# The Suslin operator in applicative theories: its proof-theoretic analysis via ordinal theories

Gerhard Jäger Dieter Probst*

**Abstract**

The Suslin operator $\mathsf{E}_1$ is a type-2 functional testing for the well-foundedness of binary relations on the natural numbers. In the context of applicative theories, its proof-theoretic strength has been analyzed in Jäger and Strahm [18]. This article provides a more direct approach to the computation of the upper bounds in question. Several theories featuring the Suslin operator are embedded into ordinal theories tailored for dealing with non-monotone inductive definitions that enable a smooth definition of the application relation.

## 1 Introduction

The so-called Suslin operator $\mathsf{E}_1$ is a type-2 functional testing for the well-foundedness of binary relations on the natural numbers. The least ordinal not recursive in $\mathsf{E}_1$ is the first recursively inaccessible ordinal $\iota_0$, its 1-section coincides with the sets of natural numbers in the constructible hierarchy up to $\iota_0$, providing, therefore, a model of $\Delta^1_2$ comprehension,

$$(\mathbb{N}, \text{1-sec}(\mathsf{E}_1), \ldots) \models (\Delta^1_2\text{-}\mathsf{CA}).$$

For more on the recursion and definability theory of $\mathsf{E}_1$ we refer to the comprehensive textbook Hinman [9].

The Suslin operator has also a natural place within the context of applicative theories. These theories are obtained by restricting systems of Feferman's explicit mathematics (see [4, 5, 6]) to their first order part and provide a natural axiomatic framework for dealing with abstract computations. This approach has been discussed, from a more general perspective, in Jäger, Kahle, and Strahm [15].

*Research partly supported by the Alexander von Humboldt Foundation.

Jäger and Strahm [18] characterizes the proof-theoretic strength of the Suslin operator in the applicative context, depending on the induction principles which are permitted. In particular, it is shown that SUS plus the schema of induction on the natural numbers for arbitrary formulas is a theory proof-theoretically equivalent to the system $\Delta^1_2$-CA of second order arithmetic with $\Delta^1_2$ comprehension,

$$\mathsf{SUS} + (\text{L-I}_\mathsf{N}) \;\equiv\; \Delta^1_2\text{-}\mathsf{CA}.$$

Subsystems with restricted forms of induction on the natural numbers have been studied in that article as well.

Simply embedding the appropriate systems of second order arithmetic into SUS plus induction takes care of the lower bounds. The determination of the upper bounds has been more demanding. Working within an extension of Kripke-Platek set theory for a recursively inaccessible universe, a $\Sigma$ definable fixed point of a specific $\Delta^1_2$ inductive definition is used to interpret the application relation of SUS. Then, in order to show that the obtained structure is indeed a model of SUS, a rather subtle "inside-outside-argument" is used in establishing a relationship between proper set-theoretic functions and operations defined in terms of this application relation.

The purpose of this article is to provide a more direct and simpler approach to the computation of the upper bounds in question. We introduce ordinal theories tailored for directly dealing with certain non-monotone inductive definitions, similar to those of Jäger [13] and Jäger and Strahm [17], and develop the required structures directly within those. Alternatively, we could also work with theories for Richter-styled combined non-monotone operators (see Jäger [14] for a more systematic proof-theoretic treatment of such systems) as originally proposed in Probst [19]. However, the line we are going to follow now seems to be the more "explicit".

## 2 The theory SUS

The following presentation of the theory SUS and its induction principles is taken from Jäger and Strahm [18]. SUS is formulated in a first order language L of partial terms with variables $a, b, c, f, g, h, u, v, w, x, y, z \ldots$ (possibly with subscripts). L includes individual constants $\mathsf{k}, \mathsf{s}$ (combinators), $\mathsf{p}, \mathsf{p_0}, \mathsf{p_1}$ (pairing and unpairing), 0 (zero), $\mathsf{s_N}$ (numerical successor), $\mathsf{p_N}$ (numerical predecessor), $\mathsf{d_N}$ (definition by numerical cases), $\mathsf{r_N}$ (primitive recursion), $\mu$ (non-constructive $\mu$ operator), and $\mathsf{E}_1$ (Suslin operator). In addition, L has a binary function symbol $\cdot$ for (partial) term application, unary relation symbols $\downarrow$ (defined) and $\mathsf{N}$ (natural numbers), as well as a binary relation symbol $=$ (equality).

The individual terms $(r, s, t, r_0, s_0, t_0, \ldots)$ of L are inductively generated as follows:

1. The individual variables and individual constants are individual terms.
2. If $s$ and $t$ are individual terms, then so also is $(s \cdot t)$.

We usually abbreviate $(s \cdot t)$ as $(st)$ or – in case that no confusion arises – simply as $st$. We also adopt the convention of association to the left so that $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2) \ldots s_n)$. Moreover, we often write $s(t_1, \ldots, t_n)$ for $st_1 \ldots t_n$. Further, we put $t' := \mathsf{s_N} t$ and $1 := 0'$. General $n$-tupling is defined by induction on $n \geq 1$ such that

$$<s_1> \; := \; s_1 \quad \text{and} \quad <s_1, \ldots, s_{n+1}> \; := \; \mathsf{p}<s_1, \ldots, s_n>s_{n+1}.$$

Finally, we frequently use the vector notation $\vec{\mathcal{Z}}$ for finite strings of objects $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ of the same sort. Whenever we write $\vec{\mathcal{Z}}$, the length of this string is either irrelevant or given by the context.

The formulas $(A, B, C, A_0, B_0, C_0, \ldots)$ of L are inductively generated as follows:

1. Each atomic formula $N(t)$, $t{\downarrow}$, and $(s = t)$ is a formula.
2. If $A$ and $B$ are formulas, then so also are $\neg A$, $(A \vee B)$, $(A \wedge B)$, and $(A \to B)$.
3. If $A$ is a formula, then so also are $\exists x A$ and $\forall x A$.

Our applicative theories are based on partial term application. Hence, it is not guaranteed that terms have a value, and $t{\downarrow}$ is read as "$t$ is defined" or "$t$ has a value". Accordingly, the partial equality relation $\simeq$ is introduced by

$$(s \simeq t) \; := \; (s{\downarrow} \vee t{\downarrow}) \; \to \; (s = t).$$

We write $(s \neq t)$ for $(s{\downarrow} \wedge t{\downarrow} \wedge \neg(s = t))$ and introduce the following abbreviations concerning the predicate $\mathsf{N}$:

$$\begin{aligned}
t \in \mathsf{N} \;&:=\; \mathsf{N}(t), \\
(\exists x \in \mathsf{N})A \;&:=\; \exists x(x \in \mathsf{N} \,\wedge\, A), \\
(\forall x \in \mathsf{N})A \;&:=\; \forall x(x \in \mathsf{N} \,\to\, A), \\
t \in (\mathsf{N} \mapsto \mathsf{N}) \;&:=\; (\forall x \in \mathsf{N})(tx \in \mathsf{N}), \\
t \in (\mathsf{N}^1 \mapsto \mathsf{N}) \;&:=\; t \in (\mathsf{N} \mapsto \mathsf{N}), \\
t \in (\mathsf{N}^{m+1} \mapsto \mathsf{N}) \;&:=\; (\forall x \in \mathsf{N})(tx \in (\mathsf{N}^m \mapsto \mathsf{N})).
\end{aligned}$$

Now we are going to recall the basic theory $\mathsf{BON}$ of operations and numbers which has been introduced in Feferman and Jäger [8]. Its underlying logic is the classical logic of partial terms due to Beeson [1, 2] with strictness and equality axioms; it is also described in Feferman [7] and Jäger, Kahle, and Strahm [15]. The non-logical axioms of $\mathsf{BON}$ are divided into the following five groups.

**I. Partial combinatory algebra.**

(1) $\mathsf{k}ab \;=\; a$,

(2) $\mathsf{s}ab{\downarrow} \;\wedge\; \mathsf{s}abc \simeq ac(bc)$.

**II. Paring and projection.**

(3) $\mathsf{p}_0{<}a,b{>} = a \;\wedge\; \mathsf{p}_1{<}a,b{>} = b$.

**III. Natural numbers.**

(4) $0 \in \mathsf{N} \;\wedge\; (\forall x \in \mathsf{N})(x' \in \mathsf{N})$,

(5) $(\forall x \in \mathsf{N})(x' \neq 0 \;\wedge\; \mathsf{p_N}(x') = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq 0 \;\rightarrow\; \mathsf{p_N}x \in \mathsf{N} \;\wedge\; (\mathsf{p_N}x)' = x)$.

**IV. Definition by numerical cases.**

(7) $u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u = v \;\rightarrow\; \mathsf{d_N}(a,b,u,v) = a$,

(8) $u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u \neq v \;\rightarrow\; \mathsf{d_N}(a,b,u,v) = b$.

**V. Primitive recursion on $\mathsf{N}$.**

(9) $f \in (\mathsf{N} \mapsto \mathsf{N}) \wedge g \in (\mathsf{N}^3 \mapsto \mathsf{N}) \;\rightarrow\; \mathsf{r_N}(f,g) \in (\mathsf{N}^2 \mapsto \mathsf{N})$,

(10) $f \in (\mathsf{N} \mapsto \mathsf{N}) \wedge g \in (\mathsf{N}^3 \mapsto \mathsf{N}) \wedge a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge h = \mathsf{r_N}(f,g) \;\rightarrow$
$h(a,0) = f(a) \;\wedge\; h(a,b') = g(a,b,h(a,b))$.

As usual, the axioms of a partial combinatory algebra allow one to define $\lambda$-abstraction and to demonstrate a recursion or fixed point theorem. For proofs of these standard results the reader is referred to Beeson [1] or Feferman [4]. The second assertion of the following lemma is a slight extension of the usual $\lambda$-abstraction which requires our axioms about pairing and projections.

**Lemma 1** *1. For each* L *term* $t$ *and all variables* $x$ *there exists an* L *term* $(\lambda x.t)$ *whose variables are those of* $t$*, excluding* $x$*, such that* BON *proves*

$$(\lambda x.t)\downarrow \text{ and } (\lambda x.t)x \simeq t.$$

*2. For each* L *term* $t$ *and all variables* $x_0, \ldots, x_{n-1}$ $(n \geq 1)$ *there exists an* L *term* $s$ *whose variables are those of* $t$*, excluding* $x_0, \ldots, x_{n-1}$*, such that* BON *proves*

$$s\downarrow \ \wedge \ s(x_0, \ldots, x_{n-1}) \simeq t.$$

*3. There exists a closed* L *term* fix *such that* BON *proves*

$$\mathsf{fix}(f)\downarrow \ \wedge \ \mathsf{fix}(f, x) \simeq f(\mathsf{fix}(f), x).$$

Next we introduce the two type-2 functionals which are to be analyzed in the context of applicative theories. The non-constructive or unbounded $\mu$ operator is characterized by the following two axioms.

**The non-constructive $\mu$ operator.**

$(\mu.1)$ $f \in (\mathsf{N} \mapsto \mathsf{N}) \ \leftrightarrow \ \mu f \in \mathsf{N}$,

$(\mu.2)$ $f \in (\mathsf{N} \mapsto \mathsf{N}) \wedge (\exists x \in \mathsf{N})(fx = 0) \ \rightarrow \ f(\mu f) = 0$.

A much stronger functional is the Suslin operator $\mathsf{E}_1$, which tests for the well-foundedness of a binary relation on $\mathsf{N}$ (given as a total operation from $\mathsf{N}^2$ to $\mathsf{N}$).

**The Suslin operator $\mathsf{E}_1$.**

$(\mathsf{E}_1.1)$ $f \in (\mathsf{N}^2 \mapsto \mathsf{N}) \ \leftrightarrow \ \mathsf{E}_1 f \in \mathsf{N}$,

$(\mathsf{E}_1.2)$ $f \in (\mathsf{N}^2 \mapsto \mathsf{N}) \ \rightarrow$

$$((\exists g \in (\mathsf{N} \mapsto \mathsf{N}))(\forall x \in \mathsf{N})(f(g(x'), g(x)) = 0) \ \leftrightarrow \ \mathsf{E}_1 f = 0).$$

The extension of BON by the two axioms for the non-constructive $\mu$ operator has been baptized $\mathsf{BON}(\mu)$, the theory SUS for the Suslin operator is $\mathsf{BON}(\mu)$ plus the two axioms for $\mathsf{E}_1$, i.e.

$$\begin{aligned} \mathsf{BON}(\mu) &:= \mathsf{BON} + (\mu.1) + (\mu.2), \\ \mathsf{SUS} &:= \mathsf{BON}(\mu) + (\mathsf{E}_1.1) + (\mathsf{E}_1.2). \end{aligned}$$

In the sequel we will be interested in three forms of complete induction on the natural numbers $\mathsf{N}$, namely *set induction*, $\mathsf{N}$ *induction*, and *formula* *induction.* Let us first recall the notion of a subset of $\mathsf{N}$ from Feferman and Jäger [8]. Sets of natural numbers are represented via their characteristic functions which are total on $\mathsf{N}$. Accordingly, we define

$$f \in \mathcal{P}(\mathsf{N}) \;:=\; (\forall x \in \mathsf{N})(fx = 0 \,\vee\, fx = 1)$$

with the intention that an object $x$ belongs to the set $f \in \mathcal{P}(\mathsf{N})$ if and only if $(fx = 0)$. The three relevant induction principles are now given as follows.

**Set induction on $\mathsf{N}$ ($\mathsf{S\text{-}I_N}$).**

$$f \in \mathcal{P}(\mathsf{N}) \wedge f0 = 0 \wedge (\forall x \in \mathsf{N})(fx = 0 \rightarrow f(x') = 0) \;\rightarrow\; (\forall x \in \mathsf{N})(fx = 0).$$

**$\mathsf{N}$ induction on $\mathsf{N}$ ($\mathsf{N\text{-}I_N}$).**

$$f0 \in \mathsf{N} \wedge (\forall x \in \mathsf{N})(fx \in \mathsf{N} \rightarrow f(x') \in \mathsf{N}) \;\rightarrow\; (\forall x \in \mathsf{N})(fx \in \mathsf{N}).$$

**Formula induction on $\mathsf{N}$ ($\mathrm{L}\text{-}\mathsf{I_N}$).** For all formulas $A[u]$ of L:

$$A[0] \wedge (\forall x \in \mathsf{N})(A[x] \rightarrow A[x']) \;\rightarrow\; (\forall x \in \mathsf{N})A[x].$$

In Jäger and Strahm [18] it is shown how $\mathsf{E_1}$ can be used to model the hyperjump in our applicative context. As a consequence, we obtain the following embedding theorem, where sets of natural numbers of second order arithmetic are represented as elements of $\mathcal{P}(\mathsf{N})$ in $\mathsf{SUS}$. As usual, $\Pi^1_1\text{-}\mathsf{CA}_0$ is the subsystem of second order arithmetic with comprehension restricted to $\Pi^1_1$ formulas and complete induction on the natural numbers restricted to sets; $\Pi^1_1\text{-}\mathsf{CA}_{<\omega^\omega}$ and $\Pi^1_1\text{-}\mathsf{CA}_{<\varepsilon_0}$ are the extensions of $\Pi^1_1\text{-}\mathsf{CA}_0$ which permit the iteration of $\Pi^1_1$ comprehension along suitable primitive recursive well-orderings of order types less than $\omega^\omega$ and $\varepsilon_0$, respectively.

**Theorem 2** *We have the following inclusions:*

1. $\Pi^1_1\text{-}\mathsf{CA}_0 \subseteq \mathsf{SUS} + (\mathsf{S\text{-}I_N})$,
2. $\Pi^1_1\text{-}\mathsf{CA}_{<\omega^\omega} \subseteq \mathsf{SUS} + (\mathsf{N\text{-}I_N})$,
3. $\Pi^1_1\text{-}\mathsf{CA}_{<\varepsilon_0} \subseteq \mathsf{SUS} + (\mathrm{L}\text{-}\mathsf{I_N})$.

## 3 The theory INA of numbers and ordinals

In this section we introduce a theory of natural numbers and ordinals, similar to those in Jäger [13] and Jäger and Strahm [17]. Our system INA allows us to formalize a variety of monotone and non-monotone inductive definitions and provides closure properties reflecting the idea that the ordinals of INA reach up to the first recursively inaccessible ordinal.

Let $\mathcal{L}_0$ denote the language of first order arithmetic, which has number variables $a, b, c, d, e, f, u, v, w, x, y, z, \ldots$ (possibly with subscripts) as well as symbols for all primitive recursive functions and relations. Number terms $(r, s, t, r_0, s_0, t_0, \ldots)$ and formulas $(A, B, C, A_0, B_0, C_0, \ldots)$ of $\mathcal{L}_0$ are defined as usual; for notational convenience, numerals are identified with the respective natural numbers.

In addition, we make use of a primitive recursive coding machinery in $\mathcal{L}_0$: $\langle \ldots \rangle$ is a standard primitive recursive function for forming $n$-tuples $\langle t_0, \ldots, t_{n-1} \rangle$; $Seq$ is the primitive recursive set of sequence numbers; $lh(t)$ denotes the length of (the sequence number coded by) $t$; $(t)_i$ is the $i$th component of (the sequence coded by) $t$ if $i < lh(t)$, i.e. $t = \langle (t)_0, \ldots, (t)_{lh(t)-1} \rangle$ if $t$ is a sequence number.

Further, let $X$ be a fresh $n$-ary relation symbol and write $\mathcal{L}_0(X)$ for the extension of $\mathcal{L}_0$ by $X$. An $\mathcal{L}_0(X)$ formula which contains at most $a_0, \ldots, a_{n-1}$ free is called an $n$-ary operator form, and we let $\mathfrak{A}[X, a_0, \ldots, a_{n-1}]$ range over such forms.

For formulating INA we extend $\mathcal{L}_0$ to a two-sorted language $\mathcal{L}^\star$ by adding a new sort of ordinal variables $\pi, \rho, \sigma, \tau, \eta, \xi, \ldots$ (possibly with subscripts), new binary relation symbols $<$ and $=$ for the less and equality relation on the ordinals[1] and a unary relation symbol $Ad$ to express that an ordinal is admissible. Moreover, $\mathcal{L}^\star$ includes an $(n+1)$-ary relation symbol $P_{\mathfrak{A}}$ for each operator form $\mathfrak{A}[X, a_0, \ldots, a_{n-1}]$.

The number terms of $\mathcal{L}^\star$ are the number terms of $\mathcal{L}_0$, the atomic formulas of $\mathcal{L}^\star$ are the atomic formulas of $\mathcal{L}_0$ plus all expressions $(\sigma < \tau)$, $(\sigma = \tau)$, $Ad(\sigma)$, and $P_{\mathfrak{A}}(\sigma, \vec{r})$ for any $n$-ary operator form $\mathfrak{A}[X, \vec{a}]$; usually, we write $P^\sigma_{\mathfrak{A}}[\vec{r}]$ for $P_{\mathfrak{A}}(\sigma, \vec{r})$.

The formulas $(A, B, C, A_0, B_0, C_0, \ldots)$ of $\mathcal{L}^\star$ are generated from the atomic $\mathcal{L}^\star$ formulas by closing under negations, disjunctions, conjunctions, implications, equivalences, quantifications over the natural numbers, bounded quantifications $(\exists \xi < \sigma)$ and $(\forall \xi < \sigma)$ over the ordinals, and unbounded quantifications over the ordinals.

[1] It will always be clear from the context whether $<$ and $=$ denote the less and equality relation on the natural numbers or on the ordinals.

An $\mathcal{L}^\star$ formula is called $\Delta_0^{\mathbb{O}}$ if it does not contain unbounded ordinal quantifiers; it is called $\Sigma^{\mathbb{O}}$ if it does not contain positive occurrences of unbounded universal ordinal quantifiers or negative occurrences of unbounded existential ordinal quantifiers. Given an $\mathcal{L}^\star$ formula $A$ and an ordinal variable $\sigma$ not occurring freely in $A$, we write $A^\sigma$ to denote the formula which is obtained from $A$ by replacing all unbounded ordinal quantifiers $Q\xi$ by bounded ordinal quantifiers $(Q\xi < \sigma)$. Hence every formula $A^\sigma$ is $\Delta_0^{\mathbb{O}}$. Additional abbreviations are

$$P_{\mathfrak{A}}^{<\sigma}[\vec{r}] \;:=\; (\exists \xi < \sigma) P_{\mathfrak{A}}^{\xi}[\vec{r}] \quad \text{and} \quad P_{\mathfrak{A}}[\vec{r}] \;:=\; \exists \xi P_{\mathfrak{A}}^{\xi}[\vec{r}].$$

The theory $\mathsf{INA}$ is formulated in classical two sorted predicate logic with equality in both sorts and contains the following non-logical axioms.

**I. Number-theoretic axioms.** The axioms of Peano arithmetic $\mathsf{PA}$ with the exception of complete induction on the natural numbers.

**II. Linearity axioms.**

$$\sigma \not< \sigma \;\wedge\; (\sigma < \tau \wedge \tau < \eta \;\rightarrow\; \sigma < \eta) \;\wedge\; (\sigma < \tau \vee \sigma = \tau \vee \tau < \sigma).$$

**III. Operator axioms.** For all operator forms $\mathfrak{A}[X, \vec{a}]$:

$$P_{\mathfrak{A}}^{\sigma}[\vec{r}] \;\leftrightarrow\; \mathfrak{A}[P_{\mathfrak{A}}^{<\sigma}, \vec{r}].$$

**IV. $\Sigma^{\mathbb{O}}$ reflection axioms.** For all $\Sigma^{\mathbb{O}}$ formulas $A$:

$$A \;\rightarrow\; \exists \xi A^{\xi}.$$

**V. Axioms for $Ad$.** For all $\Sigma^{\mathbb{O}}$ formulas $A[\vec{\tau}]$ whose free ordinal variables are from the list $\vec{\tau}$:

$(\mathsf{Ref})$ $$Ad(\sigma) \;\wedge\; \vec{\tau} < \sigma \;\wedge\; A^{\sigma}[\vec{\tau}] \;\rightarrow\; (\exists \xi < \sigma)(\vec{\tau} < \xi \;\wedge\; A^{\xi}[\vec{\tau}]),$$

$(\mathsf{Lim})$ $$\forall \eta \exists \xi (\eta < \xi \;\wedge\; Ad(\xi)).$$

**VI. Induction principles.** For all $\mathcal{L}^\star$ formulas $A[a]$ and $B[\sigma]$:

$(\mathcal{L}^\star\text{-}\mathsf{I_N})$ $$A[0] \;\wedge\; \forall x(A[x] \rightarrow A[x+1]) \;\rightarrow\; \forall x A[x],$$

$(\mathcal{L}^\star\text{-}\mathsf{I_<})$ $$\forall \eta((\forall \xi < \eta) B[\xi] \rightarrow B[\eta]) \;\rightarrow\; \forall \eta B[\eta].$$

The corresponding induction principles claiming induction for $\Delta_0^{\mathbb{O}}$ formulas only are denoted by $(\Delta_0^{\mathbb{O}}\text{-}\mathsf{I}_{\mathsf{N}})$ and $(\Delta_0^{\mathbb{O}}\text{-}\mathsf{I}_<)$, respectively. $\mathsf{INA}^{\mathsf{r}}$ is the subsystem of $\mathsf{INA}$ which we obtain if we restrict $(\mathcal{L}^\star\text{-}\mathsf{I}_{\mathsf{N}})$ to $(\Delta_0^{\mathbb{O}}\text{-}\mathsf{I}_{\mathsf{N}})$ and $(\mathcal{L}^\star\text{-}\mathsf{I}_<)$ to $(\Delta_0^{\mathbb{O}}\text{-}\mathsf{I}_<)$. Moreover, $\mathsf{INA}^{\mathsf{w}}$ is defined to be $\mathsf{INA}^{\mathsf{r}} + (\mathcal{L}^\star\text{-}\mathsf{I}_{\mathsf{N}})$ and thus permits full complete induction and induction on the ordinals for $\Delta_0^{\mathbb{O}}$ formulas.

The theories $\mathsf{INA}$, $\mathsf{INA}^{\mathsf{w}}$, and $\mathsf{INA}^{\mathsf{r}}$ are closely related to theories $\mathsf{KPi}$, $\mathsf{KPi}^{\mathsf{w}}$, and $\mathsf{KPi}^{\mathsf{r}}$ for iterated admissible sets which are studied, for example, in Jäger [10, 11, 12] and Jäger and Pohlers [16]. It is easy to show that $\mathsf{INA}$ is contained in $\mathsf{KPi}$, $\mathsf{INA}^{\mathsf{w}}$ in $\mathsf{KPi}^{\mathsf{w}}$, and $\mathsf{INA}^{\mathsf{r}}$ in $\mathsf{KPi}^{\mathsf{r}}$. Also, if we write $(\Sigma^{\mathbb{O}}\text{-}\mathsf{I}_{\mathsf{N}})$ for the schema of complete induction on the natural numbers for $\Sigma^{\mathbb{O}}$ formulas and $(\Sigma\text{-}\mathsf{I}_{\mathsf{N}})$ for the schema of complete induction on the natural numbers for $\Sigma$ formulas of the language of theories for admissible sets, then $\mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{I}_{\mathsf{N}})$ comprises $\mathsf{INA}^{\mathsf{r}} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I}_{\mathsf{N}})$.

**Theorem 3** *We have the following inclusions:*

1. $\mathsf{INA}^{\mathsf{r}} \subseteq \mathsf{KPi}^{\mathsf{r}}$,
2. $\mathsf{INA}^{\mathsf{r}} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I}_{\mathsf{N}}) \subseteq \mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{I}_{\mathsf{N}})$,
3. $\mathsf{INA}^{\mathsf{w}} \subseteq \mathsf{KPi}^{\mathsf{w}}$,
4. $\mathsf{INA} \subseteq \mathsf{KPi}$.

# 4 Modeling SUS in $\mathsf{INA}^{\mathsf{r}}$

The theory $\mathsf{INA}^{\mathsf{r}}$ provides a canonical framework for defining a model of the applicative theory $\mathsf{SUS}$. The crucial step is the interpretation of the application relation $(rs \simeq t)$. This will be achieved by the non-monotone inductive definition presented in Definition 4. It is our strategy to introduce a specific operator form $\mathfrak{A}[X, a, b, c, d]$ such that the corresponding relation symbol $P_{\mathfrak{A}}$ codes several important assertions, for example:

$P_{\mathfrak{A}}[a, b, c, 0]$ :: $a$ applied to $b$ yields $c$,

$P_{\mathfrak{A}}[a, b, 0, 1]$ :: { $b$ belongs to the accessible part of the binary relation represented by $a$,

$P_{\mathfrak{A}}[a, 0, 0, 2]$ :: { the accessible part of the relation represented by $a$ is completely built up,

$P_{\mathfrak{A}}[a, 0, 0, 3]$ :: the binary relation represented by $a$ is well-founded,

$P_{\mathfrak{A}}[a, 0, 0, 4]$ :: the binary relation represented by $a$ is not well-founded.

The following abbreviations and shorthand notations will help to make Definition 4 more readable. Let $D[f, a, b, c]$ be an $\mathcal{L}^\star$ formula with at most $f, a, b, c$ free.

**Application, totality, and functionality with respect to $D$.** For any natural number $n$ and all vectors $\vec{a} = a_0, \ldots, a_{n-1}$ and $\vec{x} = x_0, \ldots, x_{n-1}$,

$$\begin{aligned}
App^1_D[f, a_0, b] &:= D[f, a_0, b, 0],\\
App^{n+1}_D[f, \vec{a}, a_n, b] &:= \exists x(App^n_D[f, \vec{a}, x] \wedge D[x, a_n, b, 0]),\\
Tot^n_D[f] &:= \forall \vec{x} \exists y App^n_D[f, \vec{x}, y],\\
Fun^n_D[f] &:= \begin{cases} Tot^n_D[f] \wedge \\ \forall \vec{x} \forall y \forall z(App^n_D[f, \vec{x}, y] \wedge App^n_D[f, \vec{x}, z] \rightarrow y = z). \end{cases}
\end{aligned}$$

**Primitive recursion with respect to $D$.** If $f$ and $g$ represent a unary and a ternary functional operation with respect to $D$, then the following formula $Rc_D[f, g, u, v, w]$ describes the graph of the operation which is defined from $f$ and $g$ by primitive recursion with application in the sense of $D$:

$$Rc_D[f, g, u, v, w] := \begin{cases} \exists x(Seq(x) \wedge lh(x) = v + 1 \wedge D[f, u, (x)_0] \wedge \\ (\forall y < v) App^3_D[g, u, y, (x)_y, (x)_{y+1}] \wedge w = (x)_v). \end{cases}$$

Finally, for all natural numbers $n$ we set

$$Seq_n[t] := Seq(t) \wedge lh(t) = n$$

and choose pairwise different numerals $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\mathsf{p}}$, $\widehat{\mathsf{p}}_0$, $\widehat{\mathsf{p}}_1$, $\widehat{\mathsf{s}}_\mathsf{N}$, $\widehat{\mathsf{p}}_\mathsf{N}$, $\widehat{\mathsf{d}}_\mathsf{N}$, $\widehat{\mathsf{r}}_\mathsf{N}$, $\widehat{\mu}$, and $\widehat{\mathsf{E}}_1$ which do not belong to $\{0\} \cup Seq$. They serve as codes of the corresponding constants of L.

**Definition 4** *The operator form $\mathfrak{A}[X, a, b, c, d]$ is defined to be the conjunction of the formula $\forall x \neg X(a, b, x, 0)$ with the disjunction of the following formulas (1)–(28):*

*(1)* $a = \widehat{\mathsf{k}} \wedge c = \langle \widehat{\mathsf{k}}, b \rangle \wedge d = 0,$

*(2)* $Seq_2[a] \wedge (a)_0 = \widehat{\mathsf{k}} \wedge c = (a)_1 \wedge d = 0,$

*(3)* $a = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, b \rangle \wedge d = 0,$

*(4)* $Seq_2[a] \wedge (a)_0 = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, (a)_1, b \rangle \wedge d = 0,$

(5) $Seq_3[a] \ \wedge \ (a)_0 = \widehat{\mathsf{s}} \ \wedge$
$\exists x \exists y (X((a)_1, b, x, 0) \wedge X((a)_2, b, y, 0) \wedge X(x, y, c, 0)) \ \wedge \ d = 0,$

(6) $a = \widehat{\mathsf{p}} \ \wedge \ c = \langle \widehat{\mathsf{p}}, b \rangle \ \wedge \ d = 0,$

(7) $Seq_2[a] \ \wedge \ (a)_0 = \widehat{\mathsf{p}} \ \wedge \ c = \langle (a)_1, b \rangle \ \wedge \ d = 0,$

(8) $a = \widehat{\mathsf{p}}_0 \ \wedge \ \exists x (b = \langle c, x \rangle) \ \wedge \ d = 0,$

(9) $a = \widehat{\mathsf{p}}_1 \ \wedge \ \exists x (b = \langle x, c \rangle) \ \wedge \ d = 0,$

(10) $a = \widehat{\mathsf{s}}_{\mathsf{N}} \ \wedge \ c = b + 1 \ \wedge \ d = 0,$

(11) $a = \widehat{\mathsf{p}}_{\mathsf{N}} \ \wedge \ b = c + 1 \ \wedge \ d = 0,$

(12) $a = \widehat{\mathsf{d}}_{\mathsf{N}} \ \wedge \ c = \langle \widehat{\mathsf{d}}_{\mathsf{N}}, b \rangle \ \wedge \ d = 0,$

(13) $Seq_2[a] \ \wedge \ (a)_0 = \widehat{\mathsf{d}}_{\mathsf{N}} \ \wedge \ c = \langle \widehat{\mathsf{d}}_{\mathsf{N}}, (a)_1, b \rangle \ \wedge \ d = 0,$

(14) $Seq_3[a] \ \wedge \ (a)_0 = \widehat{\mathsf{d}}_{\mathsf{N}} \ \wedge \ c = \langle \widehat{\mathsf{d}}_{\mathsf{N}}, (a)_1, (a)_2, b \rangle \ \wedge \ d = 0,$

(15) $Seq_4[a] \ \wedge \ (a)_0 = \widehat{\mathsf{d}}_{\mathsf{N}} \ \wedge \ (a)_1 = (a)_2 \ \wedge \ c = (a)_3 \ \wedge \ d = 0,$

(16) $Seq_4[a] \ \wedge \ (a)_0 = \widehat{\mathsf{d}}_{\mathsf{N}} \ \wedge \ (a)_1 \neq (a)_2 \ \wedge \ c = b \ \wedge \ d = 0,$

(17) $a = \widehat{\mathsf{r}}_{\mathsf{N}} \ \wedge \ c = \langle \widehat{\mathsf{r}}_{\mathsf{N}}, b \rangle \ \wedge \ d = 0,$

(18) $Seq_2[a] \ \wedge \ (a)_0 = \widehat{\mathsf{r}}_{\mathsf{N}} \ \wedge \ c = \langle \widehat{\mathsf{r}}_{\mathsf{N}}, (a)_1, b \rangle \ \wedge \ d = 0,$

(19) $Seq_3[a] \ \wedge \ (a)_0 = \widehat{\mathsf{r}}_{\mathsf{N}} \ \wedge \ c = \langle \widehat{\mathsf{r}}_{\mathsf{N}}, (a)_1, (a)_2, b \rangle \ \wedge \ d = 0,$

(20) $Seq_4[a] \ \wedge \ (a)_0 = \widehat{\mathsf{r}}_{\mathsf{N}} \ \wedge \ Fun^1_X[(a)_1] \ \wedge \ Fun^3_X[(a)_2] \ \wedge$
$Rc_X[(a)_1, (a)_2, (a)_3, b, c] \ \wedge \ d = 0,$

(21) $a = \widehat{\mu} \ \wedge \ Fun^2_X[b] \ \wedge \ \forall x \exists y (y \neq 0 \ \wedge \ X(b, x, y, 0)) \ \wedge \ c = 0 \ \wedge \ d = 0,$

(22) $a = \widehat{\mu} \ \wedge \ Fun^2_X[b] \ \wedge \ (\forall x < c) \exists y (y \neq 0 \ \wedge \ X(b, x, y, 0)) \ \wedge \ X(b, c, 0, 0)$
$\wedge \ d = 0,$

(23) $Fun^2_X[a] \ \wedge \ \forall x (App^2_X[a, x, b, 0] \to X(a, x, 0, 1)) \ \wedge \ c = 0 \ \wedge \ d = 1,$

(24) $Fun^2_X[a] \ \wedge \ \forall x (\forall y (App^2_X[a, y, x, 0] \to X(a, y, 0, 1)) \to X(a, x, 0, 1))$
$\wedge \ b = 0 \ \wedge \ c = 0 \ \wedge \ d = 2,$

(25) $Fun^2_X[a] \ \wedge \ X(a, 0, 0, 2) \ \wedge \ \forall x X(a, x, 0, 1) \ \wedge \ b = 0 \ \wedge \ c = 0 \ \wedge \ d = 3,$

(26) $Fun^2_X[a] \ \wedge \ X(a, 0, 0, 2) \ \wedge \ \exists x \neg X(a, x, 0, 1) \ \wedge \ b = 0 \ \wedge \ c = 0 \ \wedge \ d = 4,$

(27) $a = \widehat{\mathsf{E}}_1 \wedge X(b,0,0,3) \wedge c = 1 \wedge d = 0$,

(28) $a = \widehat{\mathsf{E}}_1 \wedge X(b,0,0,4) \wedge c = 0 \wedge d = 0$.

The clauses (1)–(22) are identical to the clauses of the inductive definition used in Jäger and Strahm [18]; clauses (23)–(28) will be needed below to take care of the Suslin operator $\mathsf{E}_1$. In contrast to [18], we here have to deal with a non-monotone definition clause $\mathfrak{A}[X,a,b,c,d]$.

**Definition 5** *For all natural numbers $n$ greater than $0$, all number variables $\vec{a}, b, f$, and all ordinal variables $\sigma$ we set:*

$$\begin{array}{rclrcl} App^n_\sigma[f,\vec{a},b] & := & App^n_{P^{<\sigma}_{\mathfrak{A}}}[f,\vec{a},b], & App^n_\infty[f,\vec{a},b] & := & App^n_{P_{\mathfrak{A}}}[f,\vec{a},b], \\ Tot^n_\sigma[f] & := & Tot^n_{P^{<\sigma}_{\mathfrak{A}}}[f], & Tot^n_\infty[f] & := & Tot^n_{P_{\mathfrak{A}}}[f], \\ Fun^n_\sigma[f] & := & Fun^n_{P^{<\sigma}_{\mathfrak{A}}}[f], & Fun^n_\infty[f] & := & Fun^n_{P_{\mathfrak{A}}}[f]. \end{array}$$

The following lemma states an important extension property: if $f$ codes a function in the sense of $P^{<\sigma}_{\mathfrak{A}}$, then it does so as well in the sense of any $P^{<\tau}_{\mathfrak{A}}$ with $\sigma \leq \tau$ and in the sense of $P_{\mathfrak{A}}$. The input-output behavior of these "functions" is identical.

**Lemma 6** *For all ordinal variables $\sigma, \tau$, all number variables $f$, and all natural numbers $n \geq 1$, the theory $\mathsf{INA}^\mathsf{r}$ proves:*

1. $Fun^n_\sigma[f] \wedge \sigma \leq \tau \rightarrow Fun^n_\tau[f]$.
2. $Fun^n_\sigma[f] \wedge \sigma \leq \tau \rightarrow \forall\vec{x}\forall y(App^n_\sigma[f,\vec{x},y] \leftrightarrow App^n_\tau[f,\vec{x},y])$.
3. $Fun^n_\sigma[f] \rightarrow Fun^n_\infty[f]$.
4. $Fun^n_\sigma[f] \rightarrow \forall\vec{x}\forall y(App^n_\sigma[f,\vec{x},y] \leftrightarrow App^n_\infty[f,\vec{x},y])$.

The first two parts of this lemma directly follow from the form of our operator form which prevents adding tuples $(f,a,c,0)$ to $P^\tau_{\mathfrak{A}}$ if at an earlier stage a tuple $(f,a,b,0)$ has been included. The third and the fourth part are immediate consequences of the first and the second.

The next observation states that any $f$ which codes an $n$-ary function in the sense of $P_{\mathfrak{A}}$ does so already in the sense of an initial segment $P^{<\sigma}_{\mathfrak{A}}$ of $P_{\mathfrak{A}}$.

**Lemma 7** *For any natural number $n \geq 1$ and any number variable $f$, the theory $\mathsf{INA}^\mathsf{r}$ proves:*

1. $Tot^n_\infty[f] \rightarrow \exists\sigma\, Tot^n_\sigma[f]$.
2. $Fun^n_\infty[f] \rightarrow \exists\sigma\, Fun^n_\sigma[f]$.

PROOF Assume $Tot^n_\infty[f]$, i.e. $\forall\vec{x}\exists y App^n_\infty[f,\vec{x},y]$. Since this is a $\Sigma^{\mathbb{O}}$ formula, $\Sigma^{\mathbb{O}}$ reflection implies $\exists\sigma\forall\vec{x}\exists y App^n_\sigma[f,\vec{x},y]$. Thus we have $\exists\sigma\, Tot^n_\sigma[f]$, and the first assertion is proved. The second assertion is an immediate consequence of the first since uniqueness with respect to $P_{\mathfrak{A}}$ yields uniqueness with respect to any $P^{<\sigma}_{\mathfrak{A}}$. □

Also the next assertion is easily established, simply prove it by ($\Delta^{\mathbb{O}}_0$-$\mathsf{I_N}$) with respect to $x$.

**Lemma 8** *For any $\Delta^{\mathbb{O}}_0$ formula $D[f,a,b,c]$, the theory $\mathsf{INA^r}$ proves*

$$Fun^1_D[f] \wedge Fun^3_D[g] \rightarrow \forall a\forall x\exists! y Rc_D[f,g,a,x,y].$$

Any $f$ can be regarded as a binary relation in the sense of $P^{<\sigma}_{\mathfrak{A}}$ or $P_{\mathfrak{A}}$. If we want to do so, the notation introduced in the following definition increases readability.

**Definition 9** *For all number variables $a,b,f$ and all ordinal variables $\sigma$ we set*

$$a \prec^\sigma_f b \;:=\; App^2_\sigma[f,a,b,0] \qquad \text{and} \qquad a \prec^\infty_f b \;:=\; App^2_\infty[f,a,b,0].$$

The formula $P^\sigma_{\mathfrak{A}}[f,0,0,2]$ implies that $f$ codes a binary function, provided that application is interpreted in the sense of $P^{<\sigma}_{\mathfrak{A}}$, and that the corresponding relation $\prec^\sigma_f$ is progressive. We prove that the build up of the accessible part of $\prec^\sigma_f$ closes at $\sigma$.

**Lemma 10** *The theory $\mathsf{INA^r}$ proves:*

1. $P^\sigma_{\mathfrak{A}}[f,0,0,2] \rightarrow \forall\xi\forall x(P^\xi_{\mathfrak{A}}[f,x,0,1] \rightarrow P^{<\sigma}_{\mathfrak{A}}[f,x,0,1])$.
2. $P^\sigma_{\mathfrak{A}}[f,0,0,2] \rightarrow (\forall x(P_{\mathfrak{A}}[f,x,0,1] \leftrightarrow P^{<\sigma}_{\mathfrak{A}}[f,x,0,1])$.
3. $\neg\exists\eta\exists\xi(P^\eta_{\mathfrak{A}}[f,0,0,3] \wedge P^\xi_{\mathfrak{A}}[f,0,0,4])$.

PROOF Assume $P^\sigma_{\mathfrak{A}}[f,0,0,2]$. Then the operator axiom for $\mathfrak{A}$ implies

$$Fun^2_\sigma[f], \tag{1}$$

$$\forall x((\forall y \prec^\sigma_f x)P^{<\sigma}_{\mathfrak{A}}[f,y,0,1] \rightarrow P^{<\sigma}_{\mathfrak{A}}[f,x,0,1]). \tag{2}$$

In order to establish our first assertion, we show

$$\forall x(P^\xi_{\mathfrak{A}}[f,x,0,1] \rightarrow P^{<\sigma}_{\mathfrak{A}}[f,x,0,1])$$

by ($\Delta_0^{\mathbb{O}}$-$\mathsf{I}_<$). So pick a $\xi$ and an $x$ such that $P_{\mathfrak{A}}^{\xi}[f, x, 0, 1]$. In view of the operator axiom for $\mathfrak{A}$ we then also have

$$Fun_{\xi}^{2}[f], \tag{3}$$

$$(\forall y \prec_f^{\xi} x) P_{\mathfrak{A}}^{<\xi}[f, y, 0, 1]. \tag{4}$$

From (1), (3), (4), Lemma 6, and the induction hypothesis we conclude

$$(\forall y \prec_f^{\sigma} x) P_{\mathfrak{A}}^{<\sigma}[f, y, 0, 1]. \tag{5}$$

Hence (2) and (5) yield $P_{\mathfrak{A}}^{<\sigma}[f, x, 0, 1]$, as required.

The second assertion follows trivially from the first. For the third assertion, assume that there are $\eta$ and $\xi$ such that $P_{\mathfrak{A}}^{\eta}[f, 0, 0, 3]$ and $P_{\mathfrak{A}}^{\xi}[f, 0, 0, 4]$. Then the operator axiom for $\mathfrak{A}$ yields

$$\forall x P_{\mathfrak{A}}^{<\eta}[f, x, 0, 1], \tag{6}$$

$$\exists x \neg P_{\mathfrak{A}}^{<\xi}[f, x, 0, 1] \tag{7}$$

together with $P_{\mathfrak{A}}^{\sigma}[f, 0, 0, 2]$ and $P_{\mathfrak{A}}^{\tau}[f, 0, 0, 2]$ for some $\sigma < \eta$ and $\tau < \xi$. Hence by the second assertion

$$\forall x(P_{\mathfrak{A}}[f, x, 0, 1] \leftrightarrow P_{\mathfrak{A}}^{<\sigma}[f, x, 0, 1]) \ \text{ and } \ \forall x(P_{\mathfrak{A}}[f, x, 0, 1] \leftrightarrow P_{\mathfrak{A}}^{<\tau}[f, x, 0, 1]).$$

From these equivalences we easily conclude that

$$\forall x(P_{\mathfrak{A}}^{<\eta}[f, x, 0, 1] \leftrightarrow P_{\mathfrak{A}}^{<\xi}[f, x, 0, 1]),$$

so that either (6) or (7) has to be wrong, which is a contradiction. □

Remember that in modeling $\mathsf{SUS}$ in $\mathsf{INA}^{\mathsf{r}}$, the $\mathcal{L}^{\star}$ formula $P_{\mathfrak{A}}[a, b, c, 0]$ is intended to take care of application ($ab = c$) within L. The previous considerations set the stage for proving that the following form of functionality is satisfied, which is crucial to this approach.

**Lemma 11** *The theory* $\mathsf{INA}^{\mathsf{r}}$ *proves:*

1. $\forall a \forall b \forall x \forall y (P_{\mathfrak{A}}^{<\sigma}[a, b, x, 0] \wedge P_{\mathfrak{A}}^{<\sigma}[a, b, y, 0] \ \rightarrow \ x = y)$.
2. $\forall a \forall b \forall x \forall y (P_{\mathfrak{A}}[a, b, x, 0] \wedge P_{\mathfrak{A}}[a, b, y, 0] \ \rightarrow \ x = y)$.

Proof We show the first assertion by ($\Delta_0^{\mathbb{O}}$-$\mathsf{I}_<$). Assuming $P_{\mathfrak{A}}^{<\sigma}[a, b, x, 0]$ and $P_{\mathfrak{A}}^{<\sigma}[a, b, y, 0]$, we derive the existence of $\eta, \xi < \sigma$ such that $P_{\mathfrak{A}}^{\eta}[a, b, x, 0]$ and $P_{\mathfrak{A}}^{\xi}[a, b, y, 0]$. Moreover, in view of the operator axiom for $\mathfrak{A}$, the ordinals $\eta$

and $\xi$ have to be identical and therefore $P^{\xi}_{\mathfrak{A}}[a, b, x, 0]$ and $P^{\xi}_{\mathfrak{A}}[a, b, y, 0]$ must hold. Now we proceed by distinction of cases according to the form of $a$.

1. If $Seq_3[a]$ and $(a)_0 = \widehat{\mathsf{s}}$, then our assertion follows from the operator axiom for $\mathfrak{A}$ and the induction hypothesis.

2. If $Seq_4[a]$ and $(a)_0 = \widehat{\mathsf{r}}_{\mathsf{N}}$, then our assertion follows from the operator axiom for $\mathfrak{A}$ and Lemma 8.

3. If $a = \widehat{\mathsf{E}}_1$, then our assertion follows from the operator axiom for $\mathfrak{A}$ and Lemma 10.

4. In all other cases our assertion is trivially satisfied.

This finishes the proof of the first assertion; the second is an immediate consequence of the first. □

The embedding of $\mathsf{SUS}$ into $\mathsf{INA}^{\mathsf{r}}$ first requires to take care of the terms of L. This is achieved by associating to each L term $t$ formulas $\mathbb{V}^{\sigma}_t[u]$ and $\mathbb{V}^{\infty}_t[u]$ of $\mathcal{L}^{\star}$ expressing that $u$ is the value of $t$ under the interpretation of the application in L via the formulas $P^{<\sigma}_{\mathfrak{A}}[\cdot, \cdot, \cdot, 0]$ and $P^{\infty}_{\mathfrak{A}}[\cdot, \cdot, \cdot, 0]$, respectively.

**Definition 12** *For each* L *term $t$ we introduce formulas $\mathbb{V}^{\sigma}_t[u]$ and $\mathbb{V}^{\infty}_t[u]$ of $\mathcal{L}^{\star}$, with $u$ not occurring in $t$, which are inductively defined as follows:*

1. *If $t$ is a variable, then $\mathbb{V}^{\sigma}_t[u]$ and $\mathbb{V}^{\infty}_t[u]$ are the formula $(t = u)$.*

2. *If $t$ is a constant, then $\mathbb{V}^{\sigma}_t[u]$ and $\mathbb{V}^{\infty}_t[u]$ are the formula $(\widehat{t} = u)$.*

3. *If $t$ is the term $(rs)$, then we set*

$$\begin{aligned}\mathbb{V}^{\sigma}_t[u] &:= \exists x \exists y(\mathbb{V}^{\sigma}_r[x] \wedge \mathbb{V}^{\sigma}_s[y] \wedge P^{<\sigma}_{\mathfrak{A}}[x, y, u, 0]),\\ \mathbb{V}^{\infty}_t[u] &:= \exists x \exists y(\mathbb{V}^{\infty}_r[x] \wedge \mathbb{V}^{\infty}_s[y] \wedge P^{\infty}_{\mathfrak{A}}[x, y, u, 0]).\end{aligned}$$

This treatment of the terms of L leads to the following translations of arbitrary L formulas into formulas of $\mathcal{L}^{\star}$.

**Definition 13** *The translations of an* L *formula $A$ into the formulas $[A]^{\sigma}$ and $[A]^{\infty}$ of $\mathcal{L}^{\star}$ are inductively defined as follows:*

1. *For the atomic formulas of* L *we stipulate*

$$\begin{aligned}[\mathsf{N}(t)]^{\sigma} &:= \exists x \mathbb{V}^{\sigma}_t[x], & [\mathsf{N}(t)]^{\infty} &:= \exists x \mathbb{V}^{\infty}_t[x],\\ [t{\downarrow}]^{\sigma} &:= \exists x \mathbb{V}^{\sigma}_t[x], & [t{\downarrow}]^{\infty} &:= \exists x \mathbb{V}^{\infty}_t[x],\\ [s = t]^{\sigma} &:= \exists x(\mathbb{V}^{\sigma}_s[x] \wedge \mathbb{V}^{\sigma}_t[x]), & [s = t]^{\infty} &:= \exists x(\mathbb{V}^{\infty}_s[x] \wedge \mathbb{V}^{\infty}_t[x]).\end{aligned}$$

2. *If $A$ is a formula $\neg B$, then $[A]^{\sigma}$ is $\neg[B]^{\sigma}$ and $[A]^{\infty}$ is $\neg[B]^{\infty}$.*

3. *If $A$ is a formula $(B\,j\,C)$ for $j \in \{\vee, \wedge, \rightarrow\}$, then $[A]^\sigma$ is $([B]^\sigma\,j\,[C]^\sigma)$ and $[A]^\infty$ is $([B]^\infty\,j\,[C]^\infty)$.*

4. *If $A$ is a formula $QxB$ for $Q \in \{\exists, \forall\}$, then $[A]^\sigma$ is $Qx[B]^\sigma$ and $[A]^\infty$ is $Qx[B]^\infty$.*

In Feferman and Jäger [8] the theory $\mathsf{BON}(\mu)$ is embedded into the system $\mathsf{PA}^r_\Omega$ of ordinals over Peano arithmetic. $\mathsf{PA}^r_\Omega$ is a subsystem of $\mathsf{INA}^\mathsf{r}$, and although a slightly different inductive definition has been used, the embedding proof in [8] carries over to $\mathsf{INA}^\mathsf{r}$ without any problems. Moreover, it is easily checked that only the closure properties of admissibles are needed for this interpretation so that also the following relativized embedding is obtained.

**Theorem 14** *For all* L *formulas $A$ we have:*

1. $\mathsf{BON}(\mu) \vdash A \quad \Longrightarrow \quad \mathsf{INA}^\mathsf{r} \vdash [A]^\infty$.

2. $\mathsf{BON}(\mu) \vdash A \quad \Longrightarrow \quad \mathsf{INA}^\mathsf{r} \vdash Ad(\sigma) \rightarrow [A]^\sigma$.

It is still left to show that our translation of L formulas validates the two axioms of the Suslin operator $\mathsf{E}_1$. For doing so, the following lemma is central; it tells us that for any $f$ which codes a binary function in the sense of $P^{<\tau}_\mathfrak{A}$ or $P_\mathfrak{A}$, its accessible part is completely built up at a suitable ordinal stage.

**Lemma 15** *The theory $\mathsf{INA}^\mathsf{r}$ proves:*

1. $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^2_\tau[f] \;\rightarrow\; P^\sigma_\mathfrak{A}[f, 0, 0, 2]$.

2. $Fun^2_\infty[f] \;\rightarrow\; \exists\xi P^\xi_\mathfrak{A}[f, 0, 0, 2]$.

Proof Assume $Ad(\sigma)$, $\tau < \sigma$, and $Fun^2_\tau[f]$. By $\Sigma^{\mathbb{O}}$ reflection at $\sigma$ we thus obtain

$$\forall x((\forall y \prec^\tau_f x) P^{<\sigma}_\mathfrak{A}[f, y, 0, 1] \;\rightarrow\; P^{<\sigma}_\mathfrak{A}[f, x, 0, 1]).$$

Now, because of $Fun^2_\tau[f]$ and $\tau < \sigma$, Lemma 6 yields $Fun^2_\sigma[f]$ as well as the equivalence of the assertions $y \prec^\sigma_f x$ and $y \prec^\tau_f x$ for any $x$ and $y$. Altogether we thus have

$$Fun^2_\sigma[f] \;\wedge\; \forall x((\forall y \prec^\sigma_f x) P^{<\sigma}_\mathfrak{A}[f, y, 0, 1] \;\rightarrow\; P^{<\sigma}_\mathfrak{A}[f, x, 0, 1]).$$

Simple checking of the operator axiom for $\mathfrak{A}$ thus implies $P^\sigma_\mathfrak{A}[f, 0, 0, 2]$.

In addition, given $Fun^2_\infty[f]$, Lemma 7 tells us that there has to be a $\tau$ for which $Fun^2_\tau[f]$, and by the limit axiom ($\mathsf{Lim}$) there exists a $\sigma$ such that $Ad(\sigma)$ and $\tau < \sigma$. Now the second assertion follows from the first. □

**Lemma 16** *The theory* $\mathsf{INA}^{\mathsf{r}}$ *proves:*

1. $Ad(\rho) \wedge Ad(\sigma) \wedge \tau < \sigma < \rho \wedge Fun^2_\tau[f]$
$$\rightarrow (P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0] \vee P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 1]).$$

2. $Fun^2_\infty[f] \leftrightarrow (P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0] \vee P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 1, 0])$.

Proof For the proof of the first assertion assume $Fun^2_\tau[f]$ and let $\rho$ and $\sigma$ be admissibles with $\tau < \sigma < \rho$. By the previous lemma we have $P^{\sigma}_{\mathfrak{A}}[f, 0, 0, 2]$. Now, since $\rho$ is admissible, it is easy to see that there are $\eta$ and $\xi$ such that $\sigma < \eta < \xi < \rho$. Together with Lemma 6 this implies

$$Fun^2_\eta[f] \wedge P^{<\eta}_{\mathfrak{A}}[f, 0, 0, 2].$$

Depending on whether $\forall x P^{<\eta}_{\mathfrak{A}}[a, x, 0, 1]$ or $\exists x \neg P^{<\eta}_{\mathfrak{A}}[a, x, 0, 1]$ the operator axiom for $\mathfrak{A}$ implies $P^{\eta}_{\mathfrak{A}}[f, 0, 0, 3]$ or $P^{\eta}_{\mathfrak{A}}[f, 0, 0, 4]$, respectively. Consequently, we have $P^{<\xi}_{\mathfrak{A}}[f, 0, 0, 3]$ or $P^{<\xi}_{\mathfrak{A}}[f, 0, 0, 4]$, yielding $P^{\xi}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0]$ or $P^{\xi}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 1, 0]$ by a further use of the operator axiom for $\mathfrak{A}$. Our assertion follows immediately.

The direction from left to right of the second assertion is immediate from the first and Lemma 7. For the converse direction, we observe that the assumption $(P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0] \vee P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 1, 0])$ implies $(P^{\sigma}_{\mathfrak{A}}[f, 0, 0, 3] \vee P^{\sigma}_{\mathfrak{A}}[f, 0, 0, 4])$ for some $\sigma$. Thus, $Fun^2_\sigma[f]$ is a consequence of the operator axiom for $\mathfrak{A}$. To see that $Fun^2_\infty[f]$ it only remains to apply Lemma 6. □

**Theorem 17** *The theory* $\mathsf{INA}^{\mathsf{r}}$ *proves*

$$[f \in (\mathsf{N}^2 \mapsto \mathsf{N})]^\infty \leftrightarrow [\mathsf{E}_1 f \in \mathsf{N}]^\infty.$$

Proof According to Definition 12 and Definition 13, $[f \in (\mathsf{N}^2 \mapsto \mathsf{N})]^\infty$ is equivalent to $Tot^2_\infty[f]$ and thus, because of Lemma 11, also to $Fun^2_\infty[f]$. Applying Definition 12 and Definition 13 once more, we also obtain that $[\mathsf{E}_1 f \in \mathsf{N}]^\infty$ is equivalent to $\exists x P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, x, 0]$ which, in view of the operator axiom for $\mathfrak{A}$, is equivalent to

$$Fun^2_\infty[f] \wedge (P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0] \vee P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 1, 0]).$$

The claimed equivalence is thus an immediate consequence of Lemma 16. □

For the formulations and proofs of the following theorems some further auxiliary notations are useful. We set

$$\{f\}^\sigma(u_0, \ldots, u_{n-1}) \simeq v \quad := \quad App^n_\sigma(f, u_0, \ldots, u_{n-1}, v),$$
$$\{f\}^\infty(u_0, \ldots, u_{n-1}) \simeq v \quad := \quad App^n_\infty(f, u_0, \ldots, u_{n-1}, v)$$

and follow the standard conventions of recursion theory when working with expressions like $\{f\}^\sigma(\vec{u})$ and $\{f\}^\infty(\vec{u})$.

**Lemma 18** *The theory* $\mathsf{INA}^{\mathsf{r}}$*proves:*

1. $P_{\mathfrak{A}}^{<\sigma}[f,0,0,2] \wedge Fun_{\sigma}^{1}[g] \wedge \forall x(\{g\}^{\sigma}(x+1) \prec_f^{\sigma} \{g\}^{\sigma}(x)) \rightarrow P_{\mathfrak{A}}^{\sigma}[f,0,0,4].$

2. $Fun_{\infty}^{2}[f] \wedge Fun_{\infty}^{1}[g] \wedge \forall x(\{g\}^{\infty}(x+1) \prec_f^{\infty} \{g\}^{\infty}(x)) \rightarrow P_{\mathfrak{A}}[f,0,0,4].$

Proof To show the first assertion, let us assume that $Fun_{\sigma}^{1}[g]$ and

$$P_{\mathfrak{A}}^{<\sigma}[f,0,0,2], \tag{1}$$

$$\forall x(\{g\}^{\sigma}(x+1) \prec_f^{\sigma} \{g\}^{\sigma}(x)). \tag{2}$$

From (1), the operator axiom for $\mathfrak{A}$, and Lemma 6 we conclude that

$$Fun_{\sigma}^{2}[f] \tag{3}$$

and prove $\forall \xi \forall x \neg P_{\mathfrak{A}}^{\xi}[f,\{g\}^{\sigma}(x),0,1]$ by $(\Delta_0^{\mathbb{O}}\text{-}\mathsf{I}_<)$. Assume, for the contrary, that there exist $\xi$ and $a$ such that $P_{\mathfrak{A}}^{\xi}[f,\{g\}^{\sigma}(a),0,1]$. Then the operator axiom for $\mathfrak{A}$ implies

$$Fun_{\xi}^{2}[f], \tag{4}$$

$$(\forall x \prec_f^{\xi} \{g\}^{\sigma}(a)) P_{\mathfrak{A}}^{<\xi}[f,x,0,1]. \tag{5}$$

In view of Lemma 6 we obtain from (3), (4), and (5) that

$$(\forall x \prec_f^{\sigma} \{g\}^{\sigma}(a)) P_{\mathfrak{A}}^{<\xi}[f,x,0,1]. \tag{6}$$

Hence, because of (2), we also have

$$P_{\mathfrak{A}}^{<\xi}[f,\{g\}^{\sigma}(a+1),0,1].$$

But this contradicts the induction hypothesis, implying that our assumption was wrong, and thus we know that $\forall \xi \forall x \neg P_{\mathfrak{A}}^{\xi}[f,\{g\}^{\sigma}(x),0,1]$, in particular,

$$\forall x \neg P_{\mathfrak{A}}^{<\sigma}[f,\{g\}^{\sigma}(x),0,1]. \tag{7}$$

However, this assertion together with (1) and (3) implies $P_{\mathfrak{A}}^{\sigma}[f,0,0,4]$, as desired in the first assertion.

If we have $Fun_{\infty}^{2}[f]$, $Fun_{\infty}^{1}[g]$, and $\forall x(\{g\}^{\infty}(x+1) \prec_f^{\infty} \{g\}^{\infty}(x))$, all we must do is to apply Lemma 6, Lemma 7, and Lemma 15 in order to derive the existence of a $\sigma$ such that

$$P_{\mathfrak{A}}^{<\sigma}[f,0,0,2] \wedge Fun_{\sigma}^{1}[g] \wedge \forall x(\{g\}^{\sigma}(x+1) \prec_f^{\sigma} \{g\}^{\sigma}(x)).$$

The first assertion now yields $P_{\mathfrak{A}}^{\sigma}[f,0,0,4]$, and hence $P_{\mathfrak{A}}[f,0,0,4]$. □

**Theorem 19** *The theory* $\mathsf{INA}^{\mathsf{r}}$ *proves*

$$[f \in (\mathsf{N}^2 \mapsto \mathsf{N})]^\infty \;\rightarrow\; ([(\exists g \in (\mathsf{N} \mapsto \mathsf{N}))(\forall x \in \mathsf{N})(f(g(x'), g(x)) = 0)]^\infty \;\rightarrow\; [\mathsf{E}_1 f = 0]^\infty).$$

PROOF As in the proof of Theorem 17 one easily verifies that the formulas $[f \in (\mathsf{N}^2 \mapsto \mathsf{N})]^\infty$ and $[(\exists g \in (\mathsf{N} \mapsto \mathsf{N}))(\forall x \in \mathsf{N})(f(g(x'), g(x)) = 0)]^\infty$ imply

$$Fun^2_\infty[f] \;\wedge\; \exists g(Fun^1_\infty[g] \;\wedge\; \forall x(\{g\}^\infty(x+1) \prec^\infty_f \{g\}^\infty(x))).$$

Applying the previous lemma, we continue with $P_{\mathfrak{A}}[f, 0, 0, 4]$. But by the operator axiom for $\mathfrak{A}$ then $P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0]$; so $[\mathsf{E}_1 f = 0]^\infty$ as desired. □

This is the required translation of one direction of ($\mathsf{E}_1$.2). To prove the translation of the converse direction we follow the pattern of Jäger and Strahm [18] and convince ourselves that a suitable amount of relativized recursion theory (for example a form of $S$-$m$-$n$ theorem) can be developed within $\mathsf{INA}^{\mathsf{r}}$ in the sense of Lemma 20 to Lemma 23 below. We omit the proofs of these lemmas which – as we freely admit – are quite tedious. We only remark that everything works since primitive recursion and the non-constructive $\mu$ operator are directly built in into our inductive definition and combinatorial completeness is available due to our codings of $\mathsf{k}$ and $\mathsf{s}$.

Further, if $X$ is a fresh $n$-ary relation symbol and $A[X]$ a formula of $\mathcal{L}_0(X)$, then $A[\{f\}^\sigma]$ and $A[\{f\}^\infty]$ are the $\mathcal{L}^\star$ formulas obtained from $A[X]$ by replacing all subformulas $X(t_0, \ldots, t_{n-1})$ by $\{f\}^\sigma(t_0, \ldots, t_{n-1}) \simeq 0$ and $\{f\}^\infty(t_0, \ldots, t_{n-1}) \simeq 0$, respectively.

**Lemma 20** *Let $X$ be a fresh $m$-ary relation symbol and $A[X, \vec{v}]$ an $\mathcal{L}_0(X)$ formula with at most the variables $\vec{v} = v_0, \ldots, v_{n-1}$ free. Then there exists a number term $t$ such that* $\mathsf{INA}^{\mathsf{r}}$ *proves:*

*(R1)* $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^m_\tau[f] \;\rightarrow\; Fun^{n+1}_\sigma[t]$,

*(R2)* $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^m_\tau[f] \;\rightarrow\; (A[\{f\}^\tau, \vec{v}] \leftrightarrow \{t\}^\sigma(f, \vec{v}) \simeq 0)$.

**Lemma 21** *Let $X$ be a fresh $m$-ary relation symbol and $A[X, u, \vec{v}]$ an $\mathcal{L}_0(X)$ formula with at most the variables $u$ and $\vec{v} = v_0, \ldots, v_{n-1}$ free. Then there exists a binary primitive recursive function $\mathcal{F}$ such that* $\mathsf{INA}^{\mathsf{r}}$ *proves:*

*(R3)* $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^m_\tau[f] \;\rightarrow\; Fun^n_\sigma[\mathcal{F}(f, u)]$,

*(R4)* $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^m_\tau[f] \;\rightarrow\; (A[\{f\}^\tau, u, \vec{v}] \leftrightarrow \{\mathcal{F}(f, u)\}^\sigma(\vec{v}) \simeq 0)$.

**Lemma 22** *For every binary primitive recursive function $\mathcal{F}$ there exists a unary primitive recursive function $\mathcal{G}$ such that* $\mathsf{INA}^{\mathsf{r}}$ *proves*

*(R5)* $Ad(\sigma) \;\rightarrow\; (P^{<\sigma}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(u,v), w, 0] \;\leftrightarrow\; P^{<\sigma}_{\mathfrak{A}}[\mathcal{G}(u), v, w, 0])$.

**Lemma 23** *Let $X$ be a fresh $m$-ary and $Y$ a fresh $n$-ary relation symbol and $B[X,Y,u,v]$ an $\mathcal{L}_0(X,Y)$ formula with at most the variables $u$ and $v$ free. Further assume that* $\mathsf{INA}^{\mathsf{r}}$ *proves*

$$\forall\xi\forall f\forall g(Fun^m_\xi[f] \wedge Fun^n_\xi[g] \;\rightarrow\; \forall x\exists! yB[\{f\}^\xi, \{g\}^\xi, x, y]).$$

*Then there exists a ternary primitive recursive function $\mathcal{H}$ such that* $\mathsf{INA}^{\mathsf{r}}$ *proves:*

*(R6)* $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^m_\tau[f] \wedge Fun^n_\tau[g] \;\rightarrow\; Fun^1_\sigma[\mathcal{H}(f,g,u)]$,

*(R7)* $Ad(\sigma) \wedge \tau < \sigma \wedge Fun^m_\tau[f] \wedge Fun^n_\tau[g] \;\rightarrow$
$\{\mathcal{H}(f,g,u)\}^\sigma(0) \simeq u \;\wedge$
$\{\mathcal{H}(f,g,u)\}^\sigma(v+1) \simeq w \;\leftrightarrow\; B[\{f\}^\tau, \{g\}^\tau, \{\mathcal{H}(f,g,u)\}^\sigma(v), w]$.

After this interlude we come back to the still missing part of the treatment of the Suslin axiom ($\mathsf{E}_1$.2) in $\mathsf{INA}^{\mathsf{r}}$.

**Definition 24** *For all number variables $u, v, f$ and all ordinal variables $\sigma$ we set*

$$u \sqsubseteq^\sigma_f v \;:=\; \begin{cases} \exists x(\exists y > 0)(Seq(x) \wedge lh(x) = y \wedge (x)_0 = u \wedge \\ (x)_{y \dot{-} 1} = v \wedge (\forall z < y \dot{-} 1)((x)_z \prec^\sigma_f (x)_{z+1})). \end{cases}$$

This means that $\sqsubseteq^\sigma_f$ describes the transitive reflexive closure of the relation $\prec^\sigma_f$ introduced in Definition 9. For any codes $f, g$, numbers $u$, and ordinals $\sigma$, we say that $g$ is the restriction of $f$ to $u$ in the sense of $P^{<\sigma}_{\mathfrak{A}}$ if the formula $Rest[\sigma, f, g, u]$ is satisfied,

$$Rest[\sigma, f, g, u] \;:=\; \begin{cases} Fun^2_\sigma[f] \;\wedge\; Fun^2_\sigma[g] \;\wedge \\ \forall x\forall y(x \prec^\sigma_g y \;\leftrightarrow\; (x \prec^\sigma_f y \;\wedge\; y \sqsubseteq^\sigma_f u)). \end{cases}$$

Some important properties of restrictions are summed up in the following lemma.

**Lemma 25** *Let $D[\sigma, \tau, f, g, u]$ be short for the $\mathcal{L}^\star$ formula*

$$Ad(\sigma) \;\wedge\; \tau < \sigma \;\wedge\; Rest[\tau, f, g, u].$$

*Then the theory* $\mathsf{INA}^{\mathsf{r}}$ *proves:*

1. $D[\sigma,\tau,f,g,u] \ \rightarrow\ (\forall x \sqsubseteq^{\tau}_{f} u)(P^{<\sigma}_{\mathfrak{A}}[g,x,0,1] \leftrightarrow P^{<\sigma}_{\mathfrak{A}}[f,x,0,1])$.

2. $Ad(\rho) \wedge \sigma < \rho \wedge D[\sigma,\tau,f,g,u] \wedge (\neg P^{<\sigma}_{\mathfrak{A}}[f,u,0,1] \ \leftrightarrow\ P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1,g,0,0])$.

Proof We assume $D[\sigma,\tau,f,g,u]$ and prove the following two assertions by $\Delta^{\mathbb{O}}_0$ indiction on the ordinals and $\Sigma^{\mathbb{O}}$ reflection at $\sigma$:

$$\forall\xi(\forall x \sqsubseteq^{\tau}_{f} u)(P^{\xi}_{\mathfrak{A}}[f,x,0,1] \rightarrow P^{<\sigma}_{\mathfrak{A}}[g,x,0,1]), \tag{1}$$

$$\forall\xi(\forall x \sqsubseteq^{\tau}_{f} u)(P^{\xi}_{\mathfrak{A}}[g,x,0,1] \rightarrow P^{<\sigma}_{\mathfrak{A}}[f,x,0,1]). \tag{2}$$

The first assertion is a trivial consequence of (1) and (2). According to Lemma 15 we also have $P^{\sigma}_{\mathfrak{A}}[f,0,0,2]$ and $P^{\sigma}_{\mathfrak{A}}[g,0,0,2]$ and know that building up the accessible parts of $f$ and $g$ closes at $\sigma$. By Lemma 10, the properties of our restriction, and the definition of $\sqsubseteq^{\tau}_{f}$ we see that

$$\forall\xi\forall x(P^{\xi}_{\mathfrak{A}}[g,x,0,1] \rightarrow P^{<\sigma}_{\mathfrak{A}}[g,x,0,1], \tag{3}$$

$$(\forall x \not\sqsubseteq^{\tau}_{f} u) P^{<\sigma}_{\mathfrak{A}}[g,x,0,1], \tag{4}$$

$$P^{<\sigma}_{\mathfrak{A}}[f,v,0,1] \wedge w \sqsubseteq^{\tau}_{f} v \ \rightarrow\ P^{<\sigma}_{\mathfrak{A}}[f,w,0,1]. \tag{5}$$

If $\rho$ is an admissible greater than $\sigma$ and $\neg P^{<\sigma}_{\mathfrak{A}}[f,u,0,1]$, then the first assertion and (3) give us $\neg P^{<\xi}_{\mathfrak{A}}[g,u,0,1]$ for any $\xi$ with $\sigma < \xi < \rho$. Of course, also $Fun^{2}_{\xi}[g]$ and $P^{<\xi}_{\mathfrak{A}}[g,0,0,2]$, and so the operator axiom for $\mathfrak{A}$ yields $P^{\xi}_{\mathfrak{A}}[g,0,0,4]$ and $P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1,g,0,0])$.

On the other hand, $P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1,g,0,0])$ implies that there exists an $x$ not in the accessible part of $g$, i.e. $\neg P^{<\sigma}_{\mathfrak{A}}[g,x,0,1]$. Together with (4) and (5) we obtain $\neg P^{<\sigma}_{\mathfrak{A}}[f,u,0,1]$. □

**Lemma 26** *The theory* $\mathsf{INA}^{\mathsf{r}}$ *proves*

$$Fun^{2}_{\infty}[f] \wedge P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1,f,0,0] \ \rightarrow\ \exists g(Fun^{1}_{\infty}[g] \wedge \forall x(\{g\}^{\infty}(x+1) \prec^{\infty}_{f} \{g\}^{\infty}(x))).$$

Proof We assume $Fun^{2}_{\infty}[f]$ and $P_{\mathfrak{A}}[f,0,0,4]$ and conclude from Lemma 7 that there exists an ordinal $\xi$ such that $Fun^{2}_{\xi}[f]$. In a first step select admissibles $\pi$, $\rho$, $\sigma$, and $\tau$ for which

$$\xi < \tau < \sigma < \rho < \pi.$$

Then we pick a fresh binary relation symbol $X$, let $A[X,u,v,w]$ be the $\mathcal{L}_0(X)$ formula

$$\exists x(\exists y > 0)(Seq(x) \wedge lh(x) = y \wedge (x)_0 = u \wedge (x)_{y \dot{-} 1} = w$$
$$\wedge\ (\forall z < y \dot{-} 1) X((x)_{z+1},(x)_z)) \wedge X(v,w)$$

and make use of Lemma 21, applied to this formula, and of Lemma 22 to obtain primitive recursive functions $\mathcal{F}$ and $\mathcal{G}$ such that

$$(1) \qquad Fun^2_\tau[\mathcal{F}(f,u)]$$

for all $u$ and, in addition,

$$(2) \qquad A[\{f\}^\xi, u, v, w] \;\leftrightarrow\; \{\mathcal{F}(f,u)\}^\tau(v,w) \simeq 0,$$

$$(3) \qquad P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(f,u), v, 0] \;\leftrightarrow\; \{\mathcal{G}(f)\}^\rho(u) \simeq v$$

for all $u$, $v$, and $w$. Recalling Lemma 16,

$$(4) \qquad Fun_\rho[\mathcal{G}(f)]$$

follows from (3). A further observation, making use of (2), states

$$(5) \qquad Rest[\tau, f, \mathcal{F}(f,u), u].$$

Clearly, see Lemma 15, the accessible parts of $\{f\}^\xi$ and $\{\mathcal{F}(f,u)\}^\tau$ are completely built up at $\sigma$, i.e.

$$(6) \qquad P^\sigma_{\mathfrak{A}}[f,0,0,2] \;\wedge\; P^\sigma_{\mathfrak{A}}[\mathcal{F}(f,u),0,0,2].$$

By Lemma 10 the assumption $P_{\mathfrak{A}}[f,0,0,4]$ provides us with an $a$ not in the accessible part of $\{f\}^\xi$ and thus $\neg P^{<\sigma}_{\mathfrak{A}}[f,a,0,1]$. Because of (5) we can conclude with the previous lemma that

$$(7) \qquad P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(f,a), 0, 0].$$

A next important preliminary step, before turning to the construction of the required $g$, is to establish the following implication

$$(8) \qquad P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(f,u), 0, 0] \;\to\; (\exists x \prec^\tau_f u) P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(f,x), 0, 0]).$$

Proof of (8): Assume $P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(f,u), 0, 0]$. Then (5) and the previous lemma imply $\neg P^{<\sigma}_{\mathfrak{A}}[f,u,0,1]$. Thus, by (6) and Lemma 10, there exists an $x$ such that $x \prec^\tau_f u$ and $\neg P^{<\sigma}_{\mathfrak{A}}[f,x,0,1]$. It suffices to apply (5) and the previous lemma again to obtain our assertion.

Now pick an additional fresh unary relation symbol $Y$ and consider the $\mathcal{L}_0(X,Y)$ formula

$$B[X,Y,u,v] \;:=\; \begin{cases} (X(v,u) \;\wedge\; Y(v) \;\wedge\; (\forall w < v)(X(w,u) \to \neg Y(w))) \\ \quad \vee\; (\neg \exists w(X(w,u) \wedge Y(w)) \;\wedge\; v = 0). \end{cases}$$

In order to apply Lemma 23, we simply notice that $\mathsf{INA}^\mathsf{r}$ proves

$$\forall \eta \forall h_1 \forall h_2 (\mathit{Fun}^2_\eta[h_1] \wedge \mathit{Fun}^1_\eta[h_2] \;\to\; \forall x \exists! y B[\{h_1\}^\eta, \{h_2\}^\eta, x, y]).$$

Hence we know that there exists a ternary primitive recursive function $\mathcal{H}$ such that, for $g := \mathcal{H}(f, \mathcal{G}(f), a)$, we have $\mathit{Fun}^1_\pi[g]$ because of (1) and (4) as well as

$$\{g\}^\pi(0) \simeq a,$$
$$\{g\}^\pi(v+1) \simeq w \;\leftrightarrow\; B[\{f\}^\tau, \{\mathcal{G}(f)\}^\rho, \{g\}^\pi(v), w].$$

Our scenario has been set in a way that with (7) and (8) simple $\Delta^{\mathbb{O}}_0$ induction on the natural numbers proves

$$\forall x (P^{<\rho}_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, \mathcal{F}(f, \{g\}^\pi(x)), 0, 0] \;\wedge\; \{g\}^\pi(x+1) \prec^\tau_f \{g\}^\pi(x)).$$

By Lemma 6 this implies $\mathit{Fun}^1_\infty[g]$ and $\forall x(\{g\}^\infty(x+1) \prec^\infty_f \{g\}^\infty(x))$ as desired and finishes the proof or our theorem. $\square$

**Theorem 27** *The theory* $\mathsf{INA}^\mathsf{r}$ *proves*

$$[f \in (\mathsf{N}^2 \mapsto \mathsf{N})]^\infty \;\to$$
$$([\mathsf{E}_1 f = 0]^\infty \;\to\; [(\exists g \in (\mathsf{N} \mapsto \mathsf{N}))(\forall x \in \mathsf{N})(f(g(x'), g(x)) = 0)]^\infty).$$

PROOF As in the proof of Theorem 19 we observe that $[f \in (\mathsf{N}^2 \mapsto \mathsf{N})]^\infty$ and $[\mathsf{E}_1 f = 0]^\infty$ imply $\mathit{Fun}^1_\infty[f]$ and $P_{\mathfrak{A}}[\widehat{\mathsf{E}}_1, f, 0, 0]$. According to the previous lemma we therefore have $\exists g(\mathit{Fun}^1_\infty[g] \wedge \forall x(\{g\}^\infty(x+1) \prec^\infty_f \{g\}^\infty(x)))$. Clearly, this yields $[(\exists g \in (\mathsf{N} \mapsto \mathsf{N}))(\forall x \in \mathsf{N})(f(g(x'), g(x)) = 0)]^\infty$. $\square$

Theorem 19 and Theorem 27 provide the translations of both directions of axiom ($\mathsf{E}_1$.2) of the Suslin operator $\mathsf{E}_1$. Summing up, together with the earlier Theorem 14 we have shown that all axioms of SUS can be modeled in $\mathsf{INA}^\mathsf{r}$.

# 5 Proof theoretic equivalences

We end this article by summing up what we can conclude about the proof-theoretic strengths of the applicative theories $\mathsf{SUS} + (\mathsf{S}\text{-}\mathsf{I}_\mathsf{N})$, $\mathsf{SUS} + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$, and $\mathsf{SUS} + (\mathrm{L}\text{-}\mathsf{I}_\mathsf{N})$. Their lower bounds are provided by Theorem 2, their upper bounds can be computed via $\mathsf{INA}^\mathsf{r}$ and two of its extensions and by what we know about the bounds of those.

**Theorem 28** *For all* L *formulas* $A$ *we have:*

1. $\mathsf{SUS} + (\mathsf{S}\text{-}\mathsf{I_N}) \vdash A \quad \Longrightarrow \quad \mathsf{INA^r} \vdash [A]^\infty$.

2. $\mathsf{SUS} + (\mathsf{N}\text{-}\mathsf{I_N}) \vdash A \quad \Longrightarrow \quad \mathsf{INA^r} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I_N}) \vdash [A]^\infty$.

3. $\mathsf{SUS} + (\mathrm{L}\text{-}\mathsf{I_N}) \vdash A \quad \Longrightarrow \quad \mathsf{INA^w} \vdash [A]^\infty$.

Proof For these embedding results it is sufficient to check that the translations of all axioms of $\mathsf{SUS} + (\mathsf{S}\text{-}\mathsf{I_N})$, $\mathsf{SUS} + (\mathsf{N}\text{-}\mathsf{I_N})$, and $\mathsf{SUS} + (\mathrm{L}\text{-}\mathsf{I_N})$ are provable in $\mathsf{INA^r}$, $\mathsf{INA^r} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I_N})$, and $\mathsf{INA^w}$, respectively. Of course, Theorem 14, Theorem 17, Theorem 19, and Theorem 27 tell us already that the translations of all axioms of $\mathsf{SUS}$ are provable in $\mathsf{INA^r}$. Therefore, it only remains to have a look at the respective induction principles.

Let us begin with $(\mathsf{S}\text{-}\mathsf{I_N})$. It is easy to see that within $\mathsf{INA^r}$ the formula $[f \in \mathcal{P}(\mathsf{N})]^\infty$ is equivalent to $\forall x(P_{\mathfrak{A}}[f,x,0,0] \vee P_{\mathfrak{A}}[f,x,1,0])$. By the same argument as in the proof of Lemma 7 we can thus conclude that there exists a $\sigma$ such that $Fun^1_\sigma[f]$ and $\forall x(P^{<\sigma}_{\mathfrak{A}}[f,x,0,0] \vee P^{<\sigma}_{\mathfrak{A}}[f,x,1,0])$. Moreover, the formula

$$[f0 = 0 \;\wedge\; (\forall x \in \mathsf{N})(fx = 0 \to f(x') = 0)]^\infty$$

can be rewritten as

$$P^{<\sigma}_{\mathfrak{A}}[f,0,0,0] \;\wedge\; \forall x(P^{<\sigma}_{\mathfrak{A}}[f,x,0,0] \to P^{<\sigma}_{\mathfrak{A}}[f,x+1,0,0]).$$

Hence $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{I_N})$, which is available in $\mathsf{INA^r}$, yields $\forall x P^{<\sigma}_{\mathfrak{A}}[f,x,0,0]$ from which, see Lemma 6, $[(\forall x \in \mathsf{N})(fx = 0)]^\infty$ is an immediate consequence. So we have established that the translation of $(\mathsf{S}\text{-}\mathsf{I_N})$ is provable in $\mathsf{INA^r}$.

The translation of the premise of $(\mathsf{N}\text{-}\mathsf{I_N})$ is the formula

$$[f0 \in \mathsf{N} \;\wedge\; (\forall x \in \mathsf{N})(fx \in \mathsf{N} \to f(x') \in \mathsf{N})]^\infty$$

which is equivalent in $\mathsf{INA^r}$ to

$$\exists y P_{\mathfrak{A}}[f,0,y,0] \;\wedge\; \forall x(\exists y P_{\mathfrak{A}}[f,x,y,0] \;\to\; \exists y P_{\mathfrak{A}}[f,x+1,y,0]).$$

Since $\exists y P_{\mathfrak{A}}[f,x,y,0]$ is a $\Sigma^{\mathbb{O}}$ formula, we can now apply $(\Sigma^{\mathbb{O}}\text{-}\mathsf{I_N})$ in order to conclude $\forall x \exists y P_{\mathfrak{A}}[f,x,y,0]$, i.e. $[(\forall x \in \mathsf{N})(fx \in \mathsf{N})]^\infty$. Therefore we know that the translation of $(\mathsf{N}\text{-}\mathsf{I_N})$ is provable in $\mathsf{INA^r} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I_N})$.

Ultimately, the translation of any instance of $(\mathrm{L}\text{-}\mathsf{I_N})$ is clearly an instance of $(\mathcal{L}^\star\text{-}\mathsf{I_N})$ and therefore provable in $\mathsf{INA^w}$. □

Before presenting the central result of this paper, we want to restate an important result about the relationship between systems of second order arithmetic and theories for admissible sets. In the following theorem $\Delta^1_2$-$\mathsf{CA}$ is the usual system of second order arithmetic with the $\Delta^1_2$ comprehension axioms, and $\Delta^1_2$-$\mathsf{CR}$ is the subsystem of $\Delta^1_2$-$\mathsf{CA}$ with the $\Delta^1_2$ comprehension axioms replaced by $\Delta^1_2$ comprehension rules.

**Theorem 29** *We have the following proof-theoretic equivalences:*

1. $\Pi^1_1\text{-}\mathsf{CA}_0 \equiv \Delta^1_2\text{-}\mathsf{CA}_0 \equiv \mathsf{KPi}^{\mathsf{r}}$.
2. $\Pi^1_1\text{-}\mathsf{CA}_{<\omega^\omega} \equiv \Delta^1_2\text{-}\mathsf{CR} \equiv \mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{I_N})$.
3. $\Pi^1_1\text{-}\mathsf{CA}_{<\varepsilon_0} \equiv \Delta^1_2\text{-}\mathsf{CA} \equiv \mathsf{KPi}^{\mathsf{w}}$.

For the proofs of the first and third assertion of this theorem consult, e.g., Buchholz et al. [3] and Jäger [10, 11]. The second assertion is obtained by making use of similar techniques.

**Corollary 30** *We have the following proof-theoretic equivalences:*

1. $\mathsf{SUS} + (\mathsf{S}\text{-}\mathsf{I_N}) \equiv \Pi^1_1\text{-}\mathsf{CA}_0 \equiv \Delta^1_2\text{-}\mathsf{CA}_0 \equiv \mathsf{INA}^{\mathsf{r}} \equiv \mathsf{KPi}^{\mathsf{r}}$.
2. $\mathsf{SUS} + (\mathsf{N}\text{-}\mathsf{I_N}) \equiv \Pi^1_1\text{-}\mathsf{CA}_{<\omega^\omega} \equiv \Delta^1_2\text{-}\mathsf{CR} \equiv \mathsf{INA}^{\mathsf{r}} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I_N}) \equiv \mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{I_N})$.
3. $\mathsf{SUS} + (\mathrm{L}\text{-}\mathsf{I_N}) \equiv \Pi^1_1\text{-}\mathsf{CA}_{<\varepsilon_0} \equiv \Delta^1_2\text{-}\mathsf{CA} \equiv \mathsf{INA}^{\mathsf{w}} \equiv \mathsf{KPi}^{\mathsf{w}}$.

Proof According to Theorem 2, Theorem 3, and Theorem 28 we have the following the inclusions:

$$\Pi^1_1\text{-}\mathsf{CA}_0 \ \subseteq\ \mathsf{SUS} + (\mathsf{S}\text{-}\mathsf{I_N}) \ \subseteq\ \mathsf{INA}^{\mathsf{r}} \ \subseteq\ \mathsf{KPi}^{\mathsf{r}},$$
$$\Pi^1_1\text{-}\mathsf{CA}_{<\omega^\omega} \ \subseteq\ \mathsf{SUS} + (\mathsf{N}\text{-}\mathsf{I_N}) \ \subseteq\ \mathsf{INA}^{\mathsf{r}} + (\Sigma^{\mathbb{O}}\text{-}\mathsf{I_N}) \ \subseteq\ \mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{I_N}),$$
$$\Pi^1_1\text{-}\mathsf{CA}_{<\varepsilon_0} \ \subseteq\ \mathsf{SUS} + (\mathrm{L}\text{-}\mathsf{I_N}) \ \subseteq\ \mathsf{INA}^{\mathsf{w}} \ \subseteq\ \mathsf{KPi}^{\mathsf{w}}.$$

Therefore the claimed proof-theoretic equivalences are immediate from the previous theorem. □

This finishes the proof-theoretic analysis of the Suslin operator $\mathsf{E}_1$ in the context of applicative theories. A next and very big step is to consider the partial version $\mathsf{E}_1^{\#}$. Recursion-theoretic results indicate that a significant increase in strength is to be expected.

# References


[1] M. J. Beeson, *Foundations of Constructive Mathematics: Metamathematical Studies*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 1985.

[2] ———, *Proving programs and programming proofs*, Logic, Methodology, and Philosophy of Science VII (R. Barcan Marcus, G. J. W. Dorn, and P. Weingartner, eds.), Studies in Logic and the Foundations of Mathematics, vol. 114, North-Holland, 1986, pp. 51–82.

[3] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, Lecture Notes in Mathematics, vol. 897, Springer, 1981.

[4] S. Feferman, *A language and axioms for explicit mathematics*, Algebra and Logic (J.N. Crossley, ed.), Lecture Notes in Mathematics, vol. 450, Springer, 1975, pp. 87–139.

[5] ———, *Recursion theory and set theory: a marriage of convenience*, Generalized Recursion Theory II, Oslo 1977 (J.E. Fenstad, R.O. Gandy, and G.E. Sacks, eds.), Studies in Logic and the Foundations of Mathematics, vol. 94, North-Holland, 1978, pp. 55–98.

[6] ———, *Constructive theories of functions and classes*, Logic Colloquium '78 (M. Boffa, D. van Dalen, and K. McAloon, eds.), Studies in Logic and the Foundations of Mathematics, vol. 97, North-Holland, 1979, pp. 159–224.

[7] ———, *Definedness*, Erkenntnis **43** (1995), 295–320.

[8] S. Feferman and G. Jäger, *Systems of explicit mathematics with non-constructive $\mu$-operator. Part I*, Annals of Pure and Applied Logic **65** (1993), no. 3, 243–263.

[9] P.G. Hinman, *Recursion-Theoretic Hierarchies*, Perspectives in Mathematical Logic, vol. 9, Springer, 1978.

[10] G. Jäger, *Die konstruktible Hierarchie als Hilfsmittel zur beweistheoretischen Untersuchung von Teilsystemen der Mengenlehre und Analysis*, Ph.D. thesis, Mathematisches Institut, Univeristät München, 1979.

[11] ———, *Iterating admissibility in proof theory*, Logic Colloquium '81. Proceedings of the Herbrand Symposion (J. Stern, ed.), Studies in Logic and the Foundations of Mathematics, vol. 107, North-Holland, 1982, pp. 137–146.

[12] ———, *Theories for Admissible Sets: A Unifying Approach to Proof Theory*, Studies in Proof Theory, Lecture Notes, vol. 2, Bibliopolis, 1986.

[13] ———, *Fixed points in Peano arithmetic with ordinals*, Annals of Pure and Applied Logic **60** (1993), no. 2, 119–132.

[14] ———, *First order theories for nonmonotone inductive definitions: recursively inaccessible and Mahlo*, The Journal of Symbolic Logic **66** (2001), no. 3, 1073–1089.

[15] G. Jäger, R. Kahle, and T. Strahm, *On applicative theories*, Logic and Foundations of Mathematics (A. Cantini, E. Casari, and P. Minari, eds.), Kluwer, 1999, pp. 83–92.

[16] G. Jäger and W. Pohlers, *Eine beweistheoretische Untersuchung von* $(\Delta^1_2\text{-}\mathsf{CA}) + (\mathsf{BI})$ *und verwandter Systeme*, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse **1** (1982), 1–28.

[17] G. Jäger and T. Strahm, *Upper bounds for metapredicative Mahlo in explicit mathematics and admissible set theory*, The Journal of Symbolic Logic **66** (2001), no. 2, 935–958.

[18] ———, *The proof-theoretic strength of the Suslin operator in applicative theories*, Reflections on the Foundations of Mathematics: Essays in Honor of Solomon Feferman (W. Sieg, R. Sommer, and C. Talcott, eds.), Lecture Notes in Logic, vol. 15, Association for Symbolic Logic, 2002, pp. 270–292.

[19] D. Probst, *Suslin and* $\mathsf{FID}[POS, QF]$, 2008, Notes, Bern.

## Addresses

Gerhard Jäger (jaeger@iam.unibe.ch)
Institut für Informatik und angewandte Mathematik, Universität Bern, Neubrückstrasse 10, CH-3012 Bern, Switzerland

Dieter Probst (Dieter.Probst@mathematik.uni-muenchen.de)
Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstrasse 39, D-80333 München, Germany