma)\,(\forall z : \rho)\,((xz)\,(yz) : \tau)$$

from $y : \rho \longrightarrow \sigma$ and $z : \rho$, we get $yz : \sigma$, from $x : \rho \longrightarrow (\sigma \longrightarrow \tau)$ and $z : \rho$, we get $xz : \sigma \longrightarrow \tau$. And from $xz : \sigma \longrightarrow \tau$ and $yz : \sigma$, we get $(xz)\,(yz) : \tau$. And that is what we need.

5. $t \equiv \mathsf{0}_\Omega$, so $\mathsf{0}_\Omega : \Omega$

$$t^{\triangle} = (\mathsf{0}_\Omega)^{\triangle} = \mathsf{0}_\Omega$$

but $\mathsf{0}_\Omega : \Omega$ is an axiom of BON_Ω

6. $t \equiv \mathsf{sup}$, so $\mathsf{sup} : (\mathsf{N} \longrightarrow \Omega) \longrightarrow \Omega$

$$t^{\triangle} = (\mathsf{sup})^{\triangle} = \mathsf{sup}$$

but $\mathsf{sup} : (\mathsf{N} \longrightarrow \Omega) \longrightarrow \Omega$ is an axiom of BON_Ω

7. $t \equiv \mathsf{sup}$, so $\mathsf{sup}^{-1} : \Omega \longrightarrow (\mathsf{N} \longrightarrow \Omega)$

$$t^{\triangle} = \left(\mathsf{sup}^{-1}\right)^{\triangle} = \mathsf{sup}^{-1}$$

but $\mathsf{sup}^{-1} : \Omega \longrightarrow (\mathsf{N} \longrightarrow \Omega)$ is an axiom of BON_Ω

8. $t \equiv \mathsf{r}^\sigma$ for this case, the typedness is one half of lemma 19.

9. $t \equiv R_{\Omega,\sigma}$ for this case, the typedness is one half of lemma 20.

10. $t \equiv x$ for some variable, so $x : \sigma$

$$t^{\triangle} = (x)^{\triangle} = x$$

but $x : \sigma$ is the premise of this lemma.

11. $t \equiv \mathsf{Ap}^{\sigma,\tau}\left(a\left[\overrightarrow{x}\right], b\left[\overrightarrow{y}\right]\right)$, so $a : \sigma \longrightarrow \tau$, $b : \sigma$, $t : \tau$, $x_1 : \mu_1, \ldots, x_m : \mu_m$, $y_1 : \nu_1, \ldots y_n : \nu_n$

$$t^{\triangle} = \left(\mathsf{Ap}^{\sigma,\tau}\left(a\left[\overrightarrow{x}\right], b\left[\overrightarrow{y}\right]\right)\right) = a^{\triangle}\left[\overrightarrow{x}\right] \cdot b^{\triangle}\left[\overrightarrow{y}\right]$$

from the induction hypothesis we know, that

$$\mathsf{BON}_{\Omega} \vdash \underbrace{x_1 : \mu_1 \wedge \cdots \wedge x_m : \mu_m}_{P_1} \rightarrow a\left[\overrightarrow{x}\right] : \sigma \longrightarrow \tau$$

and

$$\mathsf{BON}_{\Omega} \vdash \underbrace{y_1 : \nu_1 \wedge \cdots \wedge y_n : \nu_n}_{P_2} \rightarrow b\left[\overrightarrow{y}\right] : \sigma$$

$$\Longrightarrow \mathsf{BON}_{\Omega} \vdash P_1 \wedge P_2 \rightarrow a\left[\overrightarrow{x}\right] : \sigma \longrightarrow \tau \wedge b\left[\overrightarrow{y}\right] : \sigma$$

$$\Longrightarrow \mathsf{BON}_{\Omega} \vdash P_1 \wedge P_2 \rightarrow (\forall z : \sigma)\left(a\left[\overrightarrow{x}\right] z : \tau\right) \wedge b\left[\overrightarrow{y}\right] : \sigma$$

$$\Longrightarrow \mathsf{BON}_{\Omega} \vdash P_1 \wedge P_2 \rightarrow a\left[\overrightarrow{x}\right] b\left[\overrightarrow{y}\right] : \tau$$

□

4.3.3 Modus ponens on translated formulae

In the embedding we do not go into detail how the basic logical rules translate, but one basic logical rule needs to be considered; the modus ponens. The problem there is, that the free variables in the conclusion are a subset of the free variables from the premises. And we need to show, that the free variables from the conclusion suffice for our embedding purposes. Therefore we show, that the modus ponens rule also works on the translated formulae:

Theorem 24. *Modus ponens on the translated formulae. Let $A\left[\overrightarrow{x}\right]$ and $B\left[\overrightarrow{y}\right]$ be $\mathcal{L}(\mathsf{BON}_{\Omega})$ formulae with all free variables exposed. Then*

$$\begin{array}{ll} & \mathsf{BON}_{\Omega} \vdash (x_1 : \sigma_1) \wedge \cdots \wedge (x_m : \sigma_m) \rightarrow A\left[\overrightarrow{x}\right]^{\triangle} \\ \& & \mathsf{BON}_{\Omega} \vdash (x_1 : \sigma_1) \wedge \cdots \wedge (x_m : \sigma_m) \wedge (y_1 : \tau_1) \wedge \cdots \wedge (y_n : \tau_n) \rightarrow \left(A\left[\overrightarrow{x}\right] \rightarrow B\left[\overrightarrow{y}\right]\right) \\ \Longrightarrow & \mathsf{BON}_{\Omega} \vdash (y_1 : \tau_1) \wedge \cdots \wedge (y_n : \tau_n) \rightarrow B\left[\overrightarrow{y}\right] \end{array}$$

Proof. We use the following abbreviations:

$$\begin{array}{rcl} X & := & (x_1 : \sigma_1) \wedge \cdots \wedge (x_m : \sigma_m) \\ Y & := & (y_1 : \tau_1) \wedge \cdots \wedge (y_n : \tau_n) \\ A & := & A\left[\overrightarrow{x}\right] \\ B & := & B\left[\overrightarrow{y}\right] \end{array}$$

so we can assume

$$X \rightarrow A \qquad X \wedge Y \rightarrow (A \rightarrow B)$$

the latter we can rewrite to $(X \to A) \to (X \wedge Y \to B)$ using some propositional reasoning. Now we can apply the modus ponens in BON_Ω:

$$\frac{X \to A \qquad (X \to A) \to (X \wedge Y \to B)}{X \wedge Y \to B}$$

now we can consider what the premise $X \wedge Y$ is:

$$(x_1 : \sigma_1) \wedge \cdots \wedge (x_m : \sigma_m) \wedge (y_1 : \tau_1) \wedge \cdots \wedge (y_n : \tau_n)$$

we introduce two new abbreviations X_c (for variables common A and B) and X_u (for variables unique to A) for x_i occurring in A and B or only A, respectively.

$$X_c := \bigwedge_{x_i \text{ free in } B} x_i : \sigma_i$$

$$X_u := \bigwedge_{x_i \text{ not free in } B} x_i : \sigma_i$$

then $X \leftrightarrow X_c \wedge X_u$ and since $X \wedge Y \to B$ can be proven, we can apply substitution. We substitute a term t_i of the type σ_i for every x_i not occurring freely in B. Such a term exist according to the lemma 13. But for each one of those terms, $\mathsf{BON}_\Omega \vdash t_i : \sigma_i$ and so we can cut them out of the premise to get

$$X_c \wedge Y \to B$$

but since all the variables from X_c are common, the terms formulae $x_i : \sigma_i$ of X_c all occur in Y. Therefore we can do a contraction to get

$$Y \to B$$

and this is what we need. □

4.4 The theorem for embedding QT_Ω into BON_Ω

For the embedding of QT_Ω into BON_Ω, we assume that the free variables be of the correct types. Then we get the correct typedness of terms using the typedness theorem and we apply the translation of formulae to prove all the axioms.

Theorem 25. *Embedding* QT_Ω *in* BON_Ω. *Let* $A[\overrightarrow{x}]$ *be an* $\mathcal{L}(\mathsf{QT}_\Omega)$ *formula with all free variables exposed. Further, let* x_1 *be of type* σ_1, x_2 *of type* σ_2, …, x_n *of type* σ_n, *respectively. Then*

$$\mathsf{QT}_\Omega \vdash A[\overrightarrow{x}] \Longrightarrow \mathsf{BON}_\Omega \vdash (x_1 : \sigma_1) \wedge (x_2 : \sigma_2) \wedge \cdots \wedge (x_n : \sigma_n) \to A[\overrightarrow{x}]^{\triangle}$$

Proof. To show, that our theory can prove any of the formulae, it suffices to show, that our theory can prove any of the axioms.

Because both theories use the usual propositional axioms and rules, we will go into detail only on the other axioms, with the exception of the modus ponens, because here we need to consider what happens with the free variables. In the following, always assume that A and B are formulae of QT_Ω.

1. Modus ponens: This is exactly the theorem 24.

2. Quantifier axioms and rules:

   (a)
   $$\left(A\left[\frac{a(\overrightarrow{y})}{x}\right]\rightarrow\exists x^\sigma A\right)^\triangle \rightsquigarrow \left(A\left[\frac{a(\overrightarrow{y})}{x}\right]\right)^\triangle\rightarrow(\exists x^\sigma A)^\triangle$$
   According to the previous lemma 22 this is the same as
   $$A^\triangle\left[\frac{(a(\overrightarrow{y}))^\triangle}{x^\triangle}\right]\rightarrow(\exists x:\sigma)\,A^\triangle$$
   We define a new formula
   $$B(z):=z:\sigma\wedge A^\triangle\left[\frac{z}{x^\triangle}\right]$$
   We now apply the axiom scheme of BON_Ω to get:
   $$B\left(\frac{(a(\overrightarrow{y}))^\triangle}{z}\right)\wedge(a(\overrightarrow{y}))^\triangle\downarrow\rightarrow\exists z B(z)$$
   $$\left((a(\overrightarrow{y}))^\triangle:\sigma\wedge A^\triangle\left[\frac{(a(\overrightarrow{y}))^\triangle}{x^\triangle}\right]\right)\wedge(a(\overrightarrow{y}))^\triangle\downarrow\rightarrow\exists z\left(z:\sigma\wedge A^\triangle\left[\frac{z}{x^\triangle}\right]\right)$$
   $$\left((a(\overrightarrow{y}))^\triangle:\sigma\wedge A^\triangle\left[\frac{(a(\overrightarrow{y}))^\triangle}{x^\triangle}\right]\right)\wedge(a(\overrightarrow{y}))^\triangle\downarrow\rightarrow(\exists z:\sigma)\,A^\triangle$$
   The typedness theorem 23 gives us that $\mathsf{BON}_\Omega\vdash(\overrightarrow{y})^\triangle:\overrightarrow{\tau}\rightarrow(a(\overrightarrow{y}))^\triangle:\sigma$. Since the free variables $\overrightarrow{y}$ of a are a subset of the free variables in the whole formula, we have the correct types of them in the premise of this theorem. Therefore we can deduce $(a(\overrightarrow{y}))^\triangle:\sigma$ and so we get the desired.

   (b) For the other quantifier rule
   $$\frac{(A\rightarrow B)^\triangle}{(\exists x^\sigma A\rightarrow B)^\triangle}\rightsquigarrow\frac{A^\triangle\rightarrow B^\triangle}{(\exists x:\sigma)\,A^\triangle\rightarrow B^\triangle}$$
   we have the following as induction hypothesis
   $$\overrightarrow{x}:\overrightarrow{\sigma}\rightarrow(A^\triangle\rightarrow B^\triangle)$$

since we do not a priori know if x occurs freely in $A^{\triangle} \to B^{\triangle}$, we have two possible cases

i. x does not occur freely and therefore it does not occur in $\overrightarrow{x} : \overrightarrow{\sigma}$. In the following tautology, set $C := \overrightarrow{x} : \overrightarrow{\sigma}$, $D := A^{\triangle}$, $E := B^{\triangle}$ and $F := x : \sigma$, respectively

$$(C \to (D \to E)) \to ((F \land D) \to (C \to E))$$

so we get

$$\left(x : \sigma \land A^{\triangle}\right) \to \left(\overrightarrow{x} : \overrightarrow{\sigma} \to B^{\triangle}\right)$$

since x does not occur freely in $A^{\triangle} \to B^{\triangle}$ and thus does not occur freely in $\overrightarrow{x} : \overrightarrow{\sigma}$, it indeed does not occur freely in $\overrightarrow{x} : \overrightarrow{\sigma} \to B^{\triangle}$ and so we can apply the quantifier rule

$$\frac{\left(x : \sigma \land A^{\triangle}\right) \to \left(\overrightarrow{x} : \overrightarrow{\sigma} \to B^{\triangle}\right)}{(\exists x : \sigma)\, A^{\triangle} \to \left(\overrightarrow{x} : \overrightarrow{\sigma} \to B^{\triangle}\right)}$$

now in the following tautology set $C := (\exists x : \sigma)\, A^{\triangle}$, $D := \overrightarrow{x} : \overrightarrow{\sigma}$ and $E := B^{\triangle}$, respectively

$$(C \to (D \to E)) \to (D \to (C \to E))$$

and this finally gives us

$$\overrightarrow{x} : \overrightarrow{\sigma} \to \left((\exists x : \sigma)\, A^{\triangle} \to B^{\triangle}\right)$$

ii. x occurs freely in $A^{\triangle} \to B^{\triangle}$, therefore $x : \sigma$ must be one of the conjuncts in $\overrightarrow{x} : \overrightarrow{\sigma}$. Without loss of generality assume, that $x : \sigma$ be the first conjunct and $\overrightarrow{x} : \overrightarrow{\sigma} \equiv x : \sigma \land \overrightarrow{x'} : \overrightarrow{\sigma'}$. In the following tautology, set $C := x : \sigma$, $D := \overrightarrow{x'} : \overrightarrow{\sigma'}$, $E := A^{\triangle}$ and $F := B^{\triangle}$, respectively.

$$((C \land D) \to (E \to F)) \to ((C \land E) \to (D \to F))$$

So we get

$$\left(x : \sigma \land A^{\triangle}\right) \to \left(\overrightarrow{x'} : \overrightarrow{\sigma'} \to B^{\triangle}\right)$$

since x does not occur freely on the right side of the implication, we can apply the quantifier axiom and get

$$\frac{\left(x : \sigma \land A^{\triangle}\right) \to \left(\overrightarrow{x'} : \overrightarrow{\sigma'} \to B^{\triangle}\right)}{(\exists x : \sigma)\, A^{\triangle} \to \left(\overrightarrow{x'} : \overrightarrow{\sigma'} \to B^{\triangle}\right)}$$

As in the previous case, now in the following tautology, set $C := (\exists x : \sigma)\, A^{\triangle}$, $D := \overrightarrow{x} : \overrightarrow{\sigma}$ and $E := B^{\triangle}$, respectively

$$(C \to (D \to E)) \to (D \to (C \to E))$$

and this finally gives us

$$\overrightarrow{x} : \overrightarrow{\sigma} \rightarrow \left((\exists x : \sigma)\, A^{\triangle} \rightarrow B^{\triangle}\right)$$

3. For equality axioms, we get the following:

   (a) $t\,[\overrightarrow{x}] =_{\sigma} t\,[\overrightarrow{x}]$

   $$(t\,[\overrightarrow{x}] =_{\sigma} t\,[\overrightarrow{x}])^{\triangle} \leftrightsquigarrow t\,[\overrightarrow{x}]^{\triangle} = t\,[\overrightarrow{x}]^{\triangle}$$

   from $\overrightarrow{x} : \overrightarrow{\sigma}$ and $t\,[\overrightarrow{x}]$ of type σ in QT_{Ω}, we get $t\,[\overrightarrow{x}] : \sigma$ and from typedness we get definedness. And therefore $t\,[\overrightarrow{x}]^{\triangle} = t\,[\overrightarrow{x}]^{\triangle}$ can be proved in BON_{Ω}. This happens by build-up of the term; for variables and constants, this is an axiom in itself and for the application of a function, we can get the equality from the induction hypothesis.

   (b) $(s =_{\sigma} t) \rightarrow (t =_{\sigma} s)$

   $$((s =_{\sigma} t) \rightarrow (t =_{\sigma} s))^{\triangle} \leftrightsquigarrow \left(s^{\triangle} = t^{\triangle}\right) \rightarrow \left(t^{\triangle} = s^{\triangle}\right)$$

   This in itself is an axiom of BON_{Ω}.

   (c) $(a =_{\sigma} b) \wedge (b =_{\sigma} c) \rightarrow (a =_{\sigma} c)$

   $$\begin{aligned} & ((a =_{\sigma} b) \wedge (b =_{\sigma} a) \rightarrow (a =_{\sigma} b))^{\triangle} \\ \leftrightsquigarrow\ & \left(a^{\triangle} = b^{\triangle}\right) \wedge \left(b^{\triangle} = a^{\triangle}\right) \rightarrow \left(a^{\triangle} = c^{\triangle}\right) \end{aligned}$$

   But this also is an axiom in itself.

   (d) $(a\,[\overrightarrow{x}] =_{\sigma} b\,[\overrightarrow{y}]) \rightarrow (c\,[\overrightarrow{z}]\,a\,[\overrightarrow{x}] =_{\tau} c\,[\overrightarrow{z}]\,b\,[\overrightarrow{y}])$

   $$\begin{aligned} & ((a\,[\overrightarrow{x}] =_{\sigma} b\,[\overrightarrow{y}]) \rightarrow (c\,[\overrightarrow{z}]\,a\,[\overrightarrow{x}] =_{\tau} c\,[\overrightarrow{z}]\,b\,[\overrightarrow{y}]))^{\triangle} \\ \leftrightsquigarrow\ & \left(a\,[\overrightarrow{x}]^{\triangle} = b\,[\overrightarrow{y}]^{\triangle}\right) \rightarrow \left(c\,[\overrightarrow{z}]^{\triangle}\,a\,[\overrightarrow{x}]^{\triangle} = c\,[\overrightarrow{z}]^{\triangle}\,b\,[\overrightarrow{y}]^{\triangle}\right) \end{aligned}$$

   in this case, the types are noteworthy: because of the nature of the typed calculus, c actually can not have just any arbitrary type, but it must be of type $\sigma \longrightarrow \tau$ for otherwise this would not even be a formula. From $a\,[\overrightarrow{x}]^{\triangle} = b\,[\overrightarrow{y}]^{\triangle}$, we immediately get $c\,[\overrightarrow{z}]^{\triangle}\,a\,[\overrightarrow{x}]^{\triangle} \simeq c\,[\overrightarrow{z}]^{\triangle}\,b\,[\overrightarrow{y}]^{\triangle}$. In order to show, that this partial equality is in fact a total one, we exploit the types: We get $a\,[\overrightarrow{x}]^{\triangle} : \sigma$, $b\,[\overrightarrow{y}]^{\triangle} : \sigma$ and also $c\,[\overrightarrow{z}]^{\triangle} : \sigma \longrightarrow \tau$ from the typedness theorem 23. As such $c\,[\overrightarrow{z}]^{\triangle}\,a\,[\overrightarrow{x}]^{\triangle} : \tau$ and therefore defined. And this gives us $c\,[\overrightarrow{z}]^{\triangle}\,a\,[\overrightarrow{x}]^{\triangle} = c\,[\overrightarrow{z}]^{\triangle}\,b\,[\overrightarrow{y}]^{\triangle}$

   (e) $(a\,[\overrightarrow{x}] =_{\sigma \longrightarrow \tau} b\,[\overrightarrow{y}]) \rightarrow (a\,[\overrightarrow{x}]\,c\,[\overrightarrow{z}] =_{\tau} b\,[\overrightarrow{y}]\,c\,[\overrightarrow{z}])$

   $$\begin{aligned} & ((a\,[\overrightarrow{x}] =_{\sigma \longrightarrow \tau} b\,[\overrightarrow{y}]) \rightarrow (a\,[\overrightarrow{x}]\,c\,[\overrightarrow{z}] =_{\tau} b\,[\overrightarrow{y}]\,c\,[\overrightarrow{z}]))^{\triangle} \\ \leftrightsquigarrow\ & \left(a\,[\overrightarrow{x}]^{\triangle} = b\,[\overrightarrow{y}]^{\triangle}\right) \rightarrow \left(a\,[\overrightarrow{x}]^{\triangle}\,c\,[\overrightarrow{z}]^{\triangle} = b\,[\overrightarrow{y}]^{\triangle}\,c\,[\overrightarrow{z}]^{\triangle}\right) \end{aligned}$$

   again, the types are important: a and b are of type $\sigma \longrightarrow \tau$ and c is of type σ, otherwise this would not even be a formula. From $a\,[\overrightarrow{x}]^{\triangle} = b\,[\overrightarrow{y}]^{\triangle}$, we get the partial equality $a\,[\overrightarrow{x}]^{\triangle}\,c\,[\overrightarrow{z}]^{\triangle} \simeq b\,[\overrightarrow{y}]^{\triangle}\,c\,[\overrightarrow{z}]^{\triangle}$. From the typedness theorem 23, we get, that $a\,[\overrightarrow{x}]^{\triangle} : \sigma \longrightarrow \tau$ and $c\,[\overrightarrow{z}]^{\triangle} : \sigma$ and so, of course $a\,[\overrightarrow{x}]^{\triangle}\,c\,[\overrightarrow{z}]^{\triangle} : \tau$. And from that, we get that the partial equality is in fact a total one.

4. now for the defining equation for the constants.

   (a)
   $$(\mathsf{k}xy =_{\sigma} x)^{\triangle} \leftrightsquigarrow (\mathsf{k}xy)^{\triangle} = x^{\triangle} \leftrightsquigarrow \mathsf{k}x^{\triangle}y^{\triangle} = x^{\triangle}$$
   and the last is an axiom of BON_{Ω}.

   (b)
   $$(\mathsf{s}xyz =_{\sigma} (xz)\,(yz))^{\triangle} \leftrightsquigarrow \mathsf{s}x^{\triangle}y^{\triangle}z^{\triangle} = \left(x^{\triangle}y^{\triangle}\right)\left(x^{\triangle}z^{\triangle}\right)$$
   notice, that in BON_{Ω} we only have the partial equality $\mathsf{s}x^{\triangle}y^{\triangle}z^{\triangle} \simeq \left(x^{\triangle}y^{\triangle}\right)\left(x^{\triangle}z^{\triangle}\right)$ from the axiom. On the other hand, from the type of s in QT_{Ω}, we know that x is of type $\rho \longrightarrow (\sigma \longrightarrow \tau)$, y of type $\rho \longrightarrow \sigma$ and z of type ρ, respectively. From the typedness theorem 23, we of course can get those types also in BON_{Ω}, apart from that, we also have the type of s, which gives us – together with the types of x, y and z – $\left(x^{\triangle}y^{\triangle}\right)\left(x^{\triangle}z^{\triangle}\right) : \sigma$. Therefore the partial equality is actually a total one.

   (c)
   $$(\mathsf{r}xy\mathsf{0_N} =_{\sigma} x)^{\triangle} \leftrightsquigarrow \mathtt{rec}_{\mathsf{N}}x^{\triangle}y^{\triangle}\mathsf{0_N} = x^{\triangle}$$
   this is an immediate consequence of lemma 19, together with the types of $x^{\triangle}$ and $y^{\triangle}$.

   (d)
   $$(\mathsf{r}xy\,(z') =_{\sigma} y\,(\mathsf{r}xyz)\,z)^{\triangle} \leftrightsquigarrow \mathtt{rec}_{\mathsf{N}}x^{\triangle}y^{\triangle}\left(z^{\triangle}\right)' = y^{\triangle}\left(\mathtt{rec}_{\mathsf{N}}x^{\triangle}y^{\triangle}z^{\triangle}\right)z^{\triangle}$$
   this is an immediate consequence of lemma 19, together with the types of $x^{\triangle}$,$y^{\triangle}$ and $z^{\triangle}$.

5. For the arithmetical axioms, consider the following:

   (a)
   $$(x' =_{\mathsf{N}} y' \rightarrow x =_{\mathsf{N}} y)^{\triangle} \leftrightsquigarrow \left(x^{\triangle}\right)' = \left(y^{\triangle}\right)' \rightarrow \left(x^{\triangle}\right) = y^{\triangle}$$
   From the typedness theorem 23, we of course get, that $x^{\triangle} : \mathsf{N}$ and $x^{\triangle} : \mathsf{N}$. The following line is an instance of an equality axiom of BON_{Ω}
   $$\left(x^{\triangle}\right)' = \left(y^{\triangle}\right)' \rightarrow \mathsf{p_N}\left(\left(x^{\triangle}\right)'\right) = \mathsf{p_N}\left(\left(y^{\triangle}\right)'\right)$$
   we immediately conclude
   $$\left(x^{\triangle}\right)' = \left(y^{\triangle}\right)' \rightarrow x^{\triangle} = y^{\triangle}$$
   because we know, that $x^{\triangle} : \mathsf{N}$ and $y^{\triangle} : \mathsf{N}$.

(b)

$$\left(0_{\mathsf{N}} \neq_{\sigma} x'\right)^{\triangle} \leftrightsquigarrow \neg\left(0_{\mathsf{N}} = \left(x^{\triangle}\right)'\right)$$

Here, the type of x' is not enough, we need to get the type of x. Luckily, in the typed system QT_{Ω}, x' can only be written, if x is of type N, for otherwise it would not be a term of the language, but then we get from the typedness theorem 23, that $x^{\triangle} : \mathsf{N}$ in BON_{Ω}. And from this we get the even stronger statement

$$0_{\mathsf{N}} \neq \left(x^{\triangle}\right)'$$

by using an axiom about the natural numbers.

(c)

$$\left(A\left(0_{\mathsf{N}}\right) \wedge \forall x^{\mathsf{N}}\left(A\left(x\right) \rightarrow A\left(x'\right)\right) \rightarrow \forall y^{\mathsf{N}} A\left(y\right)\right)^{\triangle}$$
$$\leftrightsquigarrow A^{\triangle}\left(0_{\mathsf{N}}\right) \wedge \left(\forall x : \mathsf{N}\right)\left(A^{\triangle}\left(x\right) \rightarrow A^{\triangle}\left(x'\right)\right) \rightarrow \left(\forall y : \mathsf{N}\right)\left(A^{\triangle}\left(y\right)\right)$$

and this is exactly the definition of the induction in BON_{Ω}.

6. For tree ordinals we have the following cases

(a)

$$\left(\mathsf{sup}e \neq_{\Omega} 0_{\Omega} \wedge \mathsf{sup}^{-1}\left(\mathsf{sup}e\right) =_{\mathsf{N}\longrightarrow\Omega} e\right)^{\triangle}$$
$$\leftrightsquigarrow \left(\mathsf{sup}e \neq_{\Omega} 0_{\Omega}\right)^{\triangle} \wedge \left(\mathsf{sup}^{-1}\left(\mathsf{sup}e\right) =_{\mathsf{N}\longrightarrow\Omega} e\right)^{\triangle}$$
$$\leftrightsquigarrow \neg\left(\mathsf{sup}\left(e^{\triangle}\right) = 0_{\Omega}\right) \wedge \mathsf{sup}^{-1}\left(\mathsf{sup}\left(e^{\triangle}\right)\right) = e^{\triangle}$$

From the typedness theorem 23, we get that $e^{\triangle} : \mathsf{N} \longrightarrow \Omega$ and we get from the axiom the even stronger statement

$$\mathsf{sup}e \neq 0_{\Omega} \wedge \mathsf{sup}^{-1}\left(\mathsf{sup}e\right) = e$$

(b)

$$\left(a \neq_{\Omega} 0_{\Omega} \rightarrow \mathsf{sup}\left(\mathsf{sup}^{-1}a\right) =_{\Omega} a\right)^{\triangle}$$
$$\leftrightsquigarrow \neg\left(a^{\triangle} = 0_{\Omega}\right) \rightarrow \mathsf{sup}\left(\mathsf{sup}^{-1}a^{\triangle}\right) = a^{\triangle}$$

similar to the previous case, we can get $a^{\triangle} : \Omega$ and so the stronger statement

$$a^{\triangle} \neq 0_{\Omega} \rightarrow \mathsf{sup}\left(\mathsf{sup}^{-1}a^{\triangle}\right) = a^{\triangle}$$

(c)

$$\left(\left(0_{\Omega}\right)_{x} =_{\Omega} 0_{\Omega}\right)^{\triangle} \leftrightsquigarrow \left(0_{\Omega}\right)_{x} = 0_{\Omega}$$

this is the case, provided we can show $x : \mathsf{N}$; but in QT_{Ω} x is of type N and therefore we get $x : \mathsf{N}$, so we can immediately apply the axiom and get the needed.

7. For the recursor we get the following:

(a)

$$(R_\Omega e_1 e_2 \mathsf{0}_\Omega =_\sigma e_2)^\triangle \leftrightsquigarrow \mathtt{rec}_\Omega e_1^\triangle e_2^\triangle \mathsf{0}_\Omega = e_2^\triangle$$

this is an immediate consequence of lemma 20, together with the types of $e_1^\triangle$ and $e_2^\triangle$.

(b)

$$(a \neq_\Omega \mathsf{0}_\Omega \to R_\Omega e_1 e_2 a =_\sigma e_1 a\,(\lambda x.R_\Omega e_1 e_2 a_x))^\triangle$$
$$\leftrightsquigarrow \neg\left(a^\triangle = \mathsf{0}_\Omega\right) \to \mathtt{rec}_\Omega e_1^\triangle e_2^\triangle a^\triangle = e_1^\triangle a^\triangle \left(\lambda x.\mathtt{rec}_\Omega e_1^\triangle e_2^\triangle \left(a^\triangle\right)_{x^\triangle}\right)$$

we first apply the typedness theorem 23 to e_1, e_2 and a. From that, we get the type of $a^\triangle$ and therefore, we can strengthen the statement $\neg\left(a^\triangle = \mathsf{0}_\Omega\right)$ to $a^\triangle \neq \mathsf{0}_\Omega$. This, together with the types of $e_1^\triangle$, $e_2^\triangle$ and $a^\triangle$ and together with lemma 20 gives us the needed.

8. For the induction on ordinal numbers consider the following:

$$(A\,(\mathsf{0}_\Omega) \wedge \forall a\,(a \neq_\Omega \mathsf{0}_\Omega \wedge \forall x A\,(a_x) \to A\,(a)) \to \forall a A\,(a))^\triangle$$

$$\begin{aligned} \leftrightsquigarrow A^\triangle\,(\mathsf{0}_\Omega) \quad &\wedge \quad (\forall a : \Omega)\left(\neg\,(a = \mathsf{0}_\Omega) \wedge (\forall x : \mathsf{N})\, A^\triangle\,(a_x) \to A^\triangle\,(a)\right) \\ &\to \quad (\forall a \in \Omega)\, A^\triangle\,(a) \end{aligned}$$

but in this case $\neg\,(a = \mathsf{0}_\Omega)$ is equivalent to $a \neq \mathsf{0}_\Omega$, because we already have $a : \Omega$. And so this gives us an instance of the induction scheme in BON_Ω.

□

5 The Theory $\mathsf{BON}_\Omega + \mu$

We are able to embed an extension of the theory BON_Ω into a well-known theory. By embedding this extension, we implicitly prove, that the addition of the non-constructive μ operator is harmless, in other words, it does not change the proof-theoretical strength of BON_Ω. The intended meaning of the additional axioms for $\mathsf{BON}_\Omega + \mu$ is the following

- $\mu : (\mathsf{N} \longrightarrow \mathsf{N}) \longrightarrow \mathsf{N}$

  The non-constructive μ operator is a total function that returns a natural number when applied to any function of natural numbers to natural numbers.

- $(e : \mathsf{N} \longrightarrow \mathsf{N}) \wedge (\exists x \in \mathsf{N})\,(ex = \mathsf{0}_\mathsf{N}) \to e\,(\mu e) = \mathsf{0}_\mathsf{N}$

  This formula means, that for any (coding of a) function from natural numbers to natural numbers, that has a zero, the μ operator returns a zero, i.e.

Definition 26. The theory $\mathsf{BON}_\Omega + \mu$ is an extension of BON_Ω. In addition to all the axioms and rules of BON_Ω it has the following additional axioms for the non-constructive μ operator:

$$\mu : (\mathsf{N} \longrightarrow \mathsf{N}) \longrightarrow \mathsf{N}$$

$$(e : \mathsf{N} \longrightarrow \mathsf{N}) \quad \wedge \quad (\exists x \in \mathsf{N})\,(ex = 0_{\mathsf{N}}) \rightarrow e\,(\mu e) = 0_{\mathsf{N}}$$

6 The Theory PA_Ω

The theory PA_Ω is an extension of the Peano arithmetic where inductive definitions are added. PA_Ω is slightly less minimalistic in the setting than ID_1, but it proves the same set of arithmetic statements. We will embed $\mathsf{BON}_\Omega + \mu$ into PA_Ω thus showing that $\mathsf{BON}_\Omega + \mu$ is not stronger than ID_1. When we combine the knowledge from the lower bounds with the proofs of the upper bounds, we get, that BON_Ω and $\mathsf{BON}_\Omega + \mu$ are indeed equivalent to ID_1. The typedness statements can be expressed formally and totality of typed functions can be proved using the induction principles.

In order to embed $\mathsf{BON}_\Omega + \mu$ in PA_Ω, we first define a „simulation relation". The intended purpose of it is, to simulate the behaviour of the application in $\mathsf{BON}_\Omega + \mu$. Next, we define a valuation of term from $\mathsf{BON}_\Omega + \mu$ to PA_Ω and finally a formula translation from $\mathsf{BON}_\Omega + \mu$ to PA_Ω. Other than in the lower bounds, PA_Ω immediately proves the translated $\mathsf{BON}_\Omega + \mu$ formula without the need of adding any premises, this will be proved in the embedding theorem.

6.1 Definitions of the language, terms and formulae for PA_Ω

Definition 27. The Language $\mathcal{L}\,(\mathsf{PA})$. Let $\mathcal{L}\,(\mathsf{PA})$ be the usual first-order language of arithmetic with number variables $a, b, c, u, v, w, x, y, z, \ldots$ (possibly with subscripts), the constant 0, as well as function and relation symbols for all primitive recursive functions and relations. The terms and formulae are defined as usual.

Remark 28. We assume the existence of a primitive recursive coding of sequences with $\mathsf{seq}_n\,(t)$ being fulfilled iff t is a sequence number of length n. We write $t = \langle s_0, \ldots, s_{n-1}\rangle$ to express, that t is (a coding of) the sequence $s_0, \ldots, s_{n-1}$. Furthermore, we write $(t)_i$ for the i-th component s_i of t.

In order to define the language needed in the rest of the thesis, we need an intermediate step, where we extend $\mathcal{L}\,(\mathsf{PA})$ with a new n-ary relation symbol P, not belonging to the language in order to get $\mathcal{L}\,(\mathsf{PA}, P)$. An $\mathcal{L}\,(\mathsf{PA}, P)$ formula is called *P-positive*, if each occurrence of P in the formula is positive. We call P-positive formulae which contain at most $\overrightarrow{x}$ free *inductive operator forms*, and let $A\,(P, \overrightarrow{x})$ range over such forms.

Definition 29. The language $\mathcal{L}\,(\mathsf{PA}_\Omega)$. Let $\mathcal{L}\,(\mathsf{PA}_\Omega)$ be an extension of $\mathcal{L}\,(\mathsf{PA})$. $\mathcal{L}\,(\mathsf{PA}_\Omega)$ contains a countably infinite supply of *ordinal variables* $\alpha, \beta, \gamma, \ldots$ (possibly with subscripts), a new binary relation symbol $<$ for the less relation on the ordinals and an $(n+1)$-ary relation symbol P_A for each inductive operator form $A\,(P, \overrightarrow{x})$ for which P is n-ary.

The *number terms* of $\mathcal{L}\,(\mathsf{PA}_\Omega)$ are the number terms of $\mathcal{L}\,(\mathsf{PA})$; the *ordinal terms* are the ordinal variables.

Definition 30. $\mathcal{L}(\mathsf{PA}_\Omega)$ formulae.

1. If R is an n-ary relation symbol of $\mathcal{L}(\mathsf{PA})$, then $R(s_1, \ldots, s_n)$ is an (atomic) formula.
2. $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_A(\alpha, \overrightarrow{s})$ are (atomic) formulae. We write $P_A^\alpha(\overrightarrow{s})$ for $P_A(\alpha, \overrightarrow{s})$.
3. If B and C are formulae, then $\neg B$ and $B \vee C$ are formulae.
4. If B is a formula, then $\exists x B$ are formulae.
5. If B is a formula, then $\exists \alpha B$ are formulae.
6. If B is a formula, $(\exists \alpha < \beta) B$ are formulae.

Notation 31. We use the following notations; for every $\mathcal{L}(\mathsf{PA}_\Omega)$ formula B, we write B^α to denote the formula, which is obtained by replacing all unbounded quantifiers $(\mathbb{Q}\beta)$ in B by $(\mathbb{Q}\beta < \alpha)$. Additional abbreviations are:

$$B \wedge C := \neg(\neg B \vee \neg C)$$

$$\forall x B := \neg \exists x \neg B$$
$$\forall \alpha B := \neg \exists \alpha \neg B$$
$$(\forall \alpha < \beta) B := \neg(\exists \alpha < \beta) \neg B$$
$$P_A^{<\alpha}(\overrightarrow{s}) := (\exists \beta < \alpha) P_A^\beta(\overrightarrow{s})$$
$$P_A(\overrightarrow{s}) := \exists \alpha P_A^\alpha(\overrightarrow{s})$$

Definition 32. An $\mathcal{L}(\mathsf{PA}_\Omega)$ formula is called a Σ^Ω formula if all negative existential ordinal quantifiers are bounded; correspondingly, it is called a Π^Ω formula, if all positive existential quantifiers are bounded.

6.2 Definition of PA_Ω and basic properties

The axioms of PA_Ω have the following intended meaning:

- $P_A^\alpha(\overrightarrow{s}) \leftrightarrow A(P_A^{<\alpha}, \overrightarrow{s})$
  - This means basically, that the set P_A is built up inductively, that is, an element of P_A is always added by applying the inductive operator A.
- $B \rightarrow \exists \alpha B^\alpha$ for Σ^Ω formulae
  - If a statement is true, then there must be a layer from which onwards it holds.

Definition 33. The Theory PA_Ω has the following axioms:

1. Number-theoretic axioms. These comprise the axioms of Peano Arithmetic PA with the exception of complete induction on the natural numbers.

2. Inductive operator axioms. For all inductive operator forms $A(P, \overrightarrow{x})$:

$$P_A^{\alpha}(\overrightarrow{s}) \leftrightarrow A(P_A^{<\alpha}, \overrightarrow{s})$$

3. Σ^{Ω}-reflection axioms. For every Σ^{Ω}-formula B:

$$B \to \exists \alpha B^{\alpha}$$

4. Linearity of the relation $<$ on the ordinals.

$$\alpha \not< \alpha \wedge (\alpha < \beta \wedge \beta < \gamma \to \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha)$$

5. Induction on the natural numbers. For all formulae $B(x)$:

$$B(0) \wedge (\forall x)(B(x) \to B(x')) \to (\forall x) B(x)$$

6. Induction on the ordinals. For all formulae $B(\alpha)$:

$$\forall \alpha ((\forall \beta < \alpha) B(\beta) \to B(\alpha)) \to (\forall \alpha) B(\alpha)$$

According to [8], the following fixed point theorem holds in PA_Ω, the proof of which is omitted:

Theorem 34. *For all inductive operator forms $A(P, \overrightarrow{x})$, and all formulae $B(\overrightarrow{x})$*

$$\mathsf{PA}_\Omega \vdash \forall \overrightarrow{x} (P_A(\overrightarrow{x}) \leftrightarrow A(P_A, \overrightarrow{x}))$$

$$\mathsf{PA}_\Omega \vdash \forall \overrightarrow{x} (A(B, \overrightarrow{x}) \to B(\overrightarrow{x})) \to \forall \overrightarrow{x} (P_A(\overrightarrow{x}) \to B(\overrightarrow{x}))$$

Because PA_Ω contains the induction scheme on natural numbers, the following so-called *least element principle* is provable in PA_Ω. We will use this property in the embedding theorem. The intended meaning is, that from having the existence of a number fulfilling a property, we also get, that there is a least such number.

Remark 35. For all $\mathcal{L}(\mathsf{PA}_\Omega)$ formulae A:

$$\mathsf{PA}_\Omega \vdash \exists x A(x) \to \exists x (A(x) \wedge (\forall y < x) \neg A(y))$$

7 Embedding $\mathsf{BON}_\Omega + \mu$ in PA_Ω

7.1 The inductive simulation operator and basic properties

In the following, we assume the existence of a numeral $\widehat{c}$ that is not a sequence number, for all the constants c of $\mathsf{BON}_\Omega + \mu$. Those numerals are all different, so that no clashes can occur.

This simulation relation has two parts, the „simulation“ and the definition of ordinal numbers. This is needed, because we need the *ordinal numbers* to define how the application works and vice versa. By a parallel inductive definition of both sets, we can define *one* inductive operator for both sets. The parallel induction is done by considering sets consisting of elements of the forms $(x, y, z, 0)$ (for application) and $(a, 0, 0, 1)$ (for ordinals); obviously the application part is disjoint from the ordinal part. The defining formulae, however, use the whole definition of the operator. Most of the formulae only describe how the result of the application of terms is supposed to be coded, whereas the other axioms have the following intended meaning:

- $\mathsf{seq}_2(x) \wedge (x)_0 = \hat{\mathsf{k}} \wedge (x)_1 = z$

  this is the k-combinator; $\left\langle \hat{\mathsf{k}}, s \right\rangle t$, yields s.

- $\mathsf{seq}_3(x) \wedge (x)_0 = \hat{\mathsf{s}} \wedge (\exists v, w)\left(P\left((x)_1, y, v, 0\right) \wedge P\left((x)_2, y, w, 0\right) \wedge P(v, w, z, 0)\right)$

  this is the s-combinator; $\langle \hat{\mathsf{s}}, s, t \rangle\, u$ yields $(su)(tu)$

- $\mathsf{seq}_4(x) \wedge (x)_0 = \widehat{\mathsf{d_N}} \wedge (x)_3 = y \wedge z = (x)_1$

  this is the definition by numerical cases for $u = v$; $\left\langle \widehat{\mathsf{d_N}}, s, t, u \right\rangle v$ yields s

- $\mathsf{seq}_4(x) \wedge (x)_0 = \widehat{\mathsf{d_N}} \wedge (x)_3 \neq y \wedge z = (x)_1$

  this is the definition by numerical cases for $u \neq v$; $\left\langle \widehat{\mathsf{d_N}}, s, t, u \right\rangle v$ yields t

- $\mathsf{seq}_3(x) \wedge (x)_0 = \widehat{\mathsf{d}_\Omega} \wedge y = \widehat{\mathsf{0}_\Omega} \wedge z = (x)_1$

  this is the definition by ordinal cases for $\mathsf{0}_\Omega$; $\left\langle \widehat{\mathsf{d}_\Omega}, s, t \right\rangle \widehat{\mathsf{0}_\Omega}$ yields s

- $\mathsf{seq}_3(x) \wedge (x)_0 = \widehat{\mathsf{d}_\Omega} \wedge y \neq \widehat{\mathsf{0}_\Omega} \wedge P(y, 0, 0, 1) \wedge z = (x)_2$

  this is the definition by ordinal cases for an ordinal u other than $\mathsf{0}_\Omega$; $\left\langle \widehat{\mathsf{d_N}}, s, t \right\rangle u$ yields t

- $x = \widehat{\mathsf{sup}} \wedge z = \langle \widehat{\mathsf{sup}}, y \rangle$

  sup applied to y is just simply the pair $\langle \widehat{\mathsf{sup}}, y \rangle$

- $x = \left\langle \mathsf{sup}^{-1}, \widehat{\mathsf{0}_\Omega} \right\rangle \wedge z = \widehat{\mathsf{0}_\Omega}$

  sup^{-1} applied to $\widehat{\mathsf{0}_\Omega}$ is the constant $\widehat{\mathsf{0}_\Omega}$ function.

- $x = \widehat{\mathsf{sup}^{-1}} \wedge (\exists e)\left(y = \langle \widehat{\mathsf{sup}}, e \rangle \wedge z = e\right)$

  sup^{-1} applied to a sup yields the inner term

- $x = \left\langle \mathsf{sup}^{-1}, \widehat{\mathsf{0}_\Omega} \right\rangle \wedge z = \widehat{\mathsf{0}_\Omega}$

  sup^{-1} applied to $\widehat{\mathsf{0}_\Omega}$ is the constant $\widehat{\mathsf{0}_\Omega}$ function.

- $x = \hat{\mu} \wedge \forall v \exists w\,(w \neq 0 \wedge P(y, v, w, 0)) \wedge z = 0$

  this is the first case of the non-constructive μ-operator. If the function y does not have a zero, then μ applied to it yields 0.

- $x = \hat{\mu} \wedge P(y, z, 0, 0) \wedge \forall v (v < z \to (\exists w)(w \neq 0 \wedge P(y, v, w, 0)))$

  this is the second case of the non-constructive μ-operator. The μ-operator yields a zero of the function y, that is, y applied to the result z is 0. Also it yields the smallest such value, that is, all values smaller do not yield 0 if y is applied to them.

Furthermore, we have formulae for ordinal numbers:

- $x = \widehat{0_\Omega}$

  We assume, that $\widehat{0_\Omega}$ be an ordinal number

- $\exists e (x = \langle \widehat{\mathsf{sup}}, e \rangle \wedge \forall u \exists a (P(a, 0, 0, 1) \wedge P(e, u, a, 0)))$

  if there is an e, such that $e \cdot u$ is an ordinal for arbitrary natural numbers$u$, then $\mathsf{sup}\, e$ is an ordinal.

Definition 36. We define an operator form $A(P, x, y, z, q)$, whereas P is an 4-ary relation symbol not belonging to the language. Let $A_i(P, x, y, z)$ be the formulae

$$x = \hat{\mathsf{k}} \wedge z = \left\langle \widehat{\mathsf{k}}, y \right\rangle \qquad A_1$$

$$\mathsf{seq}_2(x) \wedge (x)_0 = \hat{\mathsf{k}} \wedge (x)_1 = z \qquad A_2$$

$$x = \widehat{\mathsf{s}} \wedge z = \langle \widehat{\mathsf{s}}, y \rangle \qquad A_3$$

$$\mathsf{seq}_2(x) \wedge (x)_0 = \hat{\mathsf{s}} \wedge z = \langle \hat{\mathsf{s}}, (x)_1, y \rangle \qquad A_4$$

$$\mathsf{seq}_3(x) \wedge (x)_0 = \hat{\mathsf{s}} \wedge (\exists v, w)(P((x)_1, y, v, 0) \wedge P((x)_2, y, w, 0) \wedge P(v, w, z, 0)) \qquad A_5$$

$$x = \widehat{\mathsf{s_N}} \wedge z = y + 1 \qquad A_6$$

$$x = \widehat{\mathsf{p_N}} \wedge y = z + 1 \qquad A_7$$

$$x = \widehat{\mathsf{d_N}} \wedge z = \left\langle \widehat{\mathsf{d_N}}, y \right\rangle \qquad A_8$$

$$\mathsf{seq}_2(x) \wedge (x)_0 = \widehat{\mathsf{d_N}} \wedge z = \left\langle \widehat{\mathsf{d_N}}, (x)_1, y \right\rangle \qquad A_9$$

$$\mathsf{seq}_3(x) \wedge (x)_0 = \widehat{\mathsf{d_N}} \wedge z = \left\langle \widehat{\mathsf{d_N}}, (x)_1, (x)_2, y \right\rangle \qquad A_{10}$$

$$\mathsf{seq}_4(x) \wedge (x)_0 = \widehat{\mathsf{d_N}} \wedge (x)_3 = y \wedge z = (x)_1 \qquad A_{11}$$

$$\mathsf{seq}_4(x) \wedge (x)_0 = \widehat{\mathsf{d_N}} \wedge (x)_3 \neq y \wedge z = (x)_2 \qquad A_{12}$$

$$x = \widehat{\mathsf{d}_\Omega} \wedge z = \left\langle \widehat{\mathsf{d}_\Omega}, y \right\rangle \qquad A_{13}$$

$$\mathsf{seq}_2(x) \wedge (x)_0 = \widehat{\mathsf{d}_\Omega} \wedge z = \left\langle \widehat{\mathsf{d}_\Omega}, (x)_1, y \right\rangle \qquad A_{14}$$

$$\mathsf{seq}_3(x) \wedge (x)_0 = \widehat{\mathsf{d}_\Omega} \wedge y = \widehat{0_\Omega} \wedge z = (x)_1 \qquad A_{15}$$

$$\mathsf{seq}_3(x) \wedge (x)_0 = \widehat{\mathsf{d}_\Omega} \wedge y \neq \widehat{0_\Omega} \wedge P(y, 0, 0, 1) \wedge z = (x)_2 \qquad A_{16}$$

$$x = \widehat{\mathsf{sup}} \wedge z = \langle \widehat{\mathsf{sup}}, y \rangle \qquad A_{17}$$

$$x = \widehat{\mathsf{sup}^{-1}} \wedge y = \widehat{0_\Omega} \wedge z = \left\langle \widehat{\mathsf{sup}^{-1}}, y \right\rangle \qquad A_{18}$$

$$x = \widehat{\mathsf{sup}^{-1}} \wedge (\exists e)(y = \langle \widehat{\mathsf{sup}}, e \rangle \wedge z = e) \qquad A_{19}$$

$$x = \left\langle \mathsf{sup}^{-1}, \widehat{\mathsf{0}_\Omega} \right\rangle \wedge z = \widehat{\mathsf{0}_\Omega} \qquad A_{20}$$

$$x = \hat{\mu} \wedge \forall v \exists w \left(w \neq 0 \wedge P\left(y, v, w, 0\right)\right) \wedge z = 0 \qquad A_{21}$$

$$x = \hat{\mu} \wedge P\left(y, z, 0, 0\right) \wedge \forall v \left(v < z \to \left(\exists w\right)\left(w \neq 0 \wedge P\left(y, v, w, 0\right)\right)\right) \qquad A_{22}$$

Further let $A_\Omega\left(P, x\right)$ be the disjunction of the formulae

$$x = \widehat{\mathsf{0}_\Omega}$$

$$\exists e \left(x = \langle \widehat{\mathsf{sup}}, e \rangle \wedge \forall u \exists a \left(P\left(a, 0, 0, 1\right) \wedge P\left(e, u, a, 0\right)\right)\right)$$

Then

$$A\left(P, x, y, z, q\right) := \left(q = 0 \wedge \bigvee_i A_i\left(P, x, y, z\right)\right) \vee \left(q = 1 \wedge y = 0 \wedge z = 0 \wedge A_\Omega\left(P, x\right)\right)$$

7.1.1 The functionality of the simulation relation

The next result gives a functionality property in the last argument of the formulae $P_A^\alpha\left(x, y, z, q\right)$ and $P_A\left(x, y, z, q\right)$ which are induced by the operator form $A\left(P, x, y, z, q\right)$.

Notation 37. We use the abbreviations $\mathcal{T}\left(x, y, z\right)$ and $\mathbb{O}\left(a\right)$ for $P_A\left(x, y, z, 0\right)$ and $P_A\left(a, 0, 0, 1\right)$, respectively.

Lemma 38. PA_Ω *proves:*

$$\forall \alpha \forall x, y, u, v \Big(\mathcal{T}^\alpha\left(x, y, u\right) \wedge \mathcal{T}^\alpha\left(x, y, v\right) \quad \to \quad u = v\Big) \tag{23}$$

$$\forall x, y, u, v \Big(\mathcal{T}\left(x, y, u\right) \wedge \mathcal{T}\left(x, y, v\right) \quad \to \quad u = v\Big) \tag{24}$$

Proof. We prove the first property by induction on the levels α of the inductive definition. Let

$$\varphi\left(\gamma\right) := \forall x, y, u, v \left(\mathcal{T}^\gamma\left(x, y, u\right) \wedge \mathcal{T}^\gamma\left(x, y, v\right) \to u = v\right)$$

The induction scheme says, that

$$\forall \alpha \left(\left(\forall \beta < \alpha\right) \varphi\left(\beta\right) \to \varphi\left(\alpha\right)\right) \to \left(\forall \alpha\right)\left(\varphi\left(\alpha\right)\right)$$

Therefore, we assume that $\mathcal{T}^\beta\left(x, y, u\right) \wedge \mathcal{T}^\beta\left(x, y, v\right) \to u = v$ for all $\beta < \alpha$ and $\mathcal{T}^\alpha\left(x, y, u\right) \wedge \mathcal{T}^\alpha\left(x, y, v\right)$ for arbitrary variables x, y, u, v. We want to show $u = v$ from these assumptions, if we manage, we are done. In each defining clause of $A\left(P, x, y, z, q\right)$, we get some condition on the last argument. The condition on the last argument has different forms and we have to consider different cases:

- It is an equation (cases A_1, A_2, A_3, A_4, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{20} and A_{21}). So from the induction hypothesis we get $u = t\left[x, y\right]$ and $v = t\left[x, y\right]$. Then we get $u = v$ from the axioms about terms.

- It is $y = u + 1$ and $y = v + 1$ (cases A_6 and A_7). Then $u = v$ follows from the axioms about terms.

- It is $(\exists e_0)\,(y = \langle \widehat{\mathsf{sup}}, e_0\rangle \wedge u = e_0)$ and $(\exists e_1)\,(y = \langle \widehat{\mathsf{sup}}, e_1\rangle \wedge v = e_1)$ (case A_{19}). Because $\langle \cdot, \cdot \rangle$ is functional, we get that $e_0 = e_1$ and so $u = v$.

- It is $(\exists u_1, v_1)\,\big(\mathcal{T}^\beta(x_1, y, u_1) \wedge \mathcal{T}^\beta(x_2, y, v_1) \wedge \mathcal{T}^\beta(u_1, v_1, u)\big)$ and $(\exists u_2, v_2)\,\big(\mathcal{T}^\beta(x_1, y, u_2) \wedge \mathcal{T}^\beta(x_2, y, v_2) \wedge \mathcal{T}^\beta(u_2, v_2, v)\big)$ (case A_5).

  The induction hypothesis gives us, that $\mathcal{T}^\beta(x, y, w_1) \wedge \mathcal{T}^\beta(x, y, w_2) \to w_1 = w_2$. We can apply the induction hypothesis to $\mathcal{T}^\beta(x_1, y, u_1)$ and $\mathcal{T}^\beta(x_1, y, u_2)$, to get $u_1 = u_2$. From $\mathcal{T}^\beta(x_2, y, v_1)$ and $\mathcal{T}^\beta(x_2, y, v_2)$, we get $v_1 = v_2$. And so from $\mathcal{T}^\beta(u_1, v_1, u)$ and $\mathcal{T}^\beta(u_1, v_1, v)$, we get $u = v$.

- It is $\mathcal{T}^\beta(y, u, 0)$, $\mathcal{T}^\beta(y, v, 0)$, $\forall a\,\big(a < u \to (\exists w_1)\,\big(w_1 \neq 0 \wedge \mathcal{T}^\beta(y, a, w_1)\big)\big)$ and $\forall b\,\big(b < v \to (\exists w_2)\,\big(w_2 \neq 0 \wedge \mathcal{T}^\beta(y, b, w_2)\big)\big)$ (case A_{22}). Assume $u < v$.
  So $\exists w_1\,\big(w_1 \neq 0 \wedge \mathcal{T}^\beta(y, u, w_1)\big)$. But $\mathcal{T}^\beta(y, u, 0)$ and from induction hypothesis, we get that $w_1 = 0$ which is a contradiction. Therefore $u < v$ cannot hold. Analogously $v < u$ cannot hold and therefore $u = v$.

Putting all those together, we actually get, that our premises give us $u = v$. Therefore the premise of the induction scheme, i.e. $(\forall \beta < \alpha)\,\varphi(\beta) \to \varphi(\alpha)$, holds, and we get the conclusion

$$\forall \alpha \forall x, y, u, v\,(\mathcal{T}^\alpha(x, y, u) \wedge \mathcal{T}^\alpha(x, y, v) \to u = v)$$

So we showed the first statement of the lemma. And the second is just existential quantification over the first one. □

7.2 Valuation of terms of $\mathsf{BON}_\Omega + \mu$ in PA_Ω and basic properties thereof

Because a direct translation of $\mathsf{BON}_\Omega + \mu$ terms does not make sense, we go through an intermediate step of defining the valuation of terms. This valuation gives us a PA_Ω formula $\mathbb{V}_t(z)$ with the intended meaning: the term t has the value z. The most interesting case here is, how the application is interpreted: The term $s \cdot t$ has the value that is obtained by putting in the values of s and t into the simulation relation. So we see, that the simulation relation is directly used to define, how $\mathsf{BON}_\Omega + \mu$ terms are interpreted in PA_Ω.

Definition 39. Valuation:

1. $\mathbb{V}_x(z) \longmapsto z = x$
2. $\mathbb{V}_{\mathsf{s_N}}(z) \longmapsto z = \widehat{\mathsf{s_N}}$
3. $\mathbb{V}_{\mathsf{p_N}}(z) \longmapsto z = \widehat{\mathsf{p_N}}$
4. $\mathbb{V}_{\mathsf{s_\Omega}}(z) \longmapsto z = \widehat{\mathsf{s_\Omega}}$
5. $\mathbb{V}_{\mathsf{k}}(z) \longmapsto z = \widehat{\mathsf{k}}$

6. $\mathbb{V}_{\mathsf{s}}(z) \longmapsto z = \widehat{\mathsf{s}}$

7. $\mathbb{V}_{0_{\mathsf{N}}}(z) \longmapsto z = 0$

8. $\mathbb{V}_{\mathsf{d}_{\mathsf{N}}}(z) \longmapsto z = \widehat{\mathsf{d}_{\mathsf{N}}}$

9. $\mathbb{V}_{\mathsf{d}_{\Omega}}(z) \longmapsto \widehat{\mathsf{d}_{\Omega}}$

10. $\mathbb{V}_{0_{\Omega}}(z) \longmapsto z = \widehat{0_{\Omega}}$

11. $\mathbb{V}_{\mathsf{sup}}(z) \longmapsto z = \widehat{\mathsf{sup}}$

12. $\mathbb{V}_{\mathsf{sup}^{-1}}(z) \longmapsto z = \widehat{\mathsf{sup}^{-1}}$

13. $\mathbb{V}_{\mu}(z) \longmapsto z = \widehat{\mu}$

14. $\mathbb{V}_{s \cdot t}(z) \longmapsto \exists x \exists y \left(\mathbb{V}_s(x) \wedge \mathbb{V}_t(y) \wedge \mathcal{T}(x, y, z)\right)$

One would expect, that the value of a term should be functional, that is, if x and y are values of a term t, then $x = y$. This property is indeed true:

Remark 40. Notice, that as a direct consequence of Lemma 38, for any term t

$$\mathsf{PA}_{\Omega} \vdash \mathbb{V}_t(x) \wedge \mathbb{V}_t(y) \to x = y$$

7.2.1 Substitution lemma of the valuation

Remark 41. Furthermore one would expect, that the valuation should play nicely with substitutions. It should be possible to replace a substitution in $\mathsf{BON}_{\Omega} + \mu$ with a substitution in PA_{Ω}, whereas the term substituted in PA_{Ω} should be the value of the one substituted in $\mathsf{BON}_{\Omega} + \mu$. As is proved in the following lemma, this is indeed the case.

Lemma 42. *Let t, s be terms and x a variable of $\mathcal{L}(\mathsf{BON}_{\Omega})$. Then*

$$\mathbb{V}_t(u) \to \left(\mathbb{V}_{s[t/x]}(v) \leftrightarrow \mathbb{V}_s(v)[u/x]\right)$$

Proof. We prove this by induction on the build-up of s. □

1. $s \equiv x$

$$\mathbb{V}_{s[t/x]}(v) \Longleftrightarrow \mathbb{V}_t(v)$$

$$\mathbb{V}_s(v)[u/x] \Longleftrightarrow \mathbb{V}_x(v)[u/x] \Longleftrightarrow x = v[u/x] \Longleftrightarrow u = v$$

now consider the following

$$\mathbb{V}_t(u) \wedge \mathbb{V}_t(v) \to u = v$$

this gives us the implication from the left to the right

$$\mathbb{V}_t(u) \wedge u = v \to \mathbb{V}_t(v)$$

and this gives us the other implication

2. x does not occur freely. Then

$$\mathbb{V}_{s[t/x]}(v) \iff \mathbb{V}_s(v) \iff \mathbb{V}_s(v)[u/x]$$

because $\mathbb{V}$ does not introduce free variables.

3. $s \equiv t_1 \cdot t_2$

$$\mathbb{V}_{s[t/x]}(v) \iff (\exists y)(\exists z)\big(\mathbb{V}_{t_1[t/x]}(y) \wedge \mathbb{V}_{t_2[t/x]}(z) \wedge \mathcal{T}(y,z,v)\big)$$

by the induction hypothesis this is the equivalent to

$$\iff (\exists y)(\exists z)\left(\mathbb{V}_{t_1}(y)[u/x] \wedge \mathbb{V}_{t_2}(z)[u/x] \wedge \underbrace{\mathcal{T}(y,z,v)}_{x \text{ not free}}\right)$$

$$\iff (\exists y)(\exists z)(\mathbb{V}_{t_1}(y) \wedge \mathbb{V}_{t_2}(z) \wedge \mathcal{T}(y,z,v))[u/x]$$

$$\iff \mathbb{V}_s(v)[u/x]$$

7.3 The translation of $\mathsf{BON}_\Omega + \mu$ formulae to PA_Ω formulae and basic properties thereof

We translate BON_Ω+μ formulae to PA_Ω using the valuation defined above. The logical operators and quantifiers are simply passed through, whereas the atomic formulae are translated using the valuation. The intended meaning of each of the translations for atomic formulae is as follows:

- $(t\downarrow)^\diamond := \exists x \mathbb{V}_t(x)$

  a term is defined, if it has a value

- $(s = t)^\diamond := \exists x (\mathbb{V}_s(x) \wedge \mathbb{V}_t(x))$

  two terms are equal (remember, that equality implies definedness), if they have a common value

- $(\mathsf{N}(t))^\diamond := \exists x (\mathbb{V}_t(x))$

  a term is a natural number if it is defined and the value is a natural number. The second part can be skipped, however, since all the numbers are natural numbers anyway; the set of natural numbers is not defined inductively, but is a fixed part of the definition of PA_Ω.

- $(\Omega(t))^\diamond := \exists x (\mathbb{V}_t(x) \wedge \mathbb{O}(x))$

  a term is an ordinal number if it is defined and the value is an ordinal number. Here, the second part does indeed make sense, because we defined the set of ordinal numbers as a part of the definition of the simulation relation.

Definition 43. Translation for formulae:

1. $(t{\downarrow})^{\diamondsuit} \longmapsto \exists x \mathbb{V}_t(x)$
2. $(s = t)^{\diamondsuit} \longmapsto \exists x (\mathbb{V}_s(x) \wedge \mathbb{V}_t(x))$
3. $(\mathsf{N}(t))^{\diamondsuit} \longmapsto \exists x (\mathbb{V}_t(x))$
4. $(\Omega(t))^{\diamondsuit} \longmapsto \exists x (\mathbb{V}_t(x) \wedge \mathbb{O}(x))$
5. $(\neg A)^{\diamondsuit} \longmapsto \neg A^{\diamondsuit}$
6. $(A \vee B)^{\diamondsuit} \longmapsto A^{\diamondsuit} \vee B^{\diamondsuit}$
7. $(\exists x A)^{\diamondsuit} \longmapsto \exists x A^{\diamondsuit}$

7.3.1 Substitution lemma of the translation

One would wish for the translation of formulae to play nicely with the substitution of terms. If a term t is substituted for the variable x in $\mathsf{BON}_\Omega + \mu$, one would wish for this to be equivalent to substituting the value of t in PA_Ω instead. The following lemma yields us, that this is indeed true.

Lemma 44. *Let $A[x]$ be a formula and t be a term of $\mathcal{L}(\mathsf{BON}_\Omega)$. Then*

$$\mathsf{PA}_\Omega \vdash \mathbb{V}_t(z) \to \Big((A[t/x])^{\diamondsuit} \leftrightarrow A^{\diamondsuit}[z/x]\Big)$$

Proof. We prove this by induction on the build-up of A. Assume $\mathbb{V}_t(z)$:

1. If $A \equiv s{\downarrow}$.

$$((s{\downarrow})[t/x])^{\diamondsuit} \iff (s[t/x]{\downarrow})^{\diamondsuit} \iff \exists y \left(\mathbb{V}_{s[t/x]}(y)\right) \iff \exists y (\mathbb{V}_s(y)[z/x])$$

$$\iff (\exists y (\mathbb{V}_s(y)))[z/x] \iff (s{\downarrow})^{\diamondsuit}[z/x]$$

2. If $A \equiv s_1 = s_2$.

$$((s_1 = s_2)[t/x])^{\diamondsuit} \iff (s_1[t/x] = s_2[t/x])^{\diamondsuit} \iff \exists y \left(\mathbb{V}_{s_1[t/x]}(y) \wedge \mathbb{V}_{s_2[t/x]}(y)\right)$$

$$\exists y (\mathbb{V}_{s_1}(y)[z/x] \wedge \mathbb{V}_{s_2}(y)[z/x]) \iff (\exists y (\mathbb{V}_{s_1}(y) \wedge \mathbb{V}_{s_2}(y)))[z/x]$$

$$\iff (s_1 = s_2)^{\diamondsuit}[z/x]$$

3. If $A \equiv \mathsf{N}(s)$.

$$(\mathsf{N}(s)[t/x])^{\diamondsuit} \iff (\mathsf{N}(s[t/x]))^{\diamondsuit} \iff \exists y \left(\mathbb{V}_{s[t/x]}(y)\right) \iff \exists y (\mathbb{V}_s(y)[z/x])$$

$$\iff (\exists y (\mathbb{V}_s(y)))[z/x] \iff (\mathsf{N}(s))^{\diamondsuit}[z/x]$$

4. If $A \equiv \Omega(s)$.

$$(\Omega(s)[t/x])^\diamondsuit \iff (\Omega(s[t/x]))^\diamondsuit \iff \exists y \left(\mathbb{V}_{s[t/x]}(y) \wedge \mathbb{O}(y)\right) \iff \exists y \left(\mathbb{V}_{s[t/x]}(y) \wedge \mathbb{O}(y)\right)$$

$$\iff \exists y (\mathbb{V}_s(y)[z/x] \wedge \mathbb{O}(y)) \iff (\exists y (\mathbb{V}_s(y) \wedge \mathbb{O}(y)))[z/x] \iff (\Omega(s))^\diamondsuit [z/x]$$

5. If $A \equiv \neg B$.

$$(A[t/x])^\diamondsuit \iff ((\neg B)[t/x])^\diamondsuit \iff (\neg (B[t/x]))^\diamondsuit \iff \neg (B[t/x])^\diamondsuit$$

$$\iff \neg \left(B^\diamondsuit [z/x]\right) \iff (\neg B)^\diamondsuit [z/x] \iff (\neg B)^\diamondsuit [z/x] \iff A^\diamondsuit [z/x]$$

6. If $A \equiv B \vee C$.

$$(A[t/x])^\diamondsuit \iff ((B \vee C)[t/x])^\diamondsuit \iff (B[t/x] \vee C[t/x])^\diamondsuit$$

$$\iff \left(B^\diamondsuit [z/x] \vee C^\diamondsuit [z/x]\right) \iff \left(B^\diamondsuit \vee C^\diamondsuit\right)[z/x]$$
$$\iff (B \vee C)^\diamondsuit [z/x] \iff A^\diamondsuit [z/x]$$

7. If $A \equiv (\exists y) B$.

$$(A[t/x])^\diamondsuit \iff ((\exists y B)[t/x])^\diamondsuit \iff (\exists y (B[t/x]))^\diamondsuit \iff \exists y \left(B[t/x]^\diamondsuit\right)$$

$$\iff \exists y \left(B^\diamondsuit [z/x]\right) \iff \left(\exists y B^\diamondsuit\right)[z/x] \iff (\exists y B)^\diamondsuit [z/x]$$
$$\iff A^\diamondsuit [z/x]$$

□

7.4 The embedding theorem for $\mathsf{BON}_\Omega + \mu$ into PA_Ω

In this section we prove the embedding theorem for PA_Ω. That is, for every provable formula of $\mathsf{BON}_\Omega + \mu$ the translation can be proved in PA_Ω. This immediately yields us, that $\mathsf{BON}_\Omega + \mu$ it at most as strong as PA_Ω. Of PA_Ω we know, that it is as strong as ID_1. And so we get, that $\mathsf{BON}_\Omega + \mu$ is at most as strong as ID_1, which yields us the equivalence of $\mathsf{BON}_\Omega + \mu$ with ID_1.

Theorem 45. *Embedding $\mathsf{BON}_\Omega + \mu$ in PA_Ω. Let $A[\overrightarrow{x}]$ be an $\mathcal{L}(\mathsf{BON}_\Omega)$ formula with all free variables exposed. Then*

$$\mathsf{BON}_\Omega + \mu \vdash A[\overrightarrow{x}] \Longrightarrow \mathsf{PA}_\Omega \vdash A[\overrightarrow{x}]^\diamondsuit$$

Proof. To prove the assertion, it is enough to prove, that the translation works for every axiom of $\mathsf{BON}_\Omega + \mu$. Consider the following cases:

1. Quantifier axioms

(a) $(A[s/x] \wedge s\downarrow \to \exists x A)^{\diamond} \rightsquigarrow A[s/x]^{\diamond} \wedge (s\downarrow)^{\diamond} \to (\exists x A)^{\diamond}$.

$$\Longleftrightarrow (A[s/x])^{\diamond} \wedge \exists y \mathbb{V}_s(y) \to \exists x A^{\diamond} \tag{25}$$

Lemma 44 gives us the following, whereas we choose z to be a fresh variable:

$$\mathbb{V}_s(z) \to \left((A[s/x])^{\diamond} \leftrightarrow A^{\diamond}[z/x]\right)$$

So in particular the following holds:

$$\mathbb{V}_s(z) \to \left((A[s/x])^{\diamond} \to A^{\diamond}[z/x]\right)$$

In PA_Ω we have the quantifier axiom

$$A^{\diamond}[z/x] \to \exists x A^{\diamond}$$

those two put together yield

$$\mathbb{V}_s(z) \to \left((A[s/x])^{\diamond} \to \exists x A^{\diamond}\right)$$

since now z is not occurring freely in the conclusion, we may apply the quantifier rule of PA_Ω to get

$$(\exists y)\, \mathbb{V}_s(v) \to \left((A[s/x])^{\diamond} \to \exists x A^{\diamond}\right)$$

and then with some tautologies, we get

$$\left((A[s/x])^{\diamond} \wedge \exists y \mathbb{V}_s(y)\right) \to \exists x A^{\diamond}$$

and that is what we need.

(b)

$$\left(\frac{A \to B}{\exists x A \to B}\right)^{\diamond} \rightsquigarrow \frac{A^{\diamond} \to B^{\diamond}}{\exists x A^{\diamond} \to B^{\diamond}}$$

the latter is a rule of PA_Ω.

2. Definedness axioms

(a) $a\downarrow$ for some constant or variable a. Then $A^{\diamond} \rightsquigarrow a\downarrow^{\diamond}$

$$a\downarrow^{\diamond} \Longleftrightarrow \exists x \mathbb{V}_a(x) \Longleftrightarrow \exists x\,(x = \widehat{a})$$

and the latter holds of course because $\widehat{a}$ is such an x.

(b) $s \cdot t\downarrow \to s\downarrow \wedge t\downarrow$.

$$(s \cdot t\downarrow \to s\downarrow \wedge t\downarrow)^{\diamond} \rightsquigarrow (s \cdot t\downarrow)^{\diamond} \to (s\downarrow)^{\diamond} \wedge (t\downarrow)^{\diamond}$$

$$\rightsquigarrow \exists z \mathbb{V}_{s \cdot t}(x) \to \exists x \mathbb{V}_s(x) \wedge \exists y \mathbb{V}_t(y)$$

$$\exists z \exists u \exists v \left(\mathbb{V}_s(u) \wedge \mathbb{V}_t(v) \wedge \mathcal{T}(x,y,z)\right) \to \exists x \mathbb{V}_s(x) \wedge \exists y \mathbb{V}_t(y)$$

we get the existence of $\mathbb{V}_s$ and $\mathbb{V}_t$ in the premise, and so the right side holds as well.

(c) $(s = t) \to s{\downarrow} \wedge t{\downarrow}$

$$\left((s=t) \to s{\downarrow} \wedge t{\downarrow}\right)^{\diamondsuit} \leftrightsquigarrow \exists x \left(\mathbb{V}_s(x) \wedge \mathbb{V}_t(x)\right) \to \exists y \left(\mathbb{V}_s(y)\right) \wedge \exists z \left(\mathbb{V}_t(z)\right)$$

(d) $\mathsf{N}(t) \to t{\downarrow}$

$$\left(\mathsf{N}(t) \to t{\downarrow}\right)^{\diamondsuit} \leftrightsquigarrow \mathsf{N}(t)^{\diamondsuit} \to (t{\downarrow})^{\diamondsuit} \leftrightsquigarrow \exists x \mathbb{V}_t(x) \to \exists y \mathbb{V}_t(y)$$

(e) $\Omega(t) \to t{\downarrow}$

$$\left(\Omega(t) \to t{\downarrow}\right)^{\diamondsuit} \leftrightsquigarrow \Omega(t)^{\diamondsuit} \to (t{\downarrow})^{\diamondsuit} \leftrightsquigarrow \exists x \left(\mathbb{V}_t(x) \wedge A\langle x, 1\rangle\right) \to \exists y \mathbb{V}_t(y)$$

3. Equality axioms

(a) $t = t$ for some constant or variable.

$$(t=t)^{\diamondsuit} \leftrightsquigarrow \exists x \left(\mathbb{V}_t(x) \wedge \mathbb{V}_t(x)\right)$$

$$\iff \exists x \left(x = t \wedge x = t\right) \iff \exists x \left(x = t\right)$$

the existence is fulfilled for the x being t and so it clearly holds.

(b) $(s = t) \to (t = s)$

$$\left((s=t) \to (t=s)\right)^{\diamondsuit} \leftrightsquigarrow (s=t)^{\diamondsuit} \to (t=s)^{\diamondsuit}$$

$$\iff \exists x \left(\mathbb{V}_s(x) \wedge \mathbb{V}_t(x)\right) \to \exists y \left(\mathbb{V}_t(y) \wedge \mathbb{V}_s(y)\right)$$

(c) $(t_1 = t_2) \wedge (t_2 = t_3) \to (t_1 = t_3)$

$$\left((t_1 = t_2) \wedge (t_2 = t_3) \to (t_1 = t_3)\right)^{\diamondsuit}$$

$$\iff \exists x \left(\mathbb{V}_{t_1}(x) \wedge \mathbb{V}_{t_2}(x)\right) \wedge \exists y \left(\mathbb{V}_{t_2}(y) \wedge \mathbb{V}_{t_3}(y)\right) \to \exists z \left(\mathbb{V}_{t_1}(z) \wedge \mathbb{V}_{t_3}(z)\right)$$

from remark 40, we get, that in fact $x = y$. So we get $\exists v \left(\mathbb{V}_{t_1}(v) \wedge \mathbb{V}_{t_2}(v) \wedge \mathbb{V}_{t_3}(v)\right)$ and therefore the conclusion holds.

(d) $\mathsf{N}(s) \wedge (s = t) \to \mathsf{N}(t)$

$$\left(\mathsf{N}(s) \wedge (s=t) \to \mathsf{N}(t)\right)^{\diamondsuit} \leftrightsquigarrow \exists x \mathbb{V}_s(x) \wedge \exists y \left(\mathbb{V}_s(y) \wedge \mathbb{V}_t(y)\right) \to \exists z \left(\mathbb{V}_t(z)\right)$$

(e) $\Omega(s) \wedge (s = t) \to \Omega(t)$

$$\left(\Omega(s) \wedge (s=t) \to \Omega(t)\right)^{\diamondsuit}$$

$$\leftrightsquigarrow \exists x \left(\mathbb{V}_s(x) \wedge \mathbb{O}(x)\right) \wedge \exists y \left(\mathbb{V}_s(y) \wedge \mathbb{V}_t(y)\right) \to \exists z \left(\mathbb{V}_t(z) \wedge \mathbb{O}(z)\right)$$

we get, that $x = y = z$. So we get, that

$$\exists x\left(\mathbb{V}_s(x) \wedge \mathbb{O}(x)\right) \wedge \exists y\left(\mathbb{V}_s(y) \wedge \mathbb{V}_t(y)\right) \to \exists z\left(\mathbb{V}_s(z) \wedge \mathbb{V}_t(z) \wedge \mathbb{O}(z)\right)$$

which gives us what we need

(f) $(s_1 = t_1) \wedge (s_2 = t_2) \to (s_1 \cdot s_2 \simeq t_1 \cdot t_2)$

$$\left((s_1 = t_1) \wedge (s_2 = t_2) \to (s_1 \cdot s_2 \simeq t_1 \cdot t_2)\right)^{\diamondsuit}$$

$$\iff (s_1 = t_1) \wedge (s_2 = t_2) \to \left((s_1 \cdot s_2{\downarrow} \vee t_1 \cdot t_2{\downarrow}) \to s_1 \cdot s_2 = t_1 \cdot t_2\right)$$

$$\begin{array}{rl} & \exists x_1\left(\mathbb{V}_{s_1}(x_1) \wedge \mathbb{V}_{t_1}(x_1)\right) \\ \wedge & \exists x_2\left(\mathbb{V}_{s_2}(x_2) \wedge \mathbb{V}_{t_2}(x_2)\right) \\ \to & \exists z_3\left(\mathbb{V}_{s_1 \cdot s_2}(z_3)\right) \vee \exists z_4\left(\mathbb{V}_{t_1 \cdot t_2}(z_4)\right) \\ \to & \exists z_5\left(\mathbb{V}_{s_1 \cdot s_2}(z_5) \wedge \mathbb{V}_{t_1 \cdot t_2}(z_5)\right) \end{array}$$

given the first two clauses, the third clause (the disjunction) implies the conclusion. Therefore, we can assume them and only need to prove the conclusion. We need to show, that

$$\exists z_3\left(\mathbb{V}_{s_1 \cdot s_2}(z_3)\right) \vee \exists z_4\left(\mathbb{V}_{t_1 \cdot t_2}(z_4)\right) \to \exists z_5\left(\mathbb{V}_{s_1 \cdot s_2}(z_5) \wedge \mathbb{V}_{t_1 \cdot t_2}(z_5)\right)$$

the premise is equivalent to the following

$$\begin{array}{rl} & \exists z_3 \exists x_3 \exists y_3\left(\mathbb{V}_{s_1}(x_3) \wedge \mathbb{V}_{s_2}(y_3) \wedge \mathcal{T}(x_3, y_3, z_3)\right) \\ \vee & \exists z_4 \exists x_4 \exists y_4\left(\mathbb{V}_{t_1}(x_4) \wedge \mathbb{V}_{t_2}(y_4) \wedge \mathcal{T}(x_4, y_4, z_4)\right) \end{array}$$

from the premise we get $x_3 = x_4$ and $y_3 = y_4$ and so from $\mathcal{T}(x_3, y_{3,} z_3)$ and $\mathcal{T}(x_3, y_3, z_4)$, we get that $z_3 = z_4$. Therefore, we can join the two statements:

$$\exists z \exists x \exists y\left(\mathbb{V}_{s_1}(x) \wedge \mathbb{V}_{t_1}(x) \wedge \mathbb{V}_{s_2}(y) \wedge \mathbb{V}_{t_2}(y) \wedge \mathcal{T}(x, y, z)\right)$$

and this is equivalent to the conclusion.

4. Typedness axioms

(a)

$$(\mathsf{0_N} : \mathsf{N})^{\diamondsuit} \leftrightsquigarrow (\mathsf{N}(\mathsf{0_N}))^{\diamondsuit} \leftrightsquigarrow \exists x \mathbb{V}_{\mathsf{0_N}}(x) \iff \exists x (x = 0)$$

(b)

$$(\mathsf{suc} : \mathsf{N} \longrightarrow \mathsf{N})^{\diamondsuit} \leftrightsquigarrow ((\forall x : \mathsf{N})(\mathsf{suc}x : \mathsf{N}))^{\diamondsuit} \leftrightsquigarrow (\forall x (\mathsf{N}(x) \to \mathsf{N}(\mathsf{suc}x)))^{\diamondsuit}$$

$$\iff \forall x\left(\exists y \mathbb{V}_x(y) \to \exists y \mathbb{V}_{\mathsf{suc} \cdot x}(y)\right)$$

$$\iff \forall x\left(\exists y \mathbb{V}_x(y) \to \exists z \exists u \exists v\left(\mathbb{V}_{\mathsf{suc}}(u) \wedge \mathbb{V}_x(v) \wedge \mathcal{T}(u, v, z)\right)\right)$$

from this we get, that $v = y$ and that $u = \widehat{\mathsf{suc}}$, therefore our conclusion requires, that $\mathcal{T}(\widehat{\mathsf{suc}}, y, z)$, but that is true for $z = y + 1$.

(c)

$$(\mathsf{0}_\Omega : \Omega)^\diamondsuit \leftrightsquigarrow (\Omega(\mathsf{0}_\Omega))^\diamondsuit \leftrightsquigarrow \exists x\left(\mathbb{V}_{\mathsf{0}_\Omega}(x) \wedge \mathbb{O}(x)\right)$$

so we get, that $x = \widehat{\mathsf{0}_\Omega}$ to fulfil the left part of the conclusion. From the definition of P_A, we get, that $\mathbb{O}(\mathsf{0}_\Omega)$ holds. Therefore the whole statement holds.

(d)

$$(\mathsf{sup} : (\mathsf{N} \longrightarrow \Omega) \longrightarrow \Omega)^\diamondsuit \leftrightsquigarrow ((\forall x : \mathsf{N} \longrightarrow \Omega)(\mathsf{sup}x : \Omega))^\diamondsuit$$
$$\leftrightsquigarrow (\forall x((\forall y(y : \mathsf{N} \to xy : \Omega)) \to (\mathsf{sup}x : \Omega)))^\diamondsuit$$
$$\leftrightsquigarrow (\forall x((\forall y(\mathsf{N}(y) \to \Omega(xy))) \to \Omega(\mathsf{sup}x)))^\diamondsuit$$
$$\leftrightsquigarrow \forall x\left((\forall y(\exists v_1 \mathbb{V}_y(v_1) \to \exists v_2(\mathbb{V}_{xy}(v_2) \wedge \mathbb{O}(v_2)))) \to \exists v_3(\mathbb{V}_{\mathsf{sup}x}(v_3) \wedge \mathbb{O}(v_3))\right)$$

so we get, that $v_1 = y$, because y is a variable.

$$\begin{aligned}\iff \forall x\ &\Big(\forall y(\exists v_2 \exists v_4 \exists v_5(\mathbb{V}_x(v_4) \wedge \mathbb{V}_y(v_5) \wedge \mathcal{T}(v_4, v_5, v_2) \wedge \mathbb{O}(v_2)))\\ &\to \exists v_3(\mathbb{V}_{\mathsf{sup}x}(v_3) \wedge \mathbb{O}(v_3))\Big)\end{aligned}$$

we get, that $v_4 = x$ and $v_5 = y$

$$\iff \forall x\left((\forall y \exists v_2(\mathcal{T}(x, y, v_2) \wedge \mathbb{O}(v_2))) \to \exists v_3(\mathbb{V}_{\mathsf{sup}x}(v_3) \wedge \mathbb{O}(v_3))\right)$$

$$\begin{aligned}\iff \forall x\ &\Big((\forall y \exists v_2(\mathcal{T}(x, y, v_2) \wedge \mathbb{O}(v_2)))\\ &\to \exists v_3(\exists v_6 \exists v_7(\mathbb{V}_{\mathsf{sup}}(v_6) \wedge \mathbb{V}_x(v_7) \wedge \mathcal{T}(v_6, v_7, v_3)) \wedge \mathbb{O}(v_3))\Big)\end{aligned}$$

we get, that $v_6 = \widehat{\mathsf{sup}}$ and $v_7 = x$

$$\iff \forall x\left((\forall y \exists v_2(\mathcal{T}(x, y, v_2) \wedge \mathbb{O}(v_2))) \to \exists v_3(\mathcal{T}(\widehat{\mathsf{sup}}, x, v_3) \wedge \mathbb{O}(v_3))\right)$$

we get, that $v_3 = \langle \widehat{\mathsf{sup}}, x \rangle$

$$\iff \forall x\left((\forall y \exists v_2(\mathcal{T}(x, y, v_2) \wedge \mathbb{O}(v_2))) \to \mathbb{O}(\langle \widehat{\mathsf{sup}}, x \rangle)\right)$$

in order to prove this, we assume the premise. Consider what is $A(P_A, w, 0, 0, 1)$ for $w = \langle \widehat{\mathsf{sup}}, x \rangle$:

$$A(P_A, w, 0, 0, 1) \iff \exists w_1(w = \langle \widehat{\mathsf{sup}}, w_1 \rangle \wedge \forall w_2 \exists w_3(\mathbb{O}(w_3) \wedge \mathcal{T}(w_1, w_2, w_3)))$$

we get, that $w_1 = x$

$$\Longleftrightarrow \forall w_2 \exists w_3 \left(w = \langle \widehat{\mathsf{sup}}, x \rangle \wedge \mathbb{O}\left(w_3\right) \wedge \mathcal{T}\left(x, w_2, w_3\right)\right)$$

$$\Longleftrightarrow \forall y \exists w_3 \left(w = \langle \widehat{\mathsf{sup}}, x \rangle \wedge \mathbb{O}\left(w_3\right) \wedge \mathcal{T}\left(x, y, w_3\right)\right)$$

we get, that $w_3 = v_2$

$$\Longleftrightarrow \forall y \exists v_2 \left(w = \langle \widehat{\mathsf{sup}}, x \rangle \wedge \mathbb{O}\left(v_2\right) \wedge \mathcal{T}\left(x, y, v_2\right)\right)$$

From $\forall x \left(A\left(P_A, \langle \widehat{\mathsf{sup}}, x \rangle, 0, 0, 1\right) \to \mathbb{O}\left(\langle \widehat{\mathsf{sup}}, x \rangle\right)\right)$, we get

$$\forall y \exists v_2 \left(\mathbb{O}\left(v_2\right) \wedge \mathcal{T}\left(x, y, v_2\right)\right) \to \mathbb{O}\left(\langle \widehat{\mathsf{sup}}, x \rangle\right)$$

This is what we need.

(e)

$$(\mu : (\mathsf{N} \longrightarrow \mathsf{N}) \longrightarrow \mathsf{N})^{\diamondsuit} \leftrightsquigarrow ((\forall e : \mathsf{N} \longrightarrow \mathsf{N})\,(\mu e : \mathsf{N}))^{\diamondsuit} \leftrightsquigarrow (\forall e\,(e : \mathsf{N} \longrightarrow \mathsf{N} \to \mu e : \mathsf{N}))^{\diamondsuit}$$

$$\leftrightsquigarrow (\forall e\,((\forall x : \mathsf{N})\,(ex : \mathsf{N}) \to \mu e : \mathsf{N}))^{\diamondsuit} \leftrightsquigarrow \forall e \left(((\forall x : \mathsf{N})\,(ex : \mathsf{N}))^{\diamondsuit} \to (\mu e : \mathsf{N})^{\diamondsuit}\right)$$

$$\leftrightsquigarrow \forall e \left(\forall x \left((x : \mathsf{N})^{\diamondsuit} \to (ex : \mathsf{N})^{\diamondsuit}\right) \to (\mu e : \mathsf{N})^{\diamondsuit}\right)$$

now we apply the definition of the translation. Consider, that $(x : \mathsf{N})^{\diamondsuit}$ is true for every variable, because it translates to $\exists z \left(\mathbb{V}_x\left(z\right)\right)$ which translates to $\exists z \left(z = x\right)$ which is obviously fulfilled for $z = x$. Therefore we can replace it with $\top$.

$$\Longleftrightarrow \forall e \left(\forall x \left(\exists y \left(\mathbb{V}_{ex}\left(y\right)\right)\right) \to \exists z \left(\mathbb{V}_{\mu e}\left(z\right)\right)\right)$$

now we apply the translation of the application and get

$$\Longleftrightarrow \forall e \left(\forall x \left(\exists y \left(\exists u_1 \exists u_2 \left(\mathbb{V}_e\left(u_1\right) \wedge \mathbb{V}_x\left(u_2\right) \wedge \mathcal{T}\left(u_1, u_2, y\right)\right)\right)\right) \to \exists z \left(\mathbb{V}_{\mu e}\left(z\right)\right)\right)$$

so we get $u_2 = x$ and $u_1 = e$

$$\Longleftrightarrow \forall e \left(\forall x \exists y \left(\mathcal{T}\left(e, x, y\right)\right) \to \exists z \left(\mathbb{V}_{\mu e}\left(z\right)\right)\right)$$

according to the definition of the valuation, this is equivalent to

$$\Longleftrightarrow \forall e \left(\forall x \exists y \left(\mathcal{T}\left(e, x, y\right)\right) \to \exists z \left(\exists u_3, u_4 \left(\mathbb{V}_\mu\left(u_3\right) \wedge \mathbb{V}_e\left(u_4\right) \wedge \mathcal{T}\left(u_3, u_4, z\right)\right)\right)\right)$$

we immediately get, that $u_3 = \widehat{\mu}$ and $u_4 = e$

$$\Longleftrightarrow \forall e \left(\forall x \exists y \left(\mathcal{T}\left(e, x, y\right)\right) \to \exists z \left(\mathcal{T}\left(\widehat{\mu}, e, z\right)\right)\right)$$

in order to prove this statement, we assume $\forall x \exists y \left(\mathcal{T}\left(e, x, y\right)\right)$ for an arbitrary e and prove the conclusion. We consider two cases:

i. $\forall x \forall y \left(\mathcal{T}(e,x,y) \to y \neq 0\right)$. If we apply the fixed point theorem to $\mathcal{T}(\widehat{\mu}, e, z)$, we get that $\exists z \left(\mathcal{T}(\widehat{\mu}, e, z)\right)$ is fulfilled by $z = 0$, according to A_{21}.

ii. $\neg\left(\forall x \forall y \left(\mathcal{T}(e,x,y) \to y \neq 0\right)\right)$ and this is equivalent to $\exists x \left(\mathcal{T}(e,x,0)\right)$. By applying the fixed point theorem in this case, together with the premises, we get

$$\exists z \left(\mathcal{T}(\widehat{\mu}, e, z)\right)$$

$$\iff \exists z \left(\mathcal{T}(e,z,0) \wedge \forall v \left(v < z \to \exists w \left(w \neq 0 \wedge \mathcal{T}(e,v,w)\right)\right)\right)$$

When we apply the least element principle from remark 35 to the premise, we get

$$\exists x \left(\mathcal{T}(e,x,0) \wedge (\forall y < x)\left(\neg\mathcal{T}(e,y,0)\right)\right)$$

when we combine the premise $\forall x \exists y \left(\mathcal{T}(e,x,y)\right)$ with the previous line, we get

$$\exists x \left(\mathcal{T}(e,x,0) \wedge (\forall y < x)\left(\neg\mathcal{T}(e,y,0) \wedge \exists u \left(\mathcal{T}(e,y,u)\right)\right)\right)$$

and $\neg\mathcal{T}(e,y,0) \wedge \mathcal{T}(e,y,u)$ gives us, that $u \neq 0 \wedge \mathcal{T}(e,y,u)$ and that is what we need.

5. Defining axioms for the constants

(a) $\mathsf{k}xy = x$

$$(\mathsf{k}st = s)^{\diamond} \leftrightsquigarrow \exists z \left(\mathbb{V}_{\mathsf{k}st}(z) \wedge \mathbb{V}_s(z)\right)$$

$$\iff \exists z \left(\exists u_1 \exists u_2 \left(\mathbb{V}_{\mathsf{k}s}(u_1) \wedge \mathbb{V}_t(u_2) \wedge \mathcal{T}(u_1,u_2,z)\right) \wedge \mathbb{V}_s(z)\right)$$

$$\iff \exists z \exists u_1 \exists u_2 \left(\mathbb{V}_{\mathsf{k}s}(u_1) \wedge \mathbb{V}_t(u_2) \wedge \mathbb{V}_s(z) \wedge \mathcal{T}(u_1,u_2,z)\right)$$

$$\iff \quad \exists z \exists u_1 \exists u_2 \exists v_1 \exists v_2 \\ \left(\mathbb{V}_{\mathsf{k}}(v_1) \wedge \mathbb{V}_s(v_2) \wedge \mathbb{V}_t(u_2) \wedge \mathbb{V}_s(z) \wedge \mathcal{T}(u_1,u_2,z) \wedge \mathcal{T}(v_1,v_2,u_1)\right)$$

but this only can be true, if $v_2 = z$ and $v_1 = \widehat{\mathsf{k}}$, because we have $\mathbb{V}_s(v_2) \wedge \mathbb{V}_s(z)$ and $\mathbb{V}_{\mathsf{k}}(v_1)$.

$$\iff \exists z \exists u_1 \exists u_2 \left(\mathbb{V}_s(z) \wedge \mathbb{V}_t(u_2) \wedge \mathcal{T}(u_1,u_2,z) \wedge \mathcal{T}\left(\widehat{\mathsf{k}}, z, u_1\right)\right)$$

in order for $\mathcal{T}\left(\widehat{\mathsf{k}}, z, u_1\right)$ to be true, $u_1 = \left\langle \hat{\mathsf{k}}, z \right\rangle$. If we set u_1 thus, we do not need to require $\mathcal{T}\left(\widehat{\mathsf{k}}, z, u_1\right)$, because this is always the case and so we get

$$\iff \exists z \exists u_2 \left(\mathbb{V}_s(z) \wedge \mathbb{V}_t(u_2) \wedge \mathcal{T}\left(\left\langle \widehat{\mathsf{k}}, z \right\rangle, u_2, z\right)\right)$$

according to A_3, $\mathcal{T}\left(\left\langle \widehat{\mathsf{k}}, z\right\rangle, u_2, z\right)$ is true for any z and u_2.

$$\Longleftrightarrow \exists z \exists u_2 \left(\mathbb{V}_s(z) \wedge \mathbb{V}_t(u_2)\right)$$

$$\Longleftrightarrow (s\downarrow \wedge t\downarrow)^{\diamondsuit}$$

From the definedness axioms of $\mathsf{BON}_\Omega + \mu$ this follows from $\mathsf{k}st = s$ and since we proved the translation for the definedness axioms, this yields the required.

(b) $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq (ac)(bc)$ We can split this conjunction into two separate statements. First we show the definedness:

$$(\mathsf{s}ab\downarrow)^{\diamondsuit} \leftrightsquigarrow \exists z \left(\mathbb{V}_{\mathsf{s}ab}(z)\right)$$

$$\Longleftrightarrow \exists z \exists y_1 \exists y_2 \left(\mathbb{V}_{\mathsf{s}a}(y_1) \wedge \mathbb{V}_b(y_2) \wedge \mathcal{T}(y_1, y_2, z)\right)$$

$$\begin{aligned} \Longleftrightarrow \quad & \exists z \exists y_1 \exists y_2 \exists x_1 \exists x_2 \\ & \left(\mathbb{V}_{\mathsf{s}}(x_1) \wedge \mathbb{V}_a(x_2) \wedge \mathcal{T}(x_1, x_2, y_1) \wedge \mathbb{V}_b(y_2) \wedge \mathcal{T}(y_1, y_2, z)\right) \end{aligned}$$

from $\mathbb{V}_s(x_1)$, we get $x_1 = \widehat{\mathsf{s}}$, $x_2 = a$ and $y_2 = b$

$$\Longleftrightarrow \exists z \exists y_1 \left(\mathcal{T}(\widehat{\mathsf{s}}, a, y_1) \wedge \mathcal{T}(y_1, b, z)\right)$$

From A_3, we get that $y_1 = \langle \widehat{\mathsf{s}}, a\rangle$

$$\Longleftrightarrow \exists z \left(\mathcal{T}\left(\langle \widehat{\mathsf{s}}, a\rangle, b, z\right)\right)$$

From A_4, we get that $z = \langle \widehat{\mathsf{s}}, a, b\rangle$, whereas z is such a term, that fulfils the condition $\mathcal{T}\left(\langle \widehat{\mathsf{s}}, a\rangle, b, z\right)$, this yields the required.

$$(\mathsf{s}abc \simeq (ac)(bc))^{\diamondsuit} \leftrightsquigarrow \left((\mathsf{s}abc\downarrow \vee (ac)(bc)\downarrow) \to (\mathsf{s}abc = (ac)(bc))\right)^{\diamondsuit}$$

if neither side is defined, the statement is true, therefore we assume that at least one term is defined and we need to show the equality. Consider the following

$$\mathbb{V}_{\mathsf{s}abc}(z)$$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{s}ab}(x_1) \wedge \mathbb{V}_c(x_2) \wedge \mathcal{T}(x_1, x_2, z)\right)$$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{s}a}(x_3) \wedge \mathbb{V}_b(x_4) \wedge \mathcal{T}(x_3, x_4, x_1) \wedge \mathcal{T}(x_1, c, z)\right)$$

$$\begin{aligned} \Longleftrightarrow \quad \exists \overrightarrow{x} \quad & \Big(\mathbb{V}_{\mathsf{s}}(x_5) \wedge \mathbb{V}_a(x_6) \wedge \mathcal{T}(x_5, x_6, x_3) \\ \wedge \quad & \mathcal{T}(x_3, b, x_1) \wedge \mathcal{T}(x_1, c, z)\Big) \end{aligned}$$

We get that $x_5 = \widehat{\mathsf{s}}$

$$\iff \exists \overrightarrow{x} \left(\mathcal{T}\left(\widehat{\mathsf{s}}, a, x_3\right) \wedge \mathcal{T}\left(x_3, b, x_1\right) \wedge \mathcal{T}\left(x_1, c, z\right)\right)$$

We get that $x_3 = \langle \widehat{\mathsf{s}}, a \rangle$

$$\iff \exists \overrightarrow{x} \left(\mathcal{T}\left(\langle \widehat{\mathsf{s}}, a \rangle, b, x_1\right) \wedge \mathcal{T}\left(x_1, c, z\right)\right)$$

We get that $x_1 = \langle \widehat{\mathsf{s}}, a, b \rangle$

$$\iff \left(\mathcal{T}\left(\langle \widehat{\mathsf{s}}, a, b \rangle, c, z\right)\right)$$

The fixed point theorem yields $\mathcal{T}\left(\langle \widehat{\mathsf{s}}, a, b \rangle, c, z\right) \leftrightarrow A\left(P_A, \langle \widehat{\mathsf{s}}, a, b \rangle, c, z, 0\right)$, we can apply the operator and get the right condition of A_5 for $P \equiv P_A$. Therefore, we are allowed to do the following.

$$\iff \exists v, w \left(\mathcal{T}\left(a, c, v\right) \wedge \mathcal{T}\left(b, c, w\right) \wedge \mathcal{T}\left(v, w, z\right)\right)$$

$$\iff \exists v, w \left(\mathbb{V}_{ac}\left(v\right) \wedge \mathcal{T}\left(b, c, w\right) \wedge \mathcal{T}\left(v, w, z\right)\right)$$

$$\iff \exists v, w \left(\mathbb{V}_{ac}\left(v\right) \wedge \mathbb{V}_{bc}\left(w\right) \wedge \mathcal{T}\left(v, w, z\right)\right)$$

$$\iff \mathbb{V}_{(ac)(bc)}\left(z\right)$$

So we see that $\mathbb{V}_{\mathsf{s}abc}\left(z\right)$ is equivalent to $\mathbb{V}_{(ac)(bc)}\left(z\right)$. Now we can consider the translation

$$\left(\left(\mathsf{s}abc{\downarrow} \vee (ac)(bc){\downarrow}\right) \to \left(\mathsf{s}abc = (ac)(bc)\right)\right)^{\Diamond}$$

$$\leftrightsquigarrow \left(\exists x_1 \left(\mathbb{V}_{\mathsf{s}abc}\left(x_1\right)\right) \vee \exists x_2 \left(\mathbb{V}_{(ac)(bc)}\left(x_2\right)\right)\right) \to \left(\exists z \left(\mathbb{V}_{\mathsf{s}abc}\left(z\right) \wedge \mathbb{V}_{(ac)(bc)}\left(z\right)\right)\right)$$

because of the previous equivalence, we can rewrite the premise as

$$\exists x_1 \left(\mathbb{V}_{\mathsf{s}abc}\left(x_1\right) \wedge \mathbb{V}_{(ac)(bc)}\left(x_1\right)\right) \vee \exists x_2 \left(\mathbb{V}_{\mathsf{s}abc}\left(x_2\right) \wedge \mathbb{V}_{(ac)(bc)}\left(x_2\right)\right)$$

$$\iff \exists x \left(\mathbb{V}_{\mathsf{s}abc}\left(x\right) \wedge \mathbb{V}_{(ac)(bc)}\left(x\right)\right)$$

but that is the conclusion.

6. The axioms for natural numbers

   (a) $(\forall x \in \mathsf{N})\left(x' \neq \mathsf{0}_{\mathsf{N}} \wedge \mathsf{p}_{\mathsf{N}}\left(x'\right) = x\right)$

$$\left((\forall x \in \mathsf{N})\left(x' \neq \mathsf{0}_{\mathsf{N}} \wedge \mathsf{p}_{\mathsf{N}}\left(x'\right) = x\right)\right)^{\Diamond} \leftrightsquigarrow \left(\forall x \left(x \in \mathsf{N} \to \left(x' \neq \mathsf{0}_{\mathsf{N}} \wedge \mathsf{p}_{\mathsf{N}}\left(x'\right) = x\right)\right)\right)^{\Diamond}$$

$$\leftrightsquigarrow \left(\forall x \left(x \in \mathsf{N} \to \left(x'{\downarrow} \wedge \mathsf{0}_{\mathsf{N}}{\downarrow} \wedge \neg\left(x' = \mathsf{0}_{\mathsf{N}}\right) \wedge \mathsf{p}_{\mathsf{N}}\left(x'\right) = x\right)\right)\right)^{\Diamond}$$

this is true if the premise implies each of the conclusions separately, so we can split it into four parts:

i.

$$\left(\forall x \left(x \in \mathsf{N} \to x'{\downarrow}\right)\right)^{\Diamond}$$

$$\rightsquigarrow \forall x\left(\exists y\left(\mathbb{V}_{n}(y)\right) \rightarrow \exists z\left(\mathbb{V}_{n'}(z)\right)\right)$$

$$\Longleftrightarrow \forall x \exists y \exists z\left(\mathbb{V}_{x}(y) \rightarrow \mathbb{V}_{x'}(z)\right)$$

$$\Longleftrightarrow \forall x \exists y \exists z \exists u \exists v\left(\mathbb{V}_{x}(y) \rightarrow\left(\mathbb{V}_{\mathsf{s}_{\mathsf{N}}}(u) \wedge \mathbb{V}_{x}(v) \wedge \mathcal{T}(u, v, z)\right)\right)$$

we get that $u=\widehat{\mathsf{s}_{\mathsf{N}}}$ and $v=y$, so

$$\Longleftrightarrow \forall x \exists y \exists z\left(\mathbb{V}_{x}(y) \rightarrow \mathcal{T}\left(\widehat{\mathsf{s}_{\mathsf{N}}}, y, z\right)\right)$$

But this is fulfilled for $z=y+1$.

ii.

$$\left(\forall x\left(x \in \mathsf{N} \rightarrow 0_{\mathsf{N}} \downarrow\right)\right)^{\diamond}$$

$$\rightsquigarrow \forall x\left(\exists y\left(\mathbb{V}_{x}(y)\right) \rightarrow \exists z\left(\mathbb{V}_{0_{\mathsf{N}}}(z)\right)\right)$$

but this is true for $z=0$

iii.

$$\left(\forall x\left(x \in \mathsf{N} \rightarrow \neg\left(x'=0_{\mathsf{N}}\right)\right)\right)^{\diamond}$$

$$\rightsquigarrow \forall x\left(\exists y\left(\mathbb{V}_{x}(y)\right) \rightarrow \neg \exists z\left(\mathbb{V}_{x'}(z) \wedge \mathbb{V}_{0_{\mathsf{N}}}(z)\right)\right)$$

$$\Longleftrightarrow \forall x \exists y \exists z\left(\mathbb{V}_{x}(y) \rightarrow \neg\left(\mathbb{V}_{x'}(z) \wedge \mathbb{V}_{0_{\mathsf{N}}}(z)\right)\right)$$

from $\mathbb{V}_{0_{\mathsf{N}}}(z)$, we get that $z=0$

$$\Longleftrightarrow \forall x \exists y\left(\mathbb{V}_{n}(y) \rightarrow \neg \mathbb{V}_{x'}(0)\right)$$

$$\Longleftrightarrow \forall x \exists y \exists x_{1} \exists x_{2}\left(\mathbb{V}_{x}(y) \rightarrow \neg\left(\mathbb{V}_{\mathsf{s}_{\mathsf{N}}}\left(x_{1}\right) \wedge \mathbb{V}_{x}\left(x_{2}\right) \wedge \mathcal{T}\left(x_{1}, x_{2}, 0\right)\right)\right)$$

we get that $x_{1}=\widehat{\mathsf{s}_{\mathsf{N}}}$ and $x_{2}=y$

$$\Longleftrightarrow \forall x \exists y\left(\mathbb{V}_{x}(y) \rightarrow \neg \mathcal{T}\left(\widehat{\mathsf{s}_{\mathsf{N}}}, y, 0\right)\right)$$

and from $\mathcal{T}\left(\widehat{\mathsf{s}_{\mathsf{N}}}, y, 0\right)$, we get that $0=y+1$. But this is always false and so the conclusion is always true.

iv.

$$\left(\forall x\left(x \in \mathsf{N} \rightarrow \mathsf{p}_{\mathsf{N}}\left(x'\right)=x\right)\right)^{\diamond}$$

$$\rightsquigarrow \forall x\left(\exists y\left(\mathbb{V}_{x}(y)\right) \rightarrow \exists z\left(\mathbb{V}_{\mathsf{p}_{\mathsf{N}}(x')}(z) \wedge \mathbb{V}_{x}(z)\right)\right)$$

$$\Longleftrightarrow \forall x \exists y \exists z\left(\mathbb{V}_{x}(y) \rightarrow\left(\mathbb{V}_{\mathsf{p}_{\mathsf{N}}(x')}(z) \wedge \mathbb{V}_{x}(z)\right)\right)$$

we get, that $z=y$

$$\Longleftrightarrow \forall x \exists y\left(\mathbb{V}_{x}(y) \rightarrow \mathbb{V}_{\mathsf{p}_{\mathsf{N}}(x')}(y)\right)$$

$$\Longleftrightarrow \forall x \exists y \exists x_{1} \exists x_{2}\left(\mathbb{V}_{x}(y) \rightarrow\left(\mathbb{V}_{\mathsf{p}_{\mathsf{N}}}\left(x_{1}\right) \wedge \mathbb{V}_{x'}\left(x_{2}\right) \wedge \mathcal{T}\left(x_{1}, x_{2}, y\right)\right)\right)$$

we get, that $x_1 = \widehat{\mathsf{p}_\mathsf{N}}$

$$\begin{aligned} \iff \quad & \forall x \exists y \exists x_2 \exists u \exists v \\ & (\mathbb{V}_x(y) \to (\mathbb{V}_{\mathsf{s}_\mathsf{N}}(u) \wedge \mathbb{V}_x(v) \wedge \mathcal{T}(u, v, x_2) \wedge \mathcal{T}(\widehat{\mathsf{p}_\mathsf{N}}, x_2, y))) \end{aligned}$$

we get, that $v = y$ and $u = \widehat{\mathsf{s}_\mathsf{N}}$

$$\iff \forall x \exists y \exists x_2 (\mathbb{V}_x(y) \to (\mathcal{T}(\widehat{\mathsf{s}_\mathsf{N}}, y, x_2) \wedge \mathcal{T}(\widehat{\mathsf{p}_\mathsf{N}}, x_2, y)))$$

from $\mathcal{T}(\widehat{\mathsf{s}_\mathsf{N}}, y, x_2)$, we get, that $x_2 = y + 1$

$$\iff \forall x \exists y (\mathbb{V}_x(y) \to \mathcal{T}(\widehat{\mathsf{p}_\mathsf{N}}, y + 1, y))$$

but the conclusion is true, because $y + 1 = y + 1$

So if we put those four cases together, we get the required conclusion.

(b) $(\forall x \in \mathsf{N})\left(x \neq \mathsf{0}_\mathsf{N} \to \mathsf{p}_\mathsf{N} x \in \mathsf{N} \wedge (\mathsf{p}_\mathsf{N} x)' = x\right)$

$$\left((\forall x \in \mathsf{N})\left(x \neq \mathsf{0}_\mathsf{N} \to \mathsf{p}_\mathsf{N} x \in \mathsf{N} \wedge (\mathsf{p}_\mathsf{N} x)' = x\right)\right)^\diamond$$

$$\leftrightsquigarrow \left(\forall x \left(x \in \mathsf{N} \to \left((x\downarrow \wedge \mathsf{0}_\mathsf{N}\downarrow \wedge \neg x = \mathsf{0}_\mathsf{N}\right) \to \left(\mathsf{p}_\mathsf{N} x \in \mathsf{N} \wedge (\mathsf{p}_\mathsf{N} x)' = x\right)\right)\right)\right)^\diamond$$

this is true, if we can get the conclusions from the premises.

$$\begin{aligned} \leftrightsquigarrow \quad \forall x \exists \overrightarrow{y} \quad & \Big(\mathbb{V}_x(y_1) \\ \to \quad & \Big((\mathbb{V}_x(y_2) \wedge \mathbb{V}_{\mathsf{0}_\mathsf{N}}(y_3) \wedge \neg(\mathbb{V}_n(y_4) \wedge \mathbb{V}_{\mathsf{0}_\mathsf{N}}(y_4))) \\ \to \quad & \left(\mathbb{V}_{\mathsf{p}_\mathsf{N} x}(y_5) \wedge \mathbb{V}_{(\mathsf{p}_\mathsf{N} x)'}(y_6) \wedge \mathbb{V}_x(y_6)\right)\Big)\Big) \end{aligned}$$

we immediately get that, $y_2 = y_4 = y_6 = y_1$ and $y_4 = y_3$.

$$\iff \forall x \exists \overrightarrow{y} \left(\mathbb{V}_x(y_1) \to \left(\neg \mathbb{V}_{\mathsf{0}_\mathsf{N}}(y_1) \to \left(\mathbb{V}_{\mathsf{p}_\mathsf{N} x}(y_5) \wedge \mathbb{V}_{(\mathsf{p}_\mathsf{N} x)'}(y_1)\right)\right)\right)$$

$$\begin{aligned} \iff \quad \forall x \exists \overrightarrow{y} \quad & \Big(\mathbb{V}_x(y_1) \\ \to \quad & (\neg \mathbb{V}_{\mathsf{0}_\mathsf{N}}(y_1) \to (\mathbb{V}_{\mathsf{p}_\mathsf{N} x}(y_5) \wedge \mathbb{V}_{\mathsf{s}_\mathsf{N}}(y_7) \wedge \mathbb{V}_{\mathsf{p}_\mathsf{N} x}(y_8) \wedge \mathcal{T}(y_7, y_8, y_1)))\Big) \end{aligned}$$

we get that $y_7 = \widehat{\mathsf{s}_\mathsf{N}}$ and $y_8 = y_5$

$$\iff \forall x \exists \overrightarrow{y} (\mathbb{V}_x(y_1) \to (\neg \mathbb{V}_{\mathsf{0}_\mathsf{N}}(y_1) \to (\mathbb{V}_{\mathsf{p}_\mathsf{N} x}(y_5) \wedge \mathcal{T}(\widehat{\mathsf{s}_\mathsf{N}}, y_5, y_1))))$$

we get, that $y_1 = y_5 + 1$

$$\Longleftrightarrow \forall x \exists \overrightarrow{y} \left(\mathbb{V}_x \left(y_5 + 1 \right) \rightarrow \left(\mathbb{V}_{\mathsf{p_N} x} \left(y_5 \right) \right) \right)$$

$$\Longleftrightarrow \forall x \exists \overrightarrow{y} \left(\mathbb{V}_x \left(y_5 + 1 \right) \rightarrow \left(\mathbb{V}_{\mathsf{p_N}} \left(y_9 \right) \wedge \mathbb{V}_x \left(y_{10} \right) \wedge \mathcal{T} \left(y_9, y_{10}, y_5 \right) \right) \right)$$

we get, that $y_9 = \widehat{\mathsf{p_N}}$ and $y_{10} = y_5 + 1$

$$\Longleftrightarrow \forall x \exists \overrightarrow{y} \left(\mathbb{V}_x \left(y_5 + 1 \right) \rightarrow \mathcal{T} \left(\widehat{\mathsf{p_N}}, y_5 + 1, y_5 \right) \right)$$

and the conclusion is true, because $x_5 + 1 = x_5 + 1$

(c) $A \left(\mathsf{0_N} \right) \wedge \left(\forall x : \mathsf{N} \right) \left(A \left(x \right) \rightarrow A \left(x' \right) \right) \rightarrow \left(\forall y : \mathsf{N} \right) A \left(y \right)$

$$\left(A \left(\mathsf{0_N} \right) \wedge \left(\forall x : \mathsf{N} \right) \left(A \left(x \right) \rightarrow A \left(x' \right) \right) \rightarrow \left(\forall y : \mathsf{N} \right) A \left(y \right) \right)^{\diamondsuit}$$

$$\Longleftrightarrow \left(A \left(\mathsf{0_N} \right) \right)^{\diamondsuit} \wedge \left(\left(\forall x : \mathsf{N} \right) \left(A \left(x \right) \rightarrow A \left(x' \right) \right) \right)^{\diamondsuit} \rightarrow \left(\left(\forall y : \mathsf{N} \right) A \left(y \right) \right)^{\diamondsuit}$$

we can apply the substitution lemma 44 to $A \left(\mathsf{0_N} \right)$ to get

$$\Longleftrightarrow A^{\diamondsuit} \left(0 \right) \wedge \forall x \left(\left(x : \mathsf{N} \right)^{\diamondsuit} \rightarrow \left(A \left(x \right)^{\diamondsuit} \rightarrow A \left(x' \right)^{\diamondsuit} \right) \right) \rightarrow \forall y \left(\left(y : \mathsf{N} \right)^{\diamondsuit} \rightarrow A \left(y \right)^{\diamondsuit} \right)$$

in this case we apply the substitution lemma repeatedly to get

$$\Longleftrightarrow A^{\diamondsuit} \left(0 \right) \wedge \forall x \left(\exists u \mathbb{V}_x \left(u \right) \rightarrow \left(A^{\diamondsuit} \left(x \right) \rightarrow A^{\diamondsuit} \left(x' \right) \right) \right) \rightarrow \forall y \left(\exists v \mathbb{V}_y \left(v \right) \rightarrow A^{\diamondsuit} \left(y \right) \right)$$

but $\exists u \mathbb{V}_x \left(u \right)$ and $\exists v \mathbb{V}_y \left(v \right)$ are both always true, because x and y are variables. And so, we get

$$\Longleftrightarrow A^{\diamondsuit} \left(0 \right) \wedge \forall x \left(A^{\diamondsuit} \left(x \right) \rightarrow A^{\diamondsuit} \left(x' \right) \right) \rightarrow \forall y \left(A^{\diamondsuit} \left(y \right) \right)$$

but this is an instance of the induction scheme in PA^r_{Ω}.

7. Definition by numerical cases

(a) $u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u = v \rightarrow \mathsf{d_N} xyuv = x$

$$\left(u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u = v \rightarrow \mathsf{d_N} xyuv = x \right)^{\diamondsuit}$$

$$\leadsto \exists \overrightarrow{x} \left(\mathbb{V}_u \left(x_1 \right) \wedge \mathbb{V}_v \left(x_2 \right) \wedge \mathbb{V}_u \left(x_3 \right) \wedge \mathbb{V}_v \left(x_3 \right) \rightarrow \mathbb{V}_{\mathsf{d_N} xyuv} \left(x_4 \right) \wedge \mathbb{V}_x \left(x_4 \right) \right)$$

we get, that $x_1 = x_2 = x_3 = v = u$ and $x_4 = x$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{d_N} xyu} \left(x_5 \right) \wedge \mathbb{V}_v \left(x_6 \right) \wedge \mathcal{T} \left(x_5, x_6, x \right) \right)$$

we get, that $x_6 = u$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{d_N} xy} \left(x_7 \right) \wedge \mathbb{V}_u \left(x_8 \right) \wedge \mathcal{T} \left(x_7, x_8, x_5 \right) \wedge \mathcal{T} \left(x_5, u, x \right) \right)$$

we get, that $x_8 = u$

$$\Longleftrightarrow \exists \overrightarrow{x} \quad \Big(\quad \mathbb{V}_{\mathsf{d_N}x}\left(x_9\right) \wedge \mathbb{V}_y\left(x_{10}\right) \wedge \mathcal{T}\left(x_9, x_{10}, x_7\right) \\ \wedge \quad \mathcal{T}\left(x_7, u, x_5\right) \wedge \mathcal{T}\left(x_5, u, x\right)\Big)$$

we get, that $x_{10} = y$

$$\Longleftrightarrow \exists \overrightarrow{x} \quad \Big(\quad \mathbb{V}_{\mathsf{d_N}}\left(x_{11}\right) \wedge \mathbb{V}_x\left(x_{12}\right) \wedge \mathcal{T}\left(x_{11}, x_{12}, x_9\right) \\ \wedge \quad \mathcal{T}\left(x_9, y, x_7\right) \wedge \mathcal{T}\left(x_7, u, x_5\right) \wedge \mathcal{T}\left(x_5, u, x\right)\Big)$$

we get, that $x_{11} = \widehat{\mathsf{d_N}}$ and $x_{12} = x$

$$\Longleftrightarrow \exists \overrightarrow{x} \quad \Big(\quad \mathcal{T}\left(\widehat{\mathsf{d_N}}, x, x_9\right) \wedge \mathcal{T}\left(x_9, y, x_7\right) \\ \wedge \quad \mathcal{T}\left(x_7, u, x_5\right) \wedge \mathcal{T}\left(x_5, u, x\right)\Big)$$

we get, that $x_9 = \left\langle \widehat{\mathsf{d_N}}, x \right\rangle$

$$\Longleftrightarrow \exists \overrightarrow{x} \quad \Big(\quad \mathcal{T}\left(\left\langle \widehat{\mathsf{d_N}}, x \right\rangle, y, x_7\right) \\ \wedge \quad \mathcal{T}\left(x_7, u, x_5\right) \wedge \mathcal{T}\left(x_5, u, x\right)\Big)$$

we get, that $x_7 = \left\langle \widehat{\mathsf{d_N}}, x, y \right\rangle$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathcal{T}\left(\left\langle \widehat{\mathsf{d_N}}, x, y \right\rangle, u, x_5\right) \wedge \mathcal{T}\left(x_5, u, x\right)\right)$$

we get, that $x_5 = \left\langle \widehat{\mathsf{d_N}}, x, y, u \right\rangle$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathcal{T}\left(\left\langle \widehat{\mathsf{d_N}}, x, y, u \right\rangle, u, x\right)\right)$$

and so we get, that $\mathcal{T}\left(\left\langle \widehat{\mathsf{d_N}}, x, y, u \right\rangle, u, x\right)$ is fulfilled and so actually $\mathbb{V}_{\mathsf{d_N}xyuu}\left(x\right)$ which is the same as $\mathbb{V}_{\mathsf{d_N}xyuv}\left(x\right)$. Therefore the conclusion holds.

(b) $u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u \neq v \rightarrow \mathsf{d_N}xyuv = y$ This case is analogous to the previous one. The main difference is, that from $u \neq v$, we get, that there cannot be one variable z, that fulfils $\exists z \left(\mathbb{V}_u\left(z\right) \wedge \mathbb{V}_v\left(z\right)\right)$ and therefore we get the case A_{12} rather than A_{11}.

$$\left(m \neq n\right)^{\diamondsuit} \leftrightsquigarrow \left(m\downarrow \wedge n\downarrow \wedge \neg\left(m = n\right)\right)^{\diamondsuit}$$

$$\leftrightsquigarrow \exists x \mathbb{V}_m\left(x\right) \wedge \exists y \mathbb{V}_n\left(y\right) \wedge \neg \exists z \left(\mathbb{V}_m\left(z\right) \wedge \mathbb{V}_n\left(z\right)\right)$$

so we immediately get, that $x \neq y$. This leads to the conclusion, that

$$\exists \overrightarrow{x} \left(\mathbb{V}_s (x_1) \wedge \mathbb{V}_t (x_2) \wedge \mathbb{V}_{\mathsf{d_N} stmn} (x_2) \right)$$

This statement can be fulfilled for $x_1 = s$ and $x_2 = t$.

8. The axioms for ordinal numbers

   (a) $(e : \mathsf{N} \longrightarrow \Omega) \rightarrow \mathsf{sup} e \neq \mathsf{0}_\Omega \wedge \mathsf{sup}^{-1} (\mathsf{sup} e) = e$ We split this into the two parts of the conjunction.

      i.

$$((e : \mathsf{N} \longrightarrow \Omega) \rightarrow \mathsf{sup} e \neq \mathsf{0}_\Omega)^\diamond$$
$$\leftrightsquigarrow (\forall n (n : \mathsf{N} \rightarrow en : \Omega) \rightarrow (\mathsf{sup} e\downarrow \wedge \mathsf{0}_\Omega \downarrow \wedge \neg (\mathsf{sup} e = \mathsf{0}_\Omega)))^\diamond$$

$$\begin{aligned} \iff \quad \forall n \exists \overrightarrow{x} \quad & \Big((\mathbb{V}_n (x_1) \rightarrow \mathbb{V}_{en} (x_2) \wedge \mathbb{O} (x_2)) \\ \rightarrow \quad & (\mathbb{V}_{\mathsf{sup} e} (x_3) \wedge \mathbb{V}_{\mathsf{0}_\Omega} (x_4) \wedge \neg (\exists y) (\mathbb{V}_{\mathsf{sup} e} (y) \wedge \mathbb{V}_{\mathsf{0}_\Omega} (y))) \Big) \end{aligned}$$

we get, that $x_4 = y = \widehat{\mathsf{0}_\Omega}$

$$\iff \forall n \exists \overrightarrow{x} \left((\mathbb{V}_n (x_1) \rightarrow \mathbb{V}_{en} (x_2) \wedge \mathbb{O} (x_2)) \rightarrow \left(\mathbb{V}_{\mathsf{sup} e} (x_3) \wedge \neg \mathbb{V}_{\mathsf{sup} e} \left(\widehat{\mathsf{0}_\Omega} \right) \right) \right)$$

$$\begin{aligned} \iff \forall n \exists \overrightarrow{x} \quad \Big(\quad & \mathbb{V}_n (x_1) \wedge \mathbb{V}_e (x_4) \wedge \mathbb{V}_n (x_5) \wedge \mathcal{T} (x_4, x_5, x_2) \wedge \mathbb{O} (x_2) \\ \rightarrow \quad & \mathbb{V}_{\mathsf{sup}} (x_6) \wedge \mathbb{V}_e (x_7) \wedge \mathcal{T} (x_6, x_7, x_3) \\ \wedge \quad & \neg \left(\mathbb{V}_{\mathsf{sup}} (x_8) \wedge \mathbb{V}_e (x_9) \wedge \mathcal{T} \left(x_8, x_9, \widehat{\mathsf{0}_\Omega} \right) \right) \Big) \end{aligned}$$

we get, that $x_6 = x_8 = \widehat{\mathsf{sup}}$, $x_9 = x_1$ and $x_7 = x_9 = x_4$

$$\begin{aligned} \iff \forall n \exists \overrightarrow{x} \quad \Big(\quad & \mathbb{V}_n (x_1) \wedge \mathbb{V}_e (x_4) \wedge \mathcal{T} (x_4, x_1, x_2) \wedge \mathbb{O} (x_2) \\ \rightarrow \quad & \mathcal{T} (\widehat{\mathsf{sup}}, x_4, x_3) \wedge \neg \mathcal{T} \left(\widehat{\mathsf{sup}}, x_4, \widehat{\mathsf{0}_\Omega} \right) \Big) \end{aligned}$$

we get, that $x_3 = \langle \widehat{\mathsf{sup}}, x_4 \rangle$ and $x_9 = x_1$

$$\begin{aligned} \iff \forall n \exists \overrightarrow{x} \quad \Big(\quad & \mathbb{V}_n (x_1) \wedge \mathcal{T} (x_4, x_1, x_2) \wedge \mathbb{O} (x_2) \\ \rightarrow \quad & \neg \mathcal{T} \left(\widehat{\mathsf{sup}}, x_4, \widehat{\mathsf{0}_\Omega} \right) \Big) \end{aligned}$$

but we know, that $\mathcal{T} (\widehat{\mathsf{sup}}, x_4, z)$ is true iff $z = \langle \widehat{\mathsf{sup}}, x_4 \rangle$ and $\widehat{\mathsf{0}_\Omega} \neq \langle \widehat{\mathsf{sup}}, x_4 \rangle$.

ii.

$$\left((f : \mathsf{N} \longrightarrow \Omega) \rightarrow \mathsf{sup}^{-1}\left(\mathsf{sup}e\right) = e\right)^{\diamondsuit}$$

$$\leftrightsquigarrow \left(\forall n \left(n : \mathsf{N} \rightarrow en : \Omega\right) \rightarrow \mathsf{sup}^{-1}\left(\mathsf{sup}e\right) = e\right)^{\diamondsuit}$$

$$\leftrightsquigarrow \forall n \exists \overrightarrow{x} \left(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_{en}\left(x_2\right) \wedge \mathbb{O}\left(x_2\right) \rightarrow \mathbb{V}_{\mathsf{sup}^{-1}(\mathsf{sup}e)}\left(x_3\right) \wedge \mathbb{V}_e\left(x_3\right)\right)$$

$$\begin{aligned} \Longleftrightarrow \quad & \forall n \exists \overrightarrow{x} \quad \Big(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_{en}\left(x_2\right) \wedge \mathbb{O}\left(x_2\right) \\ & \rightarrow \quad \mathbb{V}_{\mathsf{sup}^{-1}}\left(x_4\right) \wedge \mathbb{V}_{\mathsf{sup}e}\left(x_5\right) \wedge \mathcal{T}\left(x_4, x_5, x_3\right) \wedge \mathbb{V}_e\left(x_3\right)\Big) \end{aligned}$$

we get, that $x_4 = \widehat{\mathsf{sup}^{-1}}$

$$\begin{aligned} \Longleftrightarrow \quad & \forall n \exists \overrightarrow{x} \quad \Big(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_{en}\left(x_2\right) \wedge \mathbb{O}\left(x_2\right) \\ & \rightarrow \quad \mathbb{V}_{\mathsf{sup}}\left(x_6\right) \wedge \mathbb{V}_e\left(x_7\right) \wedge \mathcal{T}\left(x_6, x_7, x_5\right) \wedge \mathcal{T}\left(\widehat{\mathsf{sup}^{-1}}, x_5, x_3\right) \wedge \mathbb{V}_e\left(x_3\right)\Big) \end{aligned}$$

we get, that $x_6 = \widehat{\mathsf{sup}}$ and $x_7 = x_3$

$$\begin{aligned} \Longleftrightarrow \quad & \forall n \exists \overrightarrow{x} \quad \Big(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_{en}\left(x_2\right) \wedge \mathbb{O}\left(x_2\right) \\ & \rightarrow \quad \mathcal{T}\left(\widehat{\mathsf{sup}}, x_3, x_5\right) \wedge \mathcal{T}\left(\widehat{\mathsf{sup}^{-1}}, x_5, x_3\right) \wedge \mathbb{V}_e\left(x_3\right)\Big) \end{aligned}$$

we get, that $x_5 = \langle \widehat{\mathsf{sup}}, x_3 \rangle$

$$\begin{aligned} \Longleftrightarrow \quad & \forall n \exists \overrightarrow{x} \quad \Big(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_{en}\left(x_2\right) \wedge \mathbb{O}\left(x_2\right) \\ & \rightarrow \quad \mathcal{T}\left(\widehat{\mathsf{sup}^{-1}}, \langle \widehat{\mathsf{sup}}, x_3 \rangle, x_3\right) \wedge \mathbb{V}_e\left(x_3\right)\Big) \end{aligned}$$

from A_{19} we get, that $\mathcal{T}\left(\widehat{\mathsf{sup}^{-1}}, \langle \widehat{\mathsf{sup}}, x_3 \rangle, x_3\right)$ holds.

$$\Longleftrightarrow \forall n \exists \overrightarrow{x} \Big(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_e\left(x_8\right) \wedge \mathbb{V}_n\left(x_9\right) \wedge \mathcal{T}\left(x_8, x_9, x_2\right) \wedge \mathbb{O}\left(x_2\right) \rightarrow \mathbb{V}_e\left(x_3\right)\Big)$$

we get, that $x_8 = x_3$ and $x_9 = x_1$

$$\Longleftrightarrow \forall n \exists \overrightarrow{x} \Big(\mathbb{V}_n\left(x_1\right) \wedge \mathbb{V}_e\left(x_3\right) \wedge \mathcal{T}\left(x_3, x_1, x_2\right) \wedge \mathbb{O}\left(x_2\right) \rightarrow \mathbb{V}_e\left(x_3\right)\Big)$$

but this is immediately true, because $\mathbb{V}_e\left(x_3\right)$ appears on both sides of the implication.

(b)

$$a : \Omega \rightarrow \left(a \neq \mathsf{0}_\Omega \rightarrow \mathsf{sup}\left(\mathsf{sup}^{-1} a\right) = a\right)$$

we assume the premises and need to show the conclusion. So we can assume $\left(a : \Omega \wedge a \neq \mathsf{0}_\Omega\right)^{\diamondsuit}$, that is

$$\exists \overrightarrow{x} \left(\mathbb{V}_a\left(x_1\right) \wedge \mathbb{O}\left(x_1\right) \wedge \mathbb{V}_a\left(x_2\right) \wedge \neg \exists y \left(\mathbb{V}_a\left(y\right) \wedge \mathbb{V}_{\mathsf{0}_\Omega}\left(y\right)\right)\right)$$

we get, that $x_2 = x_1$ and $y = \widehat{0_\Omega}$

$$\exists \overrightarrow{x} \left(\mathbb{V}_a (x_1) \wedge \mathbb{O} (x_1) \wedge \neg \mathbb{V}_a \left(\widehat{0_\Omega} \right) \right)$$

so in particular, we get that $x_1 \neq \widehat{0_\Omega}$, because otherwise, we would have $\mathbb{V}_a (x_1) \wedge \neg \mathbb{V}_a (x_1)$. Therefore the following holds

$$\exists x_1 \left(x_1 \neq \widehat{0_\Omega} \wedge \mathbb{V}_a (x_1) \wedge \mathbb{O} (x_1) \right)$$

from $x_1 \neq \widehat{0_\Omega}$ and $\mathbb{O} (x_1)$, we get according to the definition of $\mathbb{O}$, that

$$\exists e \left(x_1 = \langle \widehat{\mathsf{sup}}, e \rangle \wedge \forall u \exists a \left(\mathbb{O} (a) \wedge \mathcal{T} (e, u, a) \right) \right)$$

now consider what is the statement of the conclusion

$$\left(\mathsf{sup} \left(\mathsf{sup}^{-1} a \right) = a \right)^{\diamondsuit}$$

$$\iff \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{sup}(\mathsf{sup}^{-1} a)} (x_3) \wedge \mathbb{V}_a (x_3) \right)$$

we get, that $x_3 = x_1$

$$\iff \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{sup}} (x_4) \wedge \mathbb{V}_{\mathsf{sup}^{-1} a} (x_5) \wedge \mathcal{T} (x_4, x_5, x_1) \right)$$

we get, that $x_4 = \widehat{\mathsf{sup}}$

$$\iff \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{sup}^{-1}} (x_6) \wedge \mathbb{V}_a (x_7) \wedge \mathcal{T} (x_6, x_7, x_5) \wedge \mathcal{T} (\widehat{\mathsf{sup}}, x_5, x_1) \right)$$

we get, that $x_6 = \widehat{\mathsf{sup}^{-1}}$ and $x_7 = x_1$

$$\iff \exists \overrightarrow{x} \left(\mathcal{T} \left(\widehat{\mathsf{sup}^{-1}}, x_1, x_5 \right) \wedge \mathcal{T} (\widehat{\mathsf{sup}}, x_5, x_1) \right)$$

from the premise, we get, that $x_1 = \langle \widehat{\mathsf{sup}}, e \rangle$ for some e

$$\iff \exists \overrightarrow{x} \left(\mathcal{T} \left(\widehat{\mathsf{sup}^{-1}}, \langle \widehat{\mathsf{sup}}, e \rangle, x_5 \right) \wedge \mathcal{T} (\widehat{\mathsf{sup}}, x_5, \langle \widehat{\mathsf{sup}}, e \rangle) \right)$$

but now, we are in the case A_{19} of the inductive simulation operator and immediately get, that in order to fulfil the first conjunct, we get $x_5 = e$

$$\iff \exists \overrightarrow{x} \left(\mathcal{T} (\widehat{\mathsf{sup}}, e, \langle \widehat{\mathsf{sup}}, e \rangle) \right)$$

and this clause is true, because it is exactly the case A_{17}.

(c)

$$\left((0_\Omega)_n = 0_\Omega \right)^{\diamondsuit} \rightsquigarrow \left(\left(\mathsf{sup}^{-1} 0_\Omega \right) n = 0_\Omega \right)^{\diamondsuit} \rightsquigarrow \exists \overrightarrow{x} \left(\mathbb{V}_{(\mathsf{sup}^{-1} 0_\Omega) n} (x_1) \wedge \mathbb{V}_{0_\Omega} (x_1) \right)$$

we get, that $x_1 = \widehat{0_\Omega}$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{sup}^{-1}0_\Omega}(x_2) \wedge \mathbb{V}_n(x_3) \wedge \mathcal{T}\left(x_2, x_3, \widehat{0_\Omega}\right)\right)$$

we get, that $x_3 = n$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{sup}^{-1}}(x_4) \wedge \mathbb{V}_{0_\Omega}(x_5) \wedge \mathcal{T}(x_4, x_5, x_2) \wedge \mathcal{T}\left(x_2, n, \widehat{0_\Omega}\right)\right)$$

we get, that $x_4 = \widehat{\mathsf{sup}^{-1}}$ and $x_5 = \widehat{0_\Omega}$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathcal{T}\left(\widehat{\mathsf{sup}^{-1}}, \widehat{0_\Omega}, x_2\right) \wedge \mathcal{T}\left(x_2, n, \widehat{0_\Omega}\right)\right)$$

we get, that $x_2 = \left\langle \mathsf{sup}^{-1}, \widehat{0_\Omega}\right\rangle$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathcal{T}\left(\left\langle \mathsf{sup}^{-1}, \widehat{0_\Omega}\right\rangle, n, \widehat{0_\Omega}\right)\right)$$

and A_{20} yields the conclusion.

9. Definition of cases on ordinal numbers.

   (a)

$$(u = 0_\Omega \to \mathsf{d}_\Omega xyu = x)^\diamond$$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_u(x_1) \wedge \mathbb{V}_{0_\Omega}(x_1) \to \mathbb{V}_{\mathsf{d}_\Omega xyu}(x_2) \wedge \mathbb{V}_x(x_2)\right)$$

for $x_1 = u = \widehat{0_\Omega}$ and $x_2 = x$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{d}_\Omega fg}(x_3) \wedge \mathbb{V}_u(x_4) \wedge \mathcal{T}(x_3, x_4, x_2)\right)$$

for $x_4 = \widehat{0_\Omega}$, we get

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{d}_\Omega x}(x_5) \wedge \mathbb{V}_y(x_6) \wedge \mathcal{T}(x_5, x_6, x_3) \wedge \mathcal{T}\left(x_3, \widehat{0_\Omega}, x\right)\right)$$

for $x_6 = y$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathbb{V}_{\mathsf{d}_\Omega}(x_7) \wedge \mathbb{V}_x(x_8) \wedge \mathcal{T}(x_7, x_8, x_5) \wedge \mathcal{T}(x_5, y, x_3) \wedge \mathcal{T}\left(x_3, \widehat{0_\Omega}, x\right)\right)$$

for $x_7 = \widehat{\mathsf{d}_\Omega}$ and $x_8 = x$, we get for $x_5 = \left\langle \widehat{\mathsf{d}_\Omega}, x\right\rangle$, we get

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathcal{T}\left(\left\langle \widehat{\mathsf{d}_\Omega}, x\right\rangle, y, x_3\right) \wedge \mathcal{T}\left(x_3, \widehat{0_\Omega}, x\right)\right)$$

for $x_3 = \left\langle \widehat{\mathsf{d}_\Omega}, x, y\right\rangle$

$$\Longleftrightarrow \exists \overrightarrow{x} \left(\mathcal{T}\left(\left\langle \widehat{\mathsf{d}_\Omega}, x, y\right\rangle, \widehat{0_\Omega}, x\right)\right)$$

but this is true, according to the definition of P_A.

(b)

$$(a \in \Omega \wedge a \neq \mathsf{0}_\Omega \rightarrow \mathsf{d}_\Omega xya = y)^\diamond$$

$$\begin{aligned}\iff \quad \exists \overrightarrow{u} \quad & \Big(\mathbb{V}_a(u_1) \wedge \mathbb{O}(u_1) \wedge \mathbb{V}_a(u_2) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(u_3) \\ \wedge \quad & \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \rightarrow \mathbb{V}_{\mathsf{d}_\Omega xya}(u_4) \wedge \mathbb{V}_y(u_4)\Big)\end{aligned}$$

we get $u_1 = u_2 = a$, $u_3 = \widehat{\mathsf{0}_\Omega}$ and $u_4 = y$

$$\iff \exists \overrightarrow{u}\left(\mathbb{O}(a) \wedge \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \rightarrow \mathbb{V}_{\mathsf{d}_\Omega xy}(u_5) \wedge \mathbb{V}_a(u_6) \wedge \mathcal{T}(u_5, u_6, g)\right)$$

we get $u_6 = a$

$$\begin{aligned}\iff \quad \exists \overrightarrow{u} \quad & \Big(\mathbb{O}(a) \wedge \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \\ \rightarrow \quad & \mathbb{V}_{\mathsf{d}_\Omega x}(u_7) \wedge \mathbb{V}_y(u_8) \wedge \mathcal{T}(u_7, u_8, u_5) \wedge \mathcal{T}(u_5, a, y)\Big)\end{aligned}$$

we get $u_8 = y$

$$\begin{aligned}\iff \quad \exists \overrightarrow{u} \quad & \Big(\mathbb{O}(a) \wedge \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \rightarrow \mathbb{V}_{\mathsf{d}_\Omega}(u_9) \\ \wedge \quad & \mathbb{V}_x(u_{10}) \wedge \mathcal{T}(u_9, u_{10}, u_7) \wedge \mathcal{T}(u_7, g, u_5) \wedge \mathcal{T}(u_5, a, y)\Big)\end{aligned}$$

we get $u_9 = \widehat{\mathsf{d}_\Omega}$ and $u_{10} = x$

$$\begin{aligned}\iff \quad \exists \overrightarrow{u} \quad & \Big(\mathbb{O}(a) \wedge \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \\ \rightarrow \quad & \mathcal{T}\left(\widehat{\mathsf{d}_\Omega}, x, u_7\right) \wedge \mathcal{T}(u_7, y, u_5) \wedge \mathcal{T}(u_5, a, y)\Big)\end{aligned}$$

we get, that $u_7 = \left\langle \widehat{\mathsf{d}_\Omega}, x \right\rangle$

$$\iff \exists \overrightarrow{u}\left(\mathbb{O}(a) \wedge \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \rightarrow \mathcal{T}\left(\left\langle \widehat{\mathsf{d}_\Omega}, x \right\rangle, y, u_5\right) \wedge \mathcal{T}(u_5, a, y)\right)$$

we get, that $u_5 = \left\langle \widehat{\mathsf{d}_\Omega}, x, y \right\rangle$

$$\iff \mathbb{O}(a) \wedge \neg\exists z\left(\mathbb{V}_a(z) \wedge \mathbb{V}_{\mathsf{0}_\Omega}(z)\right) \rightarrow \mathcal{T}\left(\left\langle \widehat{\mathsf{d}_\Omega}, x, y \right\rangle, a, y\right)$$

consider, what we can get from the premise:

$$\neg\exists z\left(\mathbb{V}_y(z)\wedge\mathbb{V}_{0_\Omega}(z)\right)\Longleftrightarrow\neg\exists z\left(z=a\wedge z=\widehat{0_\Omega}\right)\Longleftrightarrow\neg\exists z\left(a=\widehat{0_\Omega}\right)\Longleftrightarrow a\neq\widehat{0_\Omega}$$

so now we can use this equivalence and continue the proof.

$$\Longleftrightarrow\mathbb{O}(a)\wedge a\neq\widehat{0_\Omega}\rightarrow\mathcal{T}\left(\left\langle\widehat{\mathsf{d}_\Omega},x,y\right\rangle,a,y\right)$$

and now we have have all the premises necessary to use case A_{12} in the definition of P_A to get, that indeed $\mathcal{T}\left(\left\langle\widehat{\mathsf{d}_\Omega},x,y\right\rangle,a,y\right)$. This concludes the proof of this case.

10. The non-constructive minimum operator.

$$((e:\mathsf{N}\longrightarrow\mathsf{N})\wedge(\exists x\in\mathsf{N})(ex=0_\mathsf{N})\rightarrow e(\mu e)=0_\mathsf{N})^\diamondsuit$$
$$\leftrightsquigarrow(e:\mathsf{N}\longrightarrow\mathsf{N})^\diamondsuit\wedge((\exists x\in\mathsf{N})(ex=0_\mathsf{N}))^\diamondsuit\rightarrow(e(\mu e)=0_\mathsf{N})^\diamondsuit$$

first, we rewrite the translations of the parts, then we prove the formula:

$$(e:\mathsf{N}\longrightarrow\mathsf{N})^\diamondsuit$$

$$\Longleftrightarrow(\forall x(x:\mathsf{N}\rightarrow ex:\mathsf{N}))^\diamondsuit$$
$$\Longleftrightarrow\forall x(\exists y(\mathbb{V}_x(y))\rightarrow\exists z(\mathbb{V}_{ex}(z)))$$

the premise is true for $y=x$

$$\Longleftrightarrow\forall x\exists z(\mathbb{V}_{ex}(z))$$

we apply the definition of the valuation

$$\Longleftrightarrow\forall x\exists z(\exists u,v(\mathbb{V}_e(u)\wedge\mathbb{V}_x(v)\wedge\mathcal{T}(u,v,z)))$$

we get, that $u=e$ and $v=x$

$$\Longleftrightarrow\forall x\exists z(\mathcal{T}(e,x,z))$$

the next part to consider is the following:

$$(\exists x\in\mathsf{N})(ex=0_\mathsf{N})^\diamondsuit$$

$$\Longleftrightarrow\exists x\left((x\in\mathsf{N})^\diamondsuit\wedge(ex=0_\mathsf{N})^\diamondsuit\right)$$
$$\Longleftrightarrow\exists x(\exists y(\mathbb{V}_x(y))\wedge\exists z(\mathbb{V}_{ex}(z)\wedge\mathbb{V}_{0_\mathsf{N}}(z)))$$

the first part $(\exists y)(\mathbb{V}_x(y))$ is fulfilled for $y=x$ and we get that $z=0$

$$\Longleftrightarrow\exists x(\mathbb{V}_{ex}(0))$$

applying the definition of the valuation, we get

$$\iff \exists x \left(\exists u, v \mathbb{V}_e(u) \wedge \mathbb{V}_x(v) \wedge \mathcal{T}(u, v, 0)\right)$$

so we get $u = e$ and $v = x$

$$\iff \exists x \left(\mathcal{T}(e, x, 0)\right)$$

The last part to consider is the conclusion:

$$\left(e(\mu e) = 0_{\mathsf{N}}\right)^{\diamondsuit}$$

with the same arguments as before, we get

$$\iff \exists y \left(\mathbb{V}_{\mu e}(y) \wedge \mathcal{T}(e, y, 0)\right)$$

and this can be further rewritten to

$$\iff \exists y \left(\mathcal{T}(\widehat{\mu}, e, y) \wedge \mathcal{T}(e, y, 0)\right)$$

so the whole statement that we need to show is the following:

$$\forall x \exists z \left(\mathcal{T}(e, x, y)\right) \wedge \exists x \left(\mathcal{T}(e, x, 0)\right) \rightarrow \exists y \left(\mathcal{T}(\widehat{\mu}, e, y) \wedge \mathcal{T}(e, y, 0)\right)$$

to do so, we apply the fixed point theorem to $\mathcal{T}(\widehat{\mu}, e, y)$. Knowing, that $\exists x \left(\mathcal{T}(e, x, 0)\right)$, we are in the case A_{22}.

$$\exists y \left(\mathcal{T}(\widehat{\mu}, e, y) \wedge \mathcal{T}(e, y, 0)\right)$$
$$\iff \exists y \left(\mathcal{T}(e, y, 0) \wedge (\forall v)\left(v < y \rightarrow \exists w \left(w \neq 0 \wedge \mathcal{T}(e, v, w)\right)\right)\right)$$

When we apply the least element principle from remark 35 to the premise, we get

$$\exists x \left(\mathcal{T}(e, x, 0) \wedge (\forall y < x)\left(\neg \mathcal{T}(e, y, 0)\right)\right)$$

when we combine the premise $\forall x \exists y \left(\mathcal{T}(e, x, y)\right)$ with the previous line, we get

$$\exists x \left(\mathcal{T}(e, x, 0) \wedge (\forall y < x)\left(\neg \mathcal{T}(e, y, 0) \wedge \exists u \left(\mathcal{T}(e, y, u)\right)\right)\right)$$

and $\neg \mathcal{T}(e, y, 0) \wedge \mathcal{T}(e, y, u)$ gives us, that $u \neq 0 \wedge \mathcal{T}(e, y, u)$ and that is what we need.

11. Transfinite induction scheme.

    Assume, we have the premises given, e.g. assume

$$A(0_{\Omega})^{\diamondsuit} \wedge \left((\forall a : \Omega)\left(a \neq 0_{\Omega} \wedge (\forall x : \mathsf{N}) A(a_x) \rightarrow A(a)\right)\right)^{\diamondsuit}$$

    If we from those premises can get the following, we are done:

$$\left((\forall a : \Omega) A(a)\right)^{\diamondsuit} \iff \left(\forall a \left(\Omega(a) \rightarrow A(a)\right)\right)^{\diamondsuit}$$

we can pull through the translation $\diamondsuit$ to the inner formula

$$\Longleftrightarrow \forall a\left((\exists x)\left(\mathbb{V}_a(x)\wedge\mathbb{O}(x)\right)\rightarrow A(a)^{\diamondsuit}\right)$$

notice, that in this formula, a is just simply a variable of PA_Ω, and therefore $\mathbb{V}_a(x)\equiv a=x$.

$$\Longleftrightarrow \forall a\left(\mathbb{O}(a)\rightarrow A(a)^{\diamondsuit}\right)$$

consider, that $\mathbb{O}(a)$ in PA_Ω is the same as $(\exists\alpha)(\mathbb{O}^\alpha(a))$

$$\Longleftrightarrow \forall a\left(\exists\alpha\left(\mathbb{O}^\alpha(a)\right)\rightarrow A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\left(\neg\exists\alpha\left(\mathbb{O}^\alpha(a)\right)\vee A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\left(\forall\alpha\left(\neg\mathbb{O}^\alpha(a)\right)\vee A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\forall\alpha\left(\mathbb{O}^\alpha(a)\rightarrow A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall\alpha\forall a\left(\mathbb{O}^\alpha(a)\rightarrow A(a)^{\diamondsuit}\right)$$

We can prove this by transfinite induction in PA_Ω. The formula to be proven by induction is the following:

$$B(\gamma):\equiv\forall a\left(\mathbb{O}^\gamma(a)\rightarrow A(a)^{\diamondsuit}\right)$$

We assume $(\forall\beta<\alpha)\,B(\beta)$. If we can prove from this, that $B(\alpha)$ for an arbitrary α, we get

$$\forall\alpha\left((\forall\beta<\alpha)\,B(\beta)\rightarrow B(\alpha)\right)$$

and we can apply the induction in PA_Ω to get $\forall\alpha B(\alpha)$, which is the same as

$$\forall\alpha\forall a\left(\mathbb{O}^\alpha(a)\rightarrow A(a)^{\diamondsuit}\right)$$

and this proves the translation. Now assume

$$(\forall\beta<\alpha)\,B(\beta)$$

$$\Longleftrightarrow (\forall\beta<\alpha)\,\forall a\left(\mathbb{O}^\beta(a)\rightarrow A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\,(\forall\beta<\alpha)\left(\mathbb{O}^\beta(a)\rightarrow A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\,(\forall\beta<\alpha)\left(\neg\mathbb{O}^\beta(a)\vee A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\left(\neg\,(\exists\beta<\alpha)\,\mathbb{O}^\beta(a)\vee A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\left((\exists\beta<\alpha)\,\mathbb{O}^\beta(a)\rightarrow A(a)^{\diamondsuit}\right)$$

$$\Longleftrightarrow \forall a\left(\mathbb{O}^{<\alpha}(a) \to A(a)^{\Diamond}\right)$$

in order to show $\forall a\left(\mathbb{O}^{\alpha}(a) \to A(a)^{\Diamond}\right)$, we assume $\mathbb{O}^{\alpha}(a)$ and need to prove $A(a)^{\Diamond}$. Consider two cases:

- a was added to P_A^{α} on a layer $< \alpha$. But then $\mathbb{O}^{<\alpha}(a)$ holds and we get $A(a)^{\Diamond}$ from $\forall a\left(\mathbb{O}^{<\alpha}(a) \to A(a)^{\Diamond}\right)$.
- a was added to P_A^{α} on the layer α.

$$\mathbb{O}^{\alpha}(a) \Longleftrightarrow A\left(P_A^{<\alpha}, a, 0, 0, 1\right)$$

$$\Longleftrightarrow a = \widehat{0_\Omega} \vee \exists e\left(\forall z \exists y\left(\mathbb{O}^{<\alpha}(y) \wedge \mathcal{T}^{<\alpha}(e, z, y)\right) \wedge a = \langle \widehat{\mathsf{sup}}, e\rangle\right)$$

if $a = \widehat{0_\Omega}$, then we get $A(a)^{\Diamond}$ from the induction hypothesis $A(0_\Omega)^{\Diamond}$ in BON_Ω and the substitution lemma 44. So now we assume, that $a \neq \widehat{0_\Omega}$. Consider the second induction hypothesis in BON_Ω:

$$\left((\forall a : \Omega)\left(a \neq 0_\Omega \wedge (\forall x : \mathsf{N})\, A(a_x) \to A(a)\right)\right)^{\Diamond}$$

$$\Longleftrightarrow (\forall a)\left(\mathbb{O}(a) \to \left(a \neq \widehat{0_\Omega} \wedge \forall x A(a_x)^{\Diamond}\right) \to A(a)^{\Diamond}\right)$$

From $\mathbb{O}^{\alpha}(a) \to \mathbb{O}(a)$ and $a \neq \widehat{0_\Omega}$, we get

$$\forall a\left(\forall x A(a_x)^{\Diamond} \to A(a)^{\Diamond}\right)$$

Consider the following formula

$$\forall a, e, z, y\left(\left(\mathbb{O}^{<\alpha}(y) \wedge \mathcal{T}^{<\alpha}(e, z, y)\right) \wedge a = \langle \widehat{\mathsf{sup}}, e\rangle \to \mathbb{V}_{(\mathsf{sup}^{-1} a) z}(y)\right)$$

Notice the universal quantifiers, i.e. we can choose all the variables arbitrarily. Assuming the premise, we can prove the conclusion:

$$\mathbb{V}_{(\mathsf{sup}^{-1} a) z}(y)$$

$$\Longleftrightarrow \exists v_1, v_2\left(\mathbb{V}_{\mathsf{sup}^{-1} a}(v_1) \wedge \mathbb{V}_z(v_2) \wedge \mathcal{T}(v_1, v_2, y)\right)$$

since z is a variable, $\mathbb{V}_z(v_2) \equiv z = v_2$.

$$\Longleftrightarrow \exists v_1, v_3, v_4\left(\mathbb{V}_{\mathsf{sup}^{-1}}(v_3) \wedge \mathbb{V}_a(v_4) \wedge \mathcal{T}(v_3, v_4, v_1) \wedge \mathcal{T}(v_1, z, y)\right)$$

the same argument yields

$$\Longleftrightarrow \exists v_1\left(\mathcal{T}\left(\widehat{\mathsf{sup}^{-1}}, a, v_1\right) \wedge \mathcal{T}(v_1, z, y)\right)$$

the premise gives us $a = \langle \widehat{\mathsf{sup}}, e \rangle$

$$\Longleftrightarrow \exists v_1 \left(\mathcal{T} \left(\widehat{\mathsf{sup}^{-1}}, \langle \widehat{\mathsf{sup}}, e \rangle, v_1 \right) \wedge \mathcal{T} (v_1, z, y) \right)$$

we get, that $v_1 = e$

$$\Longleftrightarrow \mathcal{T} (e, z, y)$$

but given $\mathcal{T}^{<\alpha} (e, z, y)$ this is true. We use this fact as follows. We know, that

$$\exists e \left(\forall z \exists y \left(\mathbb{O}^{<\alpha} (y) \wedge \mathcal{T}^{<\alpha} (e, z, y) \right) \wedge a = \langle \widehat{\mathsf{sup}}, e \rangle \right)$$

$$\Longrightarrow \exists e \left(\forall z \exists y \left(\mathbb{O}^{<\alpha} (y) \wedge \mathcal{T}^{<\alpha} (e, z, y) \wedge \mathbb{V}_{(\mathsf{sup}^{-1} a) z} (y) \right) \wedge a = \langle \widehat{\mathsf{sup}}, e \rangle \right)$$

Given the induction hypothesis, that $\forall a \left(\mathbb{O}^{<\alpha} (a) \to A (a)^{\diamondsuit} \right)$, we get

$$\Longrightarrow \exists e \left(\forall z \exists y \left(\mathbb{O}^{<\alpha} (y) \wedge \mathcal{T}^{<\alpha} (e, z, y) \wedge A (y)^{\diamondsuit} \wedge \mathbb{V}_{(\mathsf{sup}^{-1} a) z} (y) \right) \wedge a = \langle \widehat{\mathsf{sup}}, e \rangle \right)$$

the substitution lemma 44 gives us, that

$$\Longrightarrow \exists e \left(\forall z \exists y \left(\mathbb{O}^{<\alpha} (y) \wedge \mathcal{T}^{<\alpha} (e, z, y) \wedge A (a_z)^{\diamondsuit} \right) \wedge a = \langle \widehat{\mathsf{sup}}, e \rangle \right)$$

and so in particular

$$\forall z A (a_z)^{\diamondsuit}$$

this is the premise of the second induction hypothesis in BON_Ω.

$$\left((\forall a : \Omega) \left(a \neq 0_\Omega \wedge (\forall x : \mathsf{N}) A (a_x) \to A (a) \right) \right)^{\diamondsuit}$$

so we get $A (a)^{\diamondsuit}$. Now we showed that for an a added to P_A at the layer α, we get $\mathbb{O}^{\alpha} (a) \to A (a)^{\diamondsuit}$.

We therefore proved

$$(\forall \beta < \alpha) \forall a \left(\mathbb{O}^{\beta} (a) \to A (a)^{\diamondsuit} \right) \to \forall a \left(\mathbb{O}^{\alpha} (a) \to A (a)^{\diamondsuit} \right)$$

We apply the induction scheme in PA_Ω to $B (\alpha) \equiv \forall a \left(\mathbb{O}^{\alpha} (a) \to A (a)^{\diamondsuit} \right)$ and get

$$\forall \alpha \forall a \left(\mathbb{O}^{\alpha} (a) \to A (a)^{\diamondsuit} \right)$$

and this is equivalent to the following

$$\Longleftrightarrow \forall a \forall \alpha \left(\mathbb{O}^{\alpha} (a) \to A (a)^{\diamondsuit} \right)$$

$$\Longleftrightarrow \forall a \forall \alpha \left(\neg \mathbb{O}^{\alpha} (a) \vee A (a)^{\diamondsuit} \right)$$

$$\Longleftrightarrow \forall a\left(\neg\exists\alpha\mathbb{O}^{\alpha}(a)\vee A(a)^{\Diamond}\right)$$

$$\Longleftrightarrow \forall a\left(\neg\mathbb{O}(a)\vee A(a)^{\Diamond}\right)$$

$$\Longleftrightarrow (\forall a)\left(\mathbb{O}(a)\rightarrow A(a)^{\Diamond}\right)$$

$$\Longleftrightarrow ((\forall a:\Omega)\,A(a))^{\Diamond}$$

and this concludes the proof, for this is the conclusion of the transfinite induction in BON_{Ω}.

□

8 Proof-theoretical strength of BON_{Ω} and $\mathsf{BON}_{\Omega}+\mu$

In this section we put together all the main theorems from the different parts of this master thesis to get the following result about the proof theoretical strength of BON_{Ω} and $\mathsf{BON}_{\Omega}+\mu$. In the first part, we showed that

$$\mathsf{QT}_{\Omega}\vdash A\Longrightarrow \mathsf{BON}_{\Omega}\vdash A^{\triangle}$$

in the second part, we showed that

$$\mathsf{BON}_{\Omega}+\mu\vdash A\Longrightarrow \mathsf{PA}_{\Omega}\vdash A^{\Diamond}$$

according to [2], we get that

$$\mathsf{ID}_1\equiv\mathsf{QT}_{\Omega}$$

and according to [11], we get that

$$\mathsf{ID}_1\equiv\mathsf{PA}_{\Omega}$$

All those put together yield the

Theorem 46.

$$\mathsf{QT}_{\Omega}\equiv\mathsf{BON}_{\Omega}\equiv\mathsf{BON}_{\Omega}+\mu\equiv\mathsf{PA}_{\Omega}\equiv\mathsf{ID}_1$$

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