Syntactic Cut-Elimination for a Fragment of the Modal Mu-Calculus

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Abstract

For some modal fixed point logics, there are deductive systems that enjoy syntactic cut-elimination. An early example is the system in Pliuskevicius [15] for LTL. More recent examples are the systems by the authors of this paper for the logic of common knowledge [5] and by Hill and Poggiolesi for PDL [8], which are based on a form of deep inference. These logics can be seen as fragments of the modal mu-calculus. Here we are interested in how far this approach can be pushed in general. To this end, we introduce a nested sequent system with syntactic cut-elimination which is incomplete for the modal mu-calculus, but complete with respect to a restricted language that allows only fixed points of a certain form. This restricted language includes the logic of common knowledge and PDL. There is also a traditional sequent system for the modal mu-calculus by Jäger et al. [9], without a syntactic cut-elimination procedure. We embed that system into ours and vice versa, thus establishing cut-elimination also for the shallow system, when only the restricted language is considered.

1 Introduction

Modal fixed point logics occur in many different forms. For instance, we have temporal logics with an *always operator*, epistemic logics with a *common knowledge operator*, program logics with an *iteration operator*, and the propositional modal μ-calculus with *fixed points for arbitrary positive formulas.*

While the model-theoretic side of modal fixed point logics is very well investigated [3, 23], not much is known about their proof-theoretic aspects. However, it is possible to obtain syntactic cut-elimination results for logics of this kind through infinitary calculi that allow for deep applications of inference rules.

A deductive system is called infinitary if it includes inference rules with infinitely many premises. In a temporal logic, for instance, we may consider a rule like

$$\frac{A \quad \text{next } A \quad \text{next next } A \quad \cdots}{\text{always } A}$$

that has for any natural number i a premise consisting of an i-fold nesting of *next* operators applied to A.

We say that an inference rule applies deeply if it does not only apply to an outermost connective but also, in a certain sense, deeply inside formulas. Again for temporal logic, we may consider a conjunction rule that applies inside an arbitrary nesting of *next* operators. An instance of a rule of this kind is

$$\frac{\text{next next } A \qquad \text{next next } B}{\text{next next } (A \wedge B)} \quad .$$

Pliuskevicius [15] introduced an infinitary deep system for linear time temporal logic that features syntactic cut-elimination. Note that the case of linear time logic is particularly simple since it is enough to consider nestings of *next* operators only. The case of syntactic cut-elimination for modal fixed point logics over arbitrary Kripke structures turned out to require more machinery. This case requires systems with some form of deep inference such as nested sequents [4, 10] also called tree-hypersequents [16]. Brünnler and Studer [5, 6] use them to present a calculus with syntactic cut-elimination for the logic of common knowledge. Hill and Poggiolesi [8] use deep inference to establish a cut-elimination result for propositional dynamic logic.

Each of the cut-elimination results mentioned above applies only to one particular logic. In the present paper we try to identify the common core of those results and establish a general cut-elimination theorem that subsumes the previous ones. To do so, we base our results on a fairly small fragment of the modal μ-calculus that is however large enough to embed common knowledge operators as well as iteration operators from PDL. We then show that we necessarily have to restrict ourselves to a fragment: our system is incomplete for larger fragments. In particular, we cannot treat the whole propositional modal μ-calculus, not even the one-variable fragment.

Another question is the relationship of traditional shallow systems for modal fixed point logics and their deep counterparts. Brünnler and Studer [5] examine this relationship for the logic of common knowledge. They present embeddings of a shallow system into a deep system and vice versa. Again, so far this relationship has not been studied from a general perspective. Now we present embeddings of a shallow system for our fragment into the deep system and vice versa. Note that the direction from shallow into deep is straightforward in the case of common knowledge but requires much more work in our general setting. Moreover, the direction from deep into shallow as presented in [5] contains a mistake (Lemma 12 does not hold) that is fixed in the present paper.

Kozen [11] introduced the propositional modal μ-calculus together with a Hilbert-style deductive system, for which Walukiewicz [24] was able to establish completeness. This system includes an induction rule to guarantee that a formula of the form $\mu X.A$ denotes a least fixed point. This implies that also some variant of a cut rule has to be present in the system in order to make use of the power of the induction rule. Hence cut-elimination is not possible for that system.

The situation is different if we replace the induction rule by an infinitary rule that introduces greatest fixed points. Such a rule has been proposed by Kozen [12] based on the finite model property of the μ-calculus. Jäger et al. [9] later showed by *semantic* means that the cut rule is admissible in the infinitary system. In the present paper, we study *syntactic* cut-elimination for that system. However, we can only deal with a fragment where least fixed point variables do not occur in the scope of $\Box$ operators (and dually greatest fixed point variables not in the scope of $\Diamond$). Fontaine [7] showed that this syntactic restriction of least fixed point formulas characterizes the continuous fragment of the μ-calculus.

There are also syntactic cut-elimination results for the fixed point logics $\mu\mathsf{MALL}^{=}$ by Baelde and Miller [1] and Linc^{-} by Tiu and Momigliano [21]. Since these systems are based on induction rules (and thus are finitary), cut-elimination results in the loss of the subformula property. Still the cut-free proofs do have some useful structure. In a different line of research, Mints and Studer [13] were recently able to provide a syntactic transformation of Hilbert-style proofs from Kozen's original system to cut-free proofs in the infinitary system.

Our paper is organized as follows. We first recall the infinitary shallow system G_μ for the propositional modal μ-calculus introduced by Jäger et al. in [9] and show the obstacles to cut-elimination. In Section 3 we introduce our deep system D_μ for modal fixed point logic and establish syntactic cut-elimination for that system. We observe that D_μ is not complete for the modal μ-calculus. Thus we define a fragment of the μ-calculus in which we can embed PDL and the logic of common knowledge, and for which D_μ is complete. This is established in Section 4 by embedding System G_μ restricted to our fragment into System D_μ. We also present the reverse embedding of D_μ into G_μ.

Combining the embedding of G_μ into D_μ, cut-elimination of D_μ, and the embedding of D_μ into G_μ provides syntactic cut-elimination for G_μ with respect to our fragment.

2 The Shallow System

Formulas. We start with a countable set $\mathcal{P}$ of atomic propositions and one variable X. An operator form is given by the following grammar

$$O ::= X \mid p \mid \bar{p} \mid \top \mid \bot \mid (O \wedge O) \mid (O \vee O) \mid \Box O \mid \Diamond O \mid \mu X.O \mid \nu X.O$$

where $p \in \mathcal{P}$. In case there is no danger of confusion, we will omit parentheses. A *formula* is an operator form in which every occurrence of X is in the scope of a μ- or a ν-operator. That is a formula does not contain free occurrences of the variable X. Note that in other work on the μ-calculus formulas are sometimes defined such that they may also contain free variables.

As usual, we define the negation $\bar{A}$ of operator forms and formulas A inductively as follows:

1. $\bar{\bar{p}} := p$, $\bar{\top} := \bot$, $\bar{\bot} := \top$, and $\bar{X} := X$,
2. $\overline{A \wedge B} := \bar{A} \vee \bar{B}$ and $\overline{A \vee B} := \bar{A} \wedge \bar{B}$,
3. $\overline{\Box A} := \Diamond \bar{A}$ and $\overline{\Diamond A} := \Box \bar{A}$
4. $\overline{\mu X.A} := \nu X.\bar{A}$ and $\overline{\nu X.A} := \mu X.\bar{A}$.

Implication is defined as usual by $A \to B := \bar{A} \vee B$.

If B is an operator form and A a formula, then $B(A)$ denotes the formula which is given by substituting each free occurrence of X in B with A. This allows us to syntactically define finite approximations of fixed points. We set for $n \geq 0$:

$$\begin{aligned} \nu^0 X.B &:= \top & \qquad \nu^{n+1} X.B &:= B(\nu^n X.B) \\ \mu^0 X.B &:= \bot & \qquad \mu^{n+1} X.B &:= B(\mu^n X.B). \end{aligned}$$

We consider a language with only one variable since this simplifies the presentation and the proofs (see, for instance, Lemma 27). We remark that Theorem 6 holds for an arbitrary number of variables and we believe that our approach can be extended to the general case. Note, however, that the variable hierarchy of the μ-calculus is strict [2].

Later we have to restrict ourselves to a fragment that includes, for instance, PDL and the logic of common knowledge. These logics can easily be embedded using one variable only, see Remark 21.

Definition 1 (rank)**.** We now define the *rank* $\mathsf{rk}(A)$ of an operator form A inductively by:

1. $\mathsf{rk}(\top) := \mathsf{rk}(\bot) := \mathsf{rk}(p) := \mathsf{rk}(\bar{p}) := \mathsf{rk}(X) := 1$ for $p \in \mathcal{P}$

2. $\mathsf{rk}(A \wedge B) := \mathsf{rk}(A \vee B) := \max(\mathsf{rk}(A), \mathsf{rk}(B)) + 1$

3. $\mathsf{rk}(\Box A) := \mathsf{rk}(\Diamond A) := \mathsf{rk}(A) + 1$

4. $\mathsf{rk}(\mu X.A) := \mathsf{rk}(\nu X.A) := \mathsf{rk}(A) + \omega$

Lemma 2. *Let A be a formula, let B be an operator form and let $n < \omega$. We have:*

1. $\mathsf{rk}(A) = \mathsf{rk}(\bar{A})$.

2. $\mathsf{rk}(A) < \omega^2$.

3. *If* $\mathsf{rk}(B) \leq \mathsf{rk}(A)$ *then* $\mathsf{rk}(B(A)) < \mathsf{rk}(A) + \omega$.

4. $\mathsf{rk}(\nu^n X.A) < \mathsf{rk}(\nu X.A)$.

Proof. The first two statements are immediate. The third one follows from a straightforward induction on B. The fourth statement we now prove by induction on n. The case $n = 0$ is trivial. So, given $n > 0$ and A, we need to show that

$$\mathsf{rk}(\nu^n X.A) = \mathsf{rk}(A(\nu^{n-1} X.A)) < \mathsf{rk}(A) + \omega \quad .$$

By 3. we have an m_1 such that

$$\mathsf{rk}(A(\nu^{n-1} X.A)) < \mathsf{rk}(\nu^{n-1} X.A) + m_1 \quad .$$

By induction hypothesis we have an m_2 such that

$$\mathsf{rk}(\nu^{n-1} X.A) < \mathsf{rk}(A) + m_2 \quad .$$

So we have

$$\mathsf{rk}(A(\nu^{n-1} X.A)) < \mathsf{rk}(A) + m_1 + m_2 < \mathsf{rk}(A) + \omega \quad .$$

□

A *sequent* is a finite multiset of formulas. We employ $\Gamma, \Delta, ...$ to denote sequents. If Δ is a sequent, then $\Diamond\Delta$ is obtained from Δ by prefixing the connective $\Diamond$ to each formula occurrence in Δ.

$$\mathsf{id}\,\frac{}{\Gamma, p, \bar{p}} \qquad \top\,\frac{}{\Gamma, \top} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \Box\,\frac{\Gamma, A}{\Diamond\Gamma, \Box A, \Delta}$$

$$\mu\,\frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \qquad \nu\,\frac{\Gamma, \nu^k X.B \quad \text{for all } k \geq 0}{\Gamma, \nu X.B}$$

Figure 1: The Shallow System G_μ

$$\mathsf{cut}\,\frac{\Gamma, A \quad \Gamma, \bar{A}}{\Gamma} \qquad \mathsf{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \mathsf{ctr}\,\frac{\Gamma, A, A}{\Gamma, A}$$

Figure 2: Admissible Rules

Inference rules. In an instance of the inference rule ρ

$$\rho\,\frac{\Gamma_1 \quad \Gamma_2 \quad \ldots}{\Delta}$$

the sequents $\Gamma_1, \Gamma_2, \ldots$ are its *premises* and the sequent Δ is its *conclusion*. An *axiom* is a rule without premises. We will not distinguish between an axiom and its conclusion. A *system*, denoted by $\mathcal{S}$, is a set of rules. The system G_μ is defined in Figure 1. Notice that the ν-rule has infinitely many premises.

The cut rule, weakening, and contraction are shown in Figure 2. In an instance of cut, the formula A is called the *cut formula*.

Derivations and proofs. In the following, a *tree* is a tree in the graph-theoretic sense, and may be infinite. A tree is *well-founded* if it does not have an infinite path. A *derivation* in a system $\mathcal{S}$ is a directed, rooted, ordered and well-founded tree whose nodes are labeled with sequents and which is built according to the inference rules from $\mathcal{S}$. Derivations are visualised as upward-growing trees, so the root is at the bottom. The sequent at the root is the *conclusion* and the sequents at the leaves are the *premises* of the derivation. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ where all leaves are axioms. We write $\mathcal{S} \vdash \Gamma$ if there is a proof of Γ in system $\mathcal{S}$. Notice that derivations here are in general infinitely branching, thus their depth can be infinite even though each branch has to be finite.

The derivability relation. The *cut-rank* of an instance of cut as shown in

Figure 2 is the rank of its *cut formula* A. The *μ-rank* of an instance of the μ-rule as shown in Figure 1 is the rank of the formula $A(\mu X.A)$. The *cut-rank* of a derivation is the supremum of the cut-ranks of its instances of cut, and similarly for the μ-rank. For a system $\mathcal{S}$ and ordinals α, β, γ and a sequent Γ we write $\mathcal{S} \vdash^{\alpha}_{\beta,\gamma} \Gamma$ to say that there is a proof of Γ in system $\mathcal{S} + \mathsf{cut}$ with depth bounded by α, cut-rank strictly smaller than β, and μ-rank strictly smaller than γ. We write $\mathcal{S} \vdash^{\alpha}_{\beta} \Gamma$ if there exists a γ such that $\mathcal{S} \vdash^{\alpha}_{\beta,\gamma} \Gamma$ and we write $\mathcal{S} \vdash_{\beta} \Gamma$ if there exists an α such that $\mathcal{S} \vdash^{\alpha}_{\beta} \Gamma$. Thus, $\mathsf{S} \vdash_{0} \Gamma$ means that Γ can be derived in S alone—which is cut-free.

Admissibility and invertibility. An inference rule ρ is *depth- and cut-rank-preserving admissible* or, for short, *perfectly admissible* for a system S if for each instance of ρ with premises $\Gamma_1, \Gamma_2, \ldots$ and conclusion Δ, whenever $\mathsf{S} \vdash^{\alpha}_{\beta} \Gamma_i$ for each premise Γ_i, then $\mathsf{S} \vdash^{\alpha}_{\beta} \Delta$. For each rule ρ, there is its *inverse* $\overline{\rho}$, which has the conclusion of ρ as its only premise and admits each premise of ρ as its conclusion, i.e. both

$$\overline{\wedge}\,\frac{\Gamma, A \wedge B}{\Gamma, A} \qquad \text{and} \qquad \overline{\wedge}\,\frac{\Gamma, A \wedge B}{\Gamma, B}$$

are instances of the inverse of $\wedge$ and if ρ is a rule without premises, then it does not have an inverse. An inference rule ρ is *perfectly invertible* for a system S if $\overline{\rho}$ is perfectly admissible for S.

We omit the proof of the following lemma, which is standard [19, 22].

Lemma 3. *1. The rules weakening and contraction are perfectly admissible for G_μ.*

2. All rules of G_μ except $\Box$ are perfectly invertible for G_μ.

Definition 4 (Kripke structure)**.** A Kripke structure $\mathsf{K} = (S, R, \pi)$ is a triple where S is a non–empty set of states, $R \subseteq S \times S$ is the accessibility relation, and $\pi : \mathcal{P} \cup \{X\} \to \mathrm{Pow}(S)$ is the valuation function. Furthermore, given a set $T \subseteq S$, we define the Kripke structure $\mathsf{K}[X := T]$ as the triple (S, R, π'), where $\pi'(X) = T$, and $\pi'(p) = \pi(p)$ for all $p \in \mathcal{P}$.

Assume we are given a Kripke structure $\mathsf{K} = (S, R, \pi)$ and a formula A. We define the set of states $\|A\|_{\mathsf{K}}$ of S at which A holds by induction on the structure of A.

Definition 5 (denotation)**.** Let $\mathsf{K} = (S, R, \pi)$ be a Kripke structure. For every operator form and every formula A we define the set $\|A\|_{\mathsf{K}} \subseteq S$ inductively as follows:

$$\begin{aligned}
&\|p\|_{\mathsf{K}} := \pi(p) \text{ for all } p \in \mathcal{P} \cup \{X\},\\
&\|\bar{p}\|_{\mathsf{K}} := S \setminus \pi(p) \text{ for all } p \in \mathcal{P},\\
&\|\top\|_{\mathsf{K}} := S \quad \text{and} \quad \|\bot\|_{\mathsf{K}} := \varnothing,\\
&\|B \wedge C\|_{\mathsf{K}} := \|B\|_{\mathsf{K}} \cap \|C\|_{\mathsf{K}}, \quad \|B \vee C\|_{\mathsf{K}} := \|B\|_{\mathsf{K}} \cup \|C\|_{\mathsf{K}},\\
&\|\Box B\|_{\mathsf{K}} := \{w \in S : v \in \|B\|_{\mathsf{K}} \text{ for all } v \text{ such that } wRv\},\\
&\|\Diamond B\|_{\mathsf{K}} := \{w \in S : v \in \|B\|_{\mathsf{K}} \text{ for some } v \text{ such that } wRv\}.
\end{aligned}$$

For all formulas $\mu X.A$ and $\nu X.A$ we define

$$\begin{aligned}
\|\mu X.A\|_{\mathsf{K}} &:= \bigcap\{T \subseteq S : T \supseteq F_{A,\mathsf{K}}(T)\} \text{ and}\\
\|\nu X.A\|_{\mathsf{K}} &:= \bigcup\{T \subseteq S : T \subseteq F_{A,\mathsf{K}}(T)\}
\end{aligned}$$

where $F_{A,\mathsf{K}}$ is the operator on $\mathrm{Pow}(S)$ given by $F_{A,\mathsf{K}}(T) := \|A\|_{\mathsf{K}[X:=T]}$ for every subset T of S.

A formula A is called *valid* if for every Kripke structure $\mathsf{K} = (S, R, \pi)$ we have $\|A\|_{\mathsf{K}} = S$.

Making use of a canonical model construction, Jäger et al. [9] showed that System G_μ (which is cut-free) is sound and complete.

Theorem 6. *For all formulas A we have*

$$\mathsf{G}_\mu \vdash_0 A \text{ if and only if } A \text{ is valid.}$$

2.1 The Problems for Cut-Elimination

Although System G_μ is complete even without cut, the usual cut-elimination procedure does not work for G_μ. The problem is that the premises of the μ- and ν-rules do not correspond to each other. Consider the following proof:

$$\mathsf{cut}\;\dfrac{\mu\;\dfrac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \qquad \nu\;\dfrac{\cdots \quad \Gamma, \nu^k X.\bar{A} \quad \cdots}{\Gamma, \nu X.\bar{A}}}{\Gamma}$$

Here the typical transformation would push the cut above the μ- and ν-rules. However, this is not possible since $A(\mu X.A)$ is not the negation of any $\nu^k X.\bar{A}$.

A first approach to solve this problem is to consider a system G'_μ which is defined like G_μ except that the μ-rule is replaced with infinitely many rules (one for each natural number k)

$$\mu^k \frac{\Gamma, \mu X.A, \mu^k X.A}{\Gamma, \mu X.A} \quad .$$

However, in System G'_μ we cannot even derive the co-closure axiom

$$\nu X.\Box X \to \Box \nu X.\Box X \quad , \tag{1}$$

which states that $\nu X.\Box X$ is a post-fixed point of the operator given by $\lambda X.\Box X$. If we search for a proof of this formula, then we end up with the following derivation:

$$\vee \dfrac{\mu^{k+1} \dfrac{= \dfrac{\Box \dfrac{\nu \dfrac{\mu^k X.\Diamond X, \nu^l X.\Box X \quad \text{for all } l \geq 0}{\mu^k X.\Diamond X, \nu X.\Box X}}{\mu X.\Diamond X, \Diamond \mu^k X.\Diamond X, \Box \nu X.\Box X}}{\mu X.\Diamond X, \mu^{k+1} X.\Diamond X, \Box \nu X.\Box X}}{\mu X.\Diamond X, \Box \nu X.\Box X}}{\mu X.\Diamond X \vee \Box \nu X.\Box X}$$

Of the ω-many assumptions of this derivation only one, namely $l = k$, is provable in G'_μ. All the others cannot be proved. The problem with deriving (1) in G'_μ is that in a proof search procedure the rule μ^k has to be applied before ν can be applied which means that we have to choose the iteration number k too early.

This problem can be solved if we switch to nested sequents that allow the deep application of rules. Then we can first apply the ν-rule deeply inside the $\Box$ modality and then in each premise apply μ^k choosing an appropriate iteration number for that premise. This is presented in detail later in Example 11.

3 The Deep System

Definition 7. We define nested sequents and boxed sequents inductively as follows: 1) a *nested sequent* is a finite multiset of formulas and boxed sequents, and 2) a *boxed sequent* is an expression of the form [Γ] where Γ is a nested sequent.

The letters $\Gamma, \Delta, \ldots$ from now on denote nested sequents and the word *sequent* from now on refers to nested sequents, except when it is clear from the context that a sequent is shallow, such as a sequent appearing in a derivation in G_μ.

A sequent is always of the form

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]. \tag{2}$$

As usual, the comma denotes multiset union and there is no distinction between a singleton multiset and its element.

The *corresponding formula* of the sequent given in (2) is $\bot$ if $m = n = 0$ and otherwise

$$A_1 \vee \cdots \vee A_m \vee \Box D_1 \vee \cdots \vee \Box D_n$$

where $D_1, \ldots, D_n$ are the corresponding formulas of the sequents $\Delta_1, \ldots \Delta_n$. We denote the corresponding formula of the sequent Γ by $\underline{\Gamma}_F$, but sometimes we do not distinguish between the two.

We introduce the additional symbol $\{\ \}$, called the *hole*, to define *sequent contexts*, or *contexts*, for short. They are denoted by $\Gamma\{\ \}, \Delta\{\ \}, \Sigma\{\ \}$, and so on, and they follow the same notational conventions as sequents.

Definition 8 (sequent context)**.** Contexts are inductively defined as follows.

1. The singleton multiset containing the hole is a context, it is called the *empty context*.

2. If $\Gamma\{\ \}$ is a context and Σ is a sequent, then the multiset union of $\Gamma\{\ \}$ and Σ is a context.

3. If $\Gamma\{\ \}$ is a context, then the singleton multiset containing $[\Gamma\{\ \}]$ is a context.

A context has exactly one occurrence of the symbol $\{\ \}$. We can substitute sequents for this symbol as follows.

Definition 9. Let $\Gamma\{\ \}$ be a context and Δ be a sequent. The sequent $\Gamma\{\Delta\}$ is given as follows.

1. If $\Gamma\{\ \}$ is the empty context, then $\Gamma\{\Delta\}$ is Δ.

2. If $\Gamma\{\ \}$ is the multiset union of a context $\Gamma'\{\ \}$ and a sequent Σ, then $\Gamma\{\Delta\}$ is the multiset union of $\Gamma'\{\Delta\}$ and Σ.

3. If $\Gamma\{\ \}$ is the singleton multiset containing $[\Gamma'\{\ \}]$, then $\Gamma\{\Delta\}$ is the singleton multiset containing $[\Gamma'\{\Delta\}]$.

$$\mathsf{id}\ \frac{}{\Gamma\{p,\bar{p}\}} \qquad \top\ \frac{}{\Gamma\{\top\}} \qquad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

$$\mu^k\ \frac{\Gamma\{\mu X.A,\mu^k X.A\}}{\Gamma\{\mu X.A\}} \qquad \nu\ \frac{\Gamma\{\nu^k X.B\} \quad \text{for all } k \geq 0}{\Gamma\{\nu X.B\}}$$

Figure 3: The Deep System D_μ

$$\mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\varnothing\}} \qquad \mathsf{nec}\ \frac{\Delta}{[\Delta]} \qquad \mathsf{wk}\ \frac{\Gamma\{\varnothing\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\ \frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

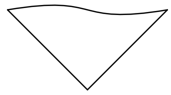
Figure 4: Admissible Rules

Definition 10 (formula context)**.** A *formula context* $C\{\ \}$ is a formula with exactly one occurrence of the special atom $\{\ \}$ which may only occur in the scope of $\vee$ and $\Box$. If $C\{\ \}$ is a formula context and A is a formula, then the formula $C\{A\}$ is obtained by replacing $\{\ \}$ in $C\{\ \}$ with A. The corresponding formula context $\underline{\Gamma}_F\{\ \}$ of a given context $\Gamma\{\ \}$ is defined by analogy with the notion of a corresponding formula.

The system D_μ is the set of axioms and inference rules in Figure 3. The rules cut, necessitation, weakening, and contraction are shown in Figure 4.

Example 11. To see System D_μ at work we will show a derivation of the co-closure axiom (1) in D_μ. Looking at it from a proof search perspective, we see that we can first apply the ν rule deeply behind the $\Box$, and then apply

μ^{k+1} with a different k in each branch.



$$\nu\ \dfrac{\cdots \qquad \mu^{k+1}\ \dfrac{=\ \dfrac{\Diamond\ \dfrac{\mu X.\Diamond X, \Diamond\mu^k X.\Diamond X, [\mu^k X.\Diamond X, \nu^k X.\Box X]}{\mu X.\Diamond X, \Diamond\mu^k X.\Diamond X, [\nu^k X.\Box X]}}{\mu X.\Diamond X, \mu^{k+1} X.\Diamond X, [\nu^k X.\Box X]}}{\mu X.\Diamond X, [\nu^k X.\Box X]} \qquad \cdots}{\vee\ \dfrac{\Box\ \dfrac{\mu X.\Diamond X, [\nu X.\Box X]}{\mu X.\Diamond X, \Box\nu X.\Box X}}{\mu X.\Diamond X \vee \Box\nu X.\Box X}}$$

The following lemma can be shown in the same way as the corresponding lemma for the logic of common knowledge [5].

Lemma 12. 1. *The rules necessitation, weakening, and contraction are perfectly admissible for* D_μ.

2. *All rules of* D_μ *are perfectly invertible for* D_μ.

3.1 Cut-Elimination

We will first give some ordinal theoretic preliminaries. For a detailed account and formal definitions of the following concepts we refer to Schütte [20]. As usual, $\alpha \# \beta$ denotes the natural sum of α and β which, in contrast to the ordinary ordinal sum, does not cancel additive components. The binary Veblen function φ is generated inductively as follows:

1. $\varphi_0\beta := \omega^\beta$,

2. if $\alpha > 0$, then $\varphi_\alpha\beta$ denotes the βth common fixed point of the functions $\lambda\xi.\varphi_\gamma\xi$ for $\gamma < \alpha$.

Lemma 13 (Reduction Lemma)**.** *For each context* $\Gamma\{\ \}$ *and each formula* A *with* $\mathsf{rk}(A) = \beta$, *we have: if (1)* $\mathsf{D}_\mu \overset{\alpha_1}{\underset{\beta}{\vdash}} \Gamma\{A\}$ *and (2)* $\mathsf{D}_\mu \overset{\alpha_2}{\underset{\beta}{\vdash}} \Gamma\{\bar{A}\}$, *then* $\mathsf{D}_\mu \overset{\alpha_1 \# \alpha_2}{\underset{\beta}{\vdash}} \Gamma\{\varnothing\}$.

Proof. As usual, by induction on $\alpha_1 \# \alpha_2$ and a case analysis on the two lowermost rules in the given proofs. We only show one case, namely the active case for $A = \nu X.B$. We have

$$\mathsf{D}_\mu \overset{\alpha_{1,k}}{\underset{\beta}{\vdash}} \Gamma\{\nu^k X.B\} \text{ for all } k \geq 0 \tag{3}$$

and

$$\mathsf{D}_\mu \vdash^{\alpha_{2,1}}_{\beta} \Gamma\{\mu X.\bar{B}, \mu^j X.\bar{B}\} \text{ for some } j. \tag{4}$$

By weakening we also have

$$\mathsf{D}_\mu \vdash^{\alpha_1}_{\beta} \Gamma\{\nu X.B, \mu^j X.\bar{B}\}.$$

The induction hypothesis together with (4) yields $\mathsf{D}_\mu \vdash^{\alpha_1 \# \alpha_{2,1}}_{\beta} \Gamma\{\mu^j X.\bar{B}\}$. Applying a cut with rank $\mathsf{rk}(\nu^k X.B) < \mathsf{rk}(A)$ to this sequent and (3) for $k = j$ results in $\mathsf{D}_\mu \vdash^{\alpha_1 \# \alpha_2}_{\beta} \Gamma\{\varnothing\}$.

All other cases are similar to the ones found in Brünnler and Studer [5]. □

From the reduction lemma we obtain the first and the second elimination theorem as usual, see for instance Pohlers [17, 18] or Schütte [20].

Theorem 14 (First Elimination Theorem)**.**
If $\mathsf{D}_\mu \vdash^{\alpha}_{\beta+1} \Gamma$, *then* $\mathsf{D}_\mu \vdash^{2^\alpha}_{\beta} \Gamma$.

Theorem 15 (Second Elimination Theorem)**.**
If $\mathsf{D}_\mu \vdash^{\alpha}_{\beta+\omega^\gamma} \Gamma$, *then* $\mathsf{D}_\mu \vdash^{\varphi_\gamma \alpha}_{\beta} \Gamma$.

Since all our cut ranks are below ω^2, we finally obtain the following cut-elimination result.

Theorem 16 (Cut-Elimination)**.**
If $\mathsf{D}_\mu \vdash^{\alpha}_{\beta} \Gamma$, *then* $\mathsf{D}_\mu \vdash^{\varphi_2 \alpha}_{0} \Gamma$.

3.2 The System is Incomplete

We are going to show that System D_μ is not complete for (the standard semantics of) the modal μ-calculus. In order to do so, we first introduce an alternative notion of validity that is based on finite approximations instead of least and greatest fixed points.

Given an operator F and an $n \geq 0$ we define F^n inductively as the identity operator for $n = 0$ and as $F \circ F^{n-1}$ for $n \geq 1$.

Definition 17. Let $\mathsf{K} = (S, R, \pi)$ be a Kripke structure. For every formula A we define the set $\lceil A \rceil_\mathsf{K} \subseteq S$ inductively like $\|A\|_\mathsf{K}$ except for the following two cases:

$$\lceil \mu X.A \rceil_\mathsf{K} := \bigcup_{n<\omega} G^n_{A,\mathsf{K}}(\varnothing) \quad \text{and} \quad \lceil \nu X.A \rceil_\mathsf{K} := \bigcap_{n<\omega} G^n_{A,\mathsf{K}}(S)$$

where $G_{A,\mathsf{K}}$ is the operator on S given by $G_{A,\mathsf{K}}(T) := \lceil A \rceil_{\mathsf{K}[X:=T]}$ for every subset T of S.

A formula A is called *valid for finite approximations* if for every Kripke structure $\mathsf{K} = (S, R, \pi)$ we have $\lceil A \rceil_{\mathsf{K}} = S$.

Soundness of D_μ with respect to finite approximations is shown as usual by induction on the depth of the derivation.

Lemma 18 (D_μ is sound for finite approximations)**.** *For all formulas A and all ordinals α, β we have that if $\mathsf{D}_\mu \mid\!\frac{\alpha}{\beta}\, A$, then A is valid for finite approximations.*

Lemma 19. *The formula $\Box(\mu X.\Box X) \to \mu X.\Box X$ is not valid for finite approximations.*

Proof. Consider some Kripke structure $\mathsf{K} = (\omega + 1 = \{\omega, \ldots, 2, 1, 0\}, >, \pi)$. Note that $G^n_{\Box X,\mathsf{K}}(\varnothing) = \{0, \ldots, n-1\}$. Expanding the definitions shows that $\omega \in \lceil \Box(\mu X.\Box X) \rceil_{\mathsf{K}} = \omega + 1$ but $\omega \notin \lceil \mu X.\Box X \rceil_{\mathsf{K}} = \omega$. $\Box$

Theorem 20. *System D_μ is not complete for (the standard semantics of) the modal μ-calculus.*

Proof. $\Box(\mu X.\Box X) \to \mu X.\Box X$ is a valid formula. However, by the two previous lemmas, we see that it is not derivable in D_μ. $\Box$

3.3 The System is Complete for a Fragment

We now define a restricted language for which System D_μ is complete as we will show later in Theorem 32. The restricted language disallows a diamond to occur between a ν and its bound variable and disallows a box to occur between a μ and its bound variable. We simultaneously define *μ-operator forms M*, *ν-operator forms N*, and *restricted formulas F* by the following grammar

$$\begin{aligned} M &::= X \mid F \mid (M \wedge M) \mid (M \vee M) \mid \Diamond M \\ N &::= X \mid F \mid (N \wedge N) \mid (N \vee N) \mid \Box N \\ F &::= p \mid \bar{p} \mid \top \mid \bot \mid (F \wedge F) \mid (F \vee F) \mid \Box F \mid \Diamond F \mid \mu X.M \mid \nu X.N \end{aligned}$$

where $p \in \mathcal{P}$. Our definition of μ-operator form corresponds to the set $CF(\{X\})$ that is given in [7, Def. 11] to syntactically capture the continuous fragment of the μ-calculus

Note that negation is well-defined for the restricted language since the negation of a μ-operator form is a ν-operator form and vice versa.

Remark 21. *We can embed in the restricted language the iteration modality from* PDL *and the common knowledge modality from epistemic logic. Namely, for* PDL *we observe that* $[p^*]A$ *corresponds to* $\nu X.(A \wedge [p]X)$ *and for common knowledge we observe that* $\mathsf{C}A$ *corresponds to* $\nu X.\mathsf{E}(A \wedge X)$ *where* E *is the* everybody knows *modality. Of course, to really embed those logics we would have to switch to a multi-modal language and maybe also include new rules for the additional modalities. However, those rules would not affect the essence of the cut-elimination procedure. Important for that are only the rules for fixed points and those are all covered by our approach.*

Note that it is not possible to embed in our restricted language all the fixed point modalities occurring in CTL*. In particular, until with universal path quantifiers does not fit:* $\forall(A\mathsf{U}B)$ *corresponds to*

$$\mu X.(B \vee (A \wedge \Box X \wedge \Diamond\top))$$

which is not a restricted formula.

4 Cut-Elimination for a Fragment of the Shallow System

Definition 22. System $\mathsf{G}_{\mu\mathsf{r}}$ is obtained from system G_μ by adding the following proviso to the μ-rule as show in Figure 1: A is a μ-operator form.

Clearly, because of the subformula property, the provability of G_μ and $\mathsf{G}_{\mu\mathsf{r}}$ (without cut) on restricted formulas is the same.

Lemma 23. *A restricted formula is provable in* $\mathsf{G}_{\mu\mathsf{r}}$ *if and only if it is provable in* G_μ*.*

4.1 Embedding Shallow into Deep

Lemma 24. *Let* $\Gamma\{\ \}$ *be a context and* A, B *be operator forms. We have that for all natural numbers* k

$$\mathsf{D}_\mu \mid\!\frac{\alpha}{\beta}\, \Gamma\{B(\nu^{k+1}X.A)\} \quad\Longrightarrow\quad \mathsf{D}_\mu \mid\!\frac{\alpha}{\beta}\, \Gamma\{B(\nu^{k}X.A)\} \quad .$$

Proof. The claim is established by an outer induction on k and an inner induction on α. The case for $k = 0$ is routine. For $k > 0$ we distiguish the following cases.

1. If $\Gamma\{B(\nu^{k+1}X.A)\}$ is an axiom, then so is $\Gamma\{B(\nu^{k}X.A)\}$.

2. If $\Gamma\{B(\nu^{k+1}X.A)\}$ is the conclusion of a rule ρ where the main connective is in Γ or in B, then we apply the inner induction hypothesis to the premises of that rule and the claim follows by an application of ρ.

3. If $B = X$, then $\Gamma\{B(\nu^{k+1}X.A)\}{=}\Gamma\{A(\nu^k X.A)\}$. By the outer induction hypothesis we obtain $\mathsf{D}_\mu \vdash^{\alpha}_{\beta} \Gamma\{A(\nu^{k-1}X.A)\}$, which is

$$\mathsf{D}_\mu \vdash^{\alpha}_{\beta} \Gamma\{B(\nu^k X.A)\}.$$

□

Definition 25 ($\mathsf{len}(A)$)**.** We define the *length* $\mathsf{len}(A)$ of an operator form A inductively as follows.

1. $\mathsf{len}(\top) := \mathsf{len}(\bot) := \mathsf{len}(p) := \mathsf{len}(\bar{p}) := 0$ for $p \in \mathcal{P}$
2. $\mathsf{len}(\mu X.A) := \mathsf{len}(\nu X.A) := 0$
3. $\mathsf{len}(X) := 1$
4. $\mathsf{len}(A \wedge B) := \mathsf{len}(A \vee B) := \mathsf{len}(A) + \mathsf{len}(B) + 1$
5. $\mathsf{len}(\Box A) := \mathsf{len}(\Diamond A) := \mathsf{len}(A) + 1$

Remark 26. *In the next lemma, the restriction of A and B to ν-operator forms is essential. Clearly the sequent $\Diamond(\nu^k X.\Diamond X), \mu X.\Box X$ is provable for every k. So without the restriction, the lemma would allow us to prove the sequent $\Diamond(\nu X.\Diamond X), \mu X.\Box X$, and thus contradict the claim in the proof of Theorem 20.*

Also note that while $\Gamma\{\ \}$ is a context, $\Gamma\{B(\{\})\}$ is not a context meaning we cannot directly apply the introduction rule for ν.

Lemma 27 (deep ν-rule)**.** *Let $\Gamma\{\ \}$ be a context and A, B be ν-operator forms. Assume we have $\mathsf{D}_\mu \vdash^{\alpha}_{\beta} \Gamma\{B(\nu^k X.A)\}$ for all natural numbers k, then we have*

$$\mathsf{D}_\mu \vdash^{\alpha+\mathsf{len}(B)}_{\beta} \Gamma\{B(\nu X.A)\} \quad .$$

Proof. By induction on the structure of B.

1. $B = \top, B = \bot, B \in \mathcal{P}$, or $\bar{B} \in \mathcal{P}$. These cases are trivial.
2. $B = X$. The claim follows by an application of ν.

3. $B = B_1 \wedge B_2$. By invertibility of $\wedge$ we obtain for all k

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha}{\beta}} \Gamma\{B_1(\nu^k X.A)\} \text{ and } \mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha}{\beta}} \Gamma\{B_2(\nu^k X.A)\}.$$

   The induction hypothesis yields

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha+\mathsf{len}(B_1)}{\beta}} \Gamma\{B_1(\nu X.A)\} \text{ and } \mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha+\mathsf{len}(B_2)}{\beta}} \Gamma\{B_2(\nu X.A)\}.$$

   Thus an application of $\wedge$ yields the claim.

4. $B = B_1 \vee B_2$. By invertibility of $\vee$ we obtain

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha}{\beta}} \Gamma\{B_1(\nu^k X.A), B_2(\nu^k X.A)\} \text{ for all } k.$$

   Iterated applications of Lemma 24 yield

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha}{\beta}} \Gamma\{B_1(\nu^{k_1} X.A), B_2(\nu^{k_2} X.A)\} \text{ for all } k_1, k_2.$$

   We apply the induction hypothesis for each k_2 to obtain

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha+\mathsf{len}(B_1)}{\beta}} \Gamma\{B_1(\nu X.A), B_2(\nu^{k_2} X.A)\} \text{ for all } k_2.$$

   Applying the induction hypothesis again yields

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha+\mathsf{len}(B_1)+\mathsf{len}(B_2)}{\beta}} \Gamma\{B_1(\nu X.A), B_2(\nu X.A)\}.$$

   Finally, the claim follows by an application of $\vee$.

5. $B = \Box B_1$. By invertibility of $\Box$ we obtain for all k

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha}{\beta}} \Gamma\{[B_1(\nu^k X.A)]\}.$$

   The induction hypothesis yields

$$\mathsf{D}_\mu \mathrel{\vert\!\frac{\alpha+\mathsf{len}(B_1)}{\beta}} \Gamma\{[B_1(\nu X.A)]\}.$$

   Thus an application of $\Box$ yields the claim.

6. $B = \Diamond B_1$. Since B is a ν-operator form, this implies that B does not contain free occurrences of X. Therefore, $B(\nu^k X.A) = B(\nu X.A)$. Thus the claim follows trivially.

7. $B = \mu X.B_1$ or $B = \nu X.B_1$. Trivial since B does not contain free occurrences of X. □

Lemma 28 (general identity axiom)**.** *Let* $\Gamma\{\ \}$ *be a context and* A *be a formula. We find that*

$$\mathsf{D}_\mu \vdash^{2\cdot \mathsf{rk}(A)}_{0} \Gamma\{A, \bar{A}\}.$$

Proof. By induction on $\mathsf{rk}(A)$ and a case distinction on the main connective of A. In all cases the claim follows from a simple derivation and the basic properties of the rank function. We just show the case where $A = \mu X.B$:

$$\nu\,\dfrac{\cdots \quad \mu\,\dfrac{\dfrac{IH}{\Gamma\{\mu X.B, \mu^k X.B, \nu^k X.\bar{B}\}}}{\Gamma\{\mu X.B, \nu^k X.\bar{B}\}} \quad \cdots}{\Gamma\{\mu X.B, \nu X.\bar{B}\}}$$

□

Lemma 29. *(restricted* μ*-unfolding) Let* $\Gamma\{\}$ *be a context and* A *be a* μ*-operator form. We have that*

$$\mathsf{D}_\mu \vdash^{\alpha}_{\beta} \Gamma\{A(\mu X.A)\} \quad \Longrightarrow \quad \mathsf{D}_\mu \vdash^{\max(\alpha+1,\omega^2)}_{\max(\beta,\mathsf{rk}(A(\mu X.A))+1)} \Gamma\{\mu X.A\}.$$

Proof. Consider the following derivation where gid is the general identity axiom and $\mathsf{d}\nu$ is the deep ν rule:

$$\mathsf{cut}\,\dfrac{\mathsf{wk}\,\dfrac{\Gamma\{A(\mu X.A)\}}{\Gamma\{\mu X.A, A(\mu X.A)\}} \qquad \mathsf{d}\nu\,\dfrac{\cdots \quad \mu\,\dfrac{=\,\dfrac{\mathsf{gid}\,\dfrac{}{\Gamma\{\mu X.A, \mu^{k+1} X.A, \nu^{k+1} X.\bar{A}\}}}{\Gamma\{\mu X.A, \mu^{k+1} X.A, \bar{A}(\nu^k X.\bar{A})\}}}{\Gamma\{\mu X.A, \bar{A}(\nu^k X.\bar{A})\}} \quad \cdots}{\Gamma\{\mu X.A, \bar{A}(\nu X.\bar{A})\}}}{\Gamma\{\mu X.A\}}$$

where $\mathsf{d}\nu$ is admissible by Lemma 27 and gid is admissible by Lemma 28. To obtain the bounds on the proof depth, we observe that Lemma 28 yields $\mathsf{D}_\mu \vdash^{2\cdot\gamma}_{0} \Gamma\{\mu X.A, \mu^{k+1}X.A, \nu^{k+1}X.\bar{A}\}$ for some $\gamma < \mathsf{rk}(\mu X.A)$. The claim follows then from $2 \cdot \mathsf{rk}(\mu X.A) + \mathsf{len}(A) + 2 < \omega^2$. □

Lemma 30 (box-rule)**.** *Let* Γ, Δ *be sequents and* A *be a formula. There exists a finite ordinal* n *such that*

$$\mathsf{D}_\mu \vdash^{\alpha}_{\beta} \Gamma, A \quad \Longrightarrow \quad \mathsf{D}_\mu \vdash^{\alpha+n}_{\beta} \Diamond\Gamma, \Box A, \Delta.$$

Proof. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A}{[\Gamma, A]}\,\mathsf{nec}}{\Diamond\Gamma, [\Gamma, A]}\,\mathsf{wk}}{\Diamond\Gamma, [A]}\,\Diamond^{\star}}{\Diamond\Gamma, \Box A, \Delta}\,\Box,\mathsf{wk}$$

□

Theorem 31 (embedding shallow into deep)**.** *Let* Γ *be a sequent. We have that*

$$\mathsf{G}_{\mu\mathsf{r}} \vdash^{\alpha}_{\beta,\gamma} \Gamma \quad \Longrightarrow \quad \mathsf{D}_{\mu} \vdash^{\omega^2\cdot\alpha}_{\max(\beta,\gamma)} \Gamma.$$

Proof. By induction on α and a case analysis of the last rule in the proof. Each rule of $\mathsf{G}_{\mu\mathsf{r}}$, except for $\Box$ and μ, is a special case of its respective rule in D_{μ}. The case of $\Box$ follows from Lemma 30. If for a μ-operator form A the sequent $\Gamma', \mu X.A$ has been derived by an application of μ, then we have $\alpha \geq 1$, $\mathsf{rk}(A(\mu X.A)) < \gamma$, and

$$\mathsf{G}_{\mu} \vdash^{\alpha'}_{\beta,\gamma} \Gamma', A(\mu X.A) \text{ for some } \alpha' < \alpha.$$

By the induction hypothesis we find

$$\mathsf{D}_{\mu} \vdash^{\omega^2\cdot\alpha'}_{\max(\beta,\gamma)} \Gamma', A(\mu X.A).$$

Lemma 29 yields

$$\mathsf{D}_{\mu} \vdash^{\max(\omega^2\cdot\alpha'+1,\omega^2)}_{\max(\beta,\gamma)} \Gamma', \mu X.A.$$

The claim follows by $\max(\omega^2 \cdot \alpha' + 1, \omega^2) \leq \max(\omega^2 \cdot \alpha, \omega^2) = \omega^2 \cdot \alpha$. □

Theorem 32. *System* D_{μ} *is complete for restricted formulas.*

Proof. By Lemma 23, we know that, with respect to restricted formulas, completeness of G_{μ} (Theorem 6) implies completeness of $\mathsf{G}_{\mu\mathsf{r}}$. The embedding of the previous theorem implies completeness of $\mathsf{D}_{\mu}+\mathsf{cut}$ and cut-elimination for D_{μ} (Theorem 16) gives us completeness of D_{μ} with respect to restricted formulas. □

4.2 Embedding Deep into Shallow

We first define a notion of saturation. Roughly, a sequent Γ is called *locally saturated* if whenever Γ is the conclusion of a certain rule, then there is a premise whose sequent is a subsequent of Γ. The rules we consider here exclude the $\Box$-rule (which in some sense changes to another world) so we call them *local*.

Definition 33 (locally saturated)**.** A shallow sequent Γ is *locally saturated* if the following are true:

$$\begin{aligned} A \vee B \in \Gamma &\implies A, B \in \Gamma \\ A \wedge B \in \Gamma &\implies A \in \Gamma \text{ or } B \in \Gamma \\ \mu X.A \in \Gamma &\implies A(\mu X.A) \in \Gamma \\ \nu X.A \in \Gamma &\implies \exists k\, \nu^k X.A \in \Gamma \end{aligned}$$

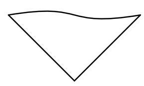
Definition 34 (canonical $\Box$-instance)**.** An instance of the rule

$$\Box \frac{\Gamma, A}{\Diamond\Gamma, \Box A, \Delta}$$

is *canonical* if Δ does not contain formulas of the form $\Diamond B$.

Lemma 35 (quasi-invertibility of the $\Box$-rule)**.** *Let Γ be a locally saturated sequent and let there be a cut-free proof of the sequent $\Box A, \Gamma$ in G_μ. Then there is a cut-free proof of the same depth in G_μ either 1) of the sequent Γ or 2) of the sequent $\Box A, \Gamma$ where the last rule instance is a canonical instance of the $\Box$-rule where the main formula is the shown formula $\Box A$.*

Proof. By induction on the depth of the given proof and a case analysis on the last rule. Most cases are trivial because of local saturation. The only non-trivial case is the $\Box$-rule. We distinguish two subcases. First, if $\Box A$ is the active formula, then the second disjunct of our lemma is either immediate or obtained via admissibility of weakening if the rule instance is not canonical. Second, if $\Box A$ is not the active formula, then the proof has the form



$$\Box \frac{\Gamma_2, B}{\Box A, \Gamma_1, \Box B}$$

where $\Box A$ has been introduced inside Δ. Thus we also get a proof of $\Gamma_1, \Box B$ which shows the first disjunct of our lemma. $\Box$

The following definition introduces a notion of deep inference in the shallow system G_μ.

Definition 36. Let $C\{\ \}$ be a formula context (see Definition 10). Given a rule ρ we define a rule $C\{\rho\}$ as follows: an instance of the rule ρ is shown on the left iff an instance of the rule $C\{\rho\}$ is shown on the right:

$$\rho\,\frac{\Gamma, A_1 \quad \cdots \quad \Gamma, A_i \quad \cdots}{\Gamma, A} \qquad\qquad C\{\rho\}\,\frac{\Gamma, C\{A_1\} \quad \cdots \quad \Gamma, C\{A_i\} \quad \cdots}{\Gamma, C\{A\}}$$

Given a rule ρ we define the rule $\breve{\rho}$ as follows: its set of instances is the union of all sets of instances of $C\{\rho\}$ where $C\{\ \}$ ranges over formula contexts.

Definition 37 (admissible, finitely admissible)**.** A rule ρ is *admissible* for System G_μ if for each instance of it with premises Γ_i and conclusion Δ the following holds: if $\mathsf{G}_\mu \vdash_0 \Gamma_i$ for all i, then $\mathsf{G}_\mu \vdash_0 \Delta$. The rule ρ is called *finitely admissible* if for each instance there exists a natural number n such that $\mathsf{G}_\mu \vdash_0^\alpha \Gamma_i$ for all i, then $\mathsf{G}_\mu \vdash_0^{\alpha+n} \Delta$.

Definition 38 (guarded formula)**.** A formula A is called *guarded* if the following holds: if $\sigma X.B(X)$ is a subformula of A where σ may be μ or ν, then every free occurrence of X in $B(X)$ is in the scope of a modality.

It is well-known that for any formula there is a semantically equivalent one that is guarded [14]. Note also that the formulas introduced in Remark 21 for embedding PDL and the logic of common knowledge are guarded.

Lemma 39 (deep applicability preserves admissibility)**.** *Let $C\{\ \}$ be a formula context.*

(i) There is an n such that for all Γ we have $\mathsf{G}_\mu \vdash_0^n \Gamma, C\{\top\}$.

(ii) There is an n such that for all Γ we have $\mathsf{G}_\mu \vdash_0^n \Gamma, C\{p \vee \bar{p}\}$.

(iii) If a rule ρ is admissible for G_μ, then $C\{\rho\}$ is also admissible for G_μ. If we consider guarded formulas only, then we have: if a rule ρ is finitely admissible for G_μ, then $C\{\rho\}$ is also finitely admissible for G_μ.

Proof. The three statements are shown by induction on the structure of $C\{\ \}$. Let us only show the case where $C\{\ \} = \Box C_1\{\ \}$ in statement (iii). We have the following situation:

$$\Box C_1\{\rho\}\,\frac{\cdots \quad \Gamma, \Box C_1\{A_k\} \quad \cdots}{\Gamma, \Box C_1\{A\}}$$

In order to apply Lemma 35, we first need to replace the shown instance of the rule $\Box C_1\{\rho\}$ by several instances of it which are applied in a context that is locally saturated. We apply invertibility of $\wedge, \vee, \mu$, and ν such that for each k there is a proof of the form

$$\wedge,\vee,\mu,\nu \; \frac{\Gamma_1, \Box C_1\{A_k\} \quad \cdots \quad \Gamma_m, \Box C_1\{A_k\}}{\Gamma, \Box C_1\{A_k\}} \tag{5}$$

where each Γ_j (for $1 \leq j \leq m$) is locally saturated. Note that m only depends on Γ.

Fix some Γ_j (where $1 \leq j \leq m$). For all k apply Lemma 35 to the proof of $\Gamma_j, \Box C_1\{A_k\}$. Either this yields a proof of Γ_j or for each k it yields a proof of some sequent $\Gamma'_j, C_1\{A_k\}$. Thus we can build either

$$\mathsf{wk}\; \frac{\Gamma_j}{\Gamma_j, \Box C_1\{A\}} \qquad \text{or} \qquad \Box\; \frac{C_1\{\rho\}\; \dfrac{\cdots \quad \Gamma'_j, C_1\{A_k\} \quad \cdots}{\Gamma'_j, C_1\{A\}}}{\Gamma_j, \Box C_1\{A\}}$$

where in the second case $C_1\{\rho\}$ is admissible by the induction hypothesis. Repeat this argument for each j with $1 \leq j \leq m$, which for each j yields of proof of $\Gamma_j, \Box C_1\{A\}$ in G_μ. From those we can derive $\Gamma, \Box C_1\{A\}$ by a derivation as in (5) where each A_k is replaced by A. This shows that $\Box C_1\{\rho\}$ is admissible.

To obtain the result about finite admissibility in the context of guarded formulas we observe that the derivation in (5) has finite depth if guarded formulas are considered only. The reason for this is that the length of the process of locally saturating Γ does not depend on the iteration number k in the case where a guarded $\nu X.B$ is is treated. $\Box$

Lemma 40. *The following rules are finitely admissible for* G_μ.

$$\mathsf{g_c}\; \frac{\Gamma, A \vee B}{\Gamma, B \vee A} \qquad \mathsf{g_a}\; \frac{\Gamma, (A \vee B) \vee C}{\Gamma, A \vee (B \vee C)}$$

$$\mathsf{g_{ctr}}\; \frac{\Gamma, A \vee A}{\Gamma, A} \qquad \mathsf{g_\Diamond}\; \frac{\Gamma, \Box(A \vee B)}{\Gamma, \Diamond A, \Box B}$$

Proof. Finite admissibility of $\mathsf{g_c}, \mathsf{g_a}$, and $\mathsf{g_{ctr}}$ follows immediately by invertibility of $\vee$. The rule $\mathsf{g_\Diamond}$ can easily be shown to be finitely admissible by induction on the given proof of the premise. $\Box$

For our translation from deep into shallow we translate nested sequents into formulas and thus fix an arbitrary order and association among elements of a sequent. The arbitrariness of this translation gets in the way, and we work around it as follows. We write

$$\mathsf{ac}\,\frac{A}{B}$$

if the formula B can be derived from the formula A by the rules $\breve{\mathsf{g}}_\mathsf{c}$ and $\breve{\mathsf{g}}_\mathsf{a}$. Note that since $\breve{\mathsf{g}}_\mathsf{c}$ and $\breve{\mathsf{g}}_\mathsf{a}$ are admissible for G_μ, so is ac. It is even finitely admissible if guarded formulas are considered only.

Lemma 41. *Let Γ be a sequent, $A(X)$ be μ-operator form, and $B(X)$ be any operator form. We have that*

$$\mathsf{G}_\mu \mathrel{\vert\!\frac{\alpha}{0}} \Gamma, B(\mu^k X.A) \quad \Longrightarrow \quad \mathsf{G}_\mu \mathrel{\vert\!\frac{2\cdot\alpha+1}{0}} \Gamma, B(\mu X.A).$$

Proof. By induction on α and a case analysis on the last rule.

1. $\Gamma, B(\mu^k X.A)$ is an axiom. If Γ or $\Gamma, B(X)$ is an axiom, then the claim trivially holds. Otherwise, we find $B = X$ and A is an atomic proposition, a negation of one, or $\top$. Thus $\Gamma, A(\mu X.A)$ is also an axiom and the claim follows by an application of μ.

2. $\wedge$-rule. We distinguish the cases for the position of the active conjunction.

   (a) If the active operator is in Γ, then we have

   $$\mathsf{G}_\mu \mathrel{\vert\!\frac{\alpha'}{0}} \Gamma_1, B(\mu^k X.A) \text{ and } \mathsf{G}_\mu \mathrel{\vert\!\frac{\alpha'}{0}} \Gamma_2, B(\mu^k X.A)$$

   for $\alpha' < \alpha$. By the induction hypothesis we obtain

   $$\mathsf{G}_\mu \mathrel{\vert\!\frac{2\cdot\alpha'+1}{0}} \Gamma_1, B(\mu X.A) \text{ and } \mathsf{G}_\mu \mathrel{\vert\!\frac{2\cdot\alpha'+1}{0}} \Gamma_2, B(\mu X.A).$$

   An application of $\wedge$ yields $\mathsf{G}_\mu \mathrel{\vert\!\frac{2\cdot\alpha+1}{0}} \Gamma, B(\mu X.A)$.

   (b) The case where the active operator is in $B(X)$ is analogous to the previous case.

   (c) $B = X$ and $\Gamma, \mu^k X.A$ has been derived from

   $$\mathsf{G}_\mu \mathrel{\vert\!\frac{\alpha'}{0}} \Gamma, A_1(\mu^{k-1} X.A) \text{ and } \mathsf{G}_\mu \mathrel{\vert\!\frac{\alpha'}{0}} \Gamma, A_2(\mu^{k-1} X.A)$$

for suitable A_1, A_2 and $\alpha' < \alpha$. By the induction hypothesis we obtain

$$\mathsf{G}_\mu \mid\frac{2\cdot\alpha'+1}{0} \Gamma, A_1(\mu X.A) \text{ and } \mathsf{G}_\mu \mid\frac{2\cdot\alpha'+1}{0} \Gamma, A_2(\mu X.A).$$

An application of $\wedge$ yields $\mathsf{G}_\mu \mid\frac{2\cdot\alpha'+2}{0} \Gamma, A(\mu X.A)$. Then,

$$\mathsf{G}_\mu \mid\frac{2\cdot\alpha+1}{0} \Gamma, \mu X.A$$

follows from an application of μ and $2 \cdot \alpha' + 3 \leq 2 \cdot \alpha + 1$.

3. $\vee$-rule. This case is similar to the case of $\wedge$.

4. $\Box$-rule. We distinguish:

   (a) If $B(\mu^k X.A)$ has been introduced by the built-in weakening in $\Box$, then the claim trivially holds.

   (b) Otherwise, if $B \neq X$, then we have $\mathsf{G}_\mu \mid\frac{\alpha'}{0} \Gamma', B'(\mu^k X.A)$ for some $\alpha' < \alpha$. By the induction hypothesis we get

   $$\mathsf{G}_\mu \mid\frac{2\cdot\alpha'+1}{0} \Gamma', B'(\mu X.A).$$

   An application of $\Box$ yields the claim.

   (c) If $B = X$, similar to the previous case we have

   $$\mathsf{G}_\mu \mid\frac{\alpha'}{0} \Gamma', A'(\mu^{k-1} X.A)$$

   from which we get by the induction hypothesis and an application of $\Box$

   $$\mathsf{G}_\mu \mid\frac{2\cdot\alpha'+2}{0} \Gamma, A(\mu X.A)$$

   for some $\alpha' < \alpha$. Then, $\mathsf{G}_\mu \mid\frac{2\cdot\alpha+1}{0} \Gamma, \mu X.A$ follows from an application of μ and $2 \cdot \alpha' + 3 \leq 2 \cdot \alpha + 1$.

5. μ-rule. We distinguish the cases for the position of the active μ-operator.

   (a) If the active operator is in Γ, then the claim follows easily from the induction hypothesis and an application of μ.

   (b) If the active operator is in B, then X is not free in B and the claim trivially holds.

(c) If $B = X$ and $A(\mu^{k-1}X.A)$ is the active formula, then A must be of the form $\mu X.C$. Thus X is not free in A and we have $A(\mu^{k-1}X.A) = A(\mu X.A)$. The claim follows immediately from an application of μ.

6. ν-rule. We distinguish the cases for the position of the active ν-operator.

   (a) If the active operator is in Γ, then for all $i < \omega$ there are $\alpha_i < \alpha$ with
   $$\mathsf{G}_\mu \overset{\alpha_i}{\underset{0}{\vdash}} \Gamma_i, B(\mu^k X.A).$$
   By the induction hypothesis we obtain for all $i < \omega$
   $$\mathsf{G}_\mu \overset{2\cdot\alpha_i+1}{\underset{0}{\vdash}} \Gamma_i, B(\mu X.A).$$
   An application on ν yields $\mathsf{G}_\mu \overset{2\cdot\alpha+1}{\underset{0}{\vdash}} \Gamma, B(\mu X.A)$.

   (b) If the active operator is in B, then X is not free in B and the claim trivially holds.

   (c) If $B = X$ and $A(\mu^{k-1}X.A)$ is the active formula, then A must be of the form $\nu X.C$. Thus X is not free in A and we have $A(\mu^{k-1}X.A) = A(\mu X.A)$. The claim follows immediately from an application of μ. □

Theorem 42. *Assume* $\mathsf{D}_\mu \overset{\alpha}{\underset{0}{\vdash}} \Gamma$. *We then have* $\mathsf{G}_\mu \underset{0}{\vdash} \underline{\Gamma}_F$. *If we consider guarded formulas only, then we have* $\mathsf{G}_\mu \overset{\omega\cdot(\alpha+1)}{\underset{0}{\vdash}} \underline{\Gamma}_F$.

Proof. By induction on α and a case analysis on the last rule.

1. If the endsequent of the given proof is axiomatic, say it is of the form $\Gamma\{p, \bar{p}\}$, then for some n we have $\mathsf{G}_\mu \overset{n}{\underset{0}{\vdash}} \underline{\Gamma}_F\{p \vee \bar{p}\}$ by Lemma 39. Admissibility of ac gives $\mathsf{G}_\mu \underset{0}{\vdash} \underline{\Gamma\{p, \bar{p}\}}_F$. If we consider guarded formulas only, ac is finitely admissible which gives $\mathsf{G}_\mu \overset{n}{\underset{0}{\vdash}} \underline{\Gamma\{p, \bar{p}\}}_F$ for some n. The case where the endsequent is of the form $\Gamma\{\top\}$ is similar.

2. If the last rule is an instance of $\vee$, then an application of ac proves our claim.

3. The case of the $\Box$-rule is trivial since the corresponding formula for the premise is the corresponding formula of the conclusion.

4. For the ν-rule, we apply the following transformation:

$$\frac{\cdots \quad \Gamma\{\nu^k X.A\} \quad \cdots}{\Gamma\{\nu X.A\}} \quad \rightsquigarrow \quad \underline{\Gamma}_F\{\nu\}\ \dfrac{\cdots \quad \mathsf{ac}\ \dfrac{\underline{\Gamma\{\nu^k X.A\}}_F}{\underline{\Gamma}_F\{\nu^k X.A\}} \quad \cdots}{\mathsf{ac}\ \dfrac{\underline{\Gamma}_F\{\nu X.A\}}{\underline{\Gamma\{\nu X.A\}}_F}}$$

To obtain the claim about the case of guarded formulas, let the depth of the proof on the left be α and the depth of the premises be α_k. The depths of the ac derivations are the same in all branches since they do not depend on the iteration number k. Thus they are bounded by a finite ordinal m. Then by finite admissibility of the rule $\underline{\Gamma}_F\{\nu\}$, there is a finite ordinal n such that the proof on the right has the depth

$$\begin{aligned}&\sup(\omega \cdot (\alpha_k + 1) + m + 1) + n + m \leq \sup(\omega \cdot (\alpha_k + 1)) + \omega \\ &\leq \omega \cdot \sup(\alpha_k + 1) + \omega \leq \omega \cdot \alpha + \omega = \omega \cdot (\alpha + 1),\end{aligned}$$

remember that α_k is the depth of the derivation of $\Gamma\{\nu^k X.A\}$.

5. The case for the $\wedge$-rule is similar.

6. For the $\Diamond$-rule, we apply the following transformation:

$$\frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \rightsquigarrow \quad \begin{array}{r}\mathsf{ac}\ \dfrac{\underline{\Gamma\{\Diamond A, [A, \Delta]\}}_F}{\underline{\Gamma}_F\{\Diamond A \vee \Box(A \vee \underline{\Delta}_F)\}} \\ \underline{\Gamma}_F\{\Diamond A \vee \mathsf{g}_\Diamond\}\ \overline{\underline{\Gamma}_F\{\Diamond A \vee (\Diamond A \vee \Box\underline{\Delta}_F)\}} \\ \mathsf{ac}\ \overline{\underline{\Gamma}_F\{(\Diamond A \vee \Diamond A) \vee \Box\underline{\Delta}_F\}} \\ \underline{\Gamma}_F\{\mathsf{g}_{\mathsf{ctr}} \vee \Box\underline{\Delta}_F\}\ \overline{\underline{\Gamma}_F\{\Diamond A \vee \Box\underline{\Delta}_F\}} \\ \mathsf{ac}\ \overline{\underline{\Gamma\{\Diamond A, [\Delta]\}}_F}\end{array}$$

Note that here a rule like $C\{\rho \vee A\}$ means rule ρ applied in the context $C\{\{\ \} \vee A\}$.

7. For the μ^k-rule, we apply the following transformation, where $\Gamma'\{\ \}$ is such that $\Gamma'\{\mu^k X.A\} = \Gamma\{\mu X.A, \mu^k X.A\}$:

$$\frac{\Gamma\{\mu X.A, \mu^k X.A\}}{\Gamma\{\mu X.A\}} \quad \rightsquigarrow \quad \begin{array}{r}\mathsf{ac}\ \dfrac{\underline{\Gamma\{\mu X.A, \mu^k X.A\}}_F}{\underline{\Gamma'}_F\{\mu^k X.A\}} \\ \text{L. 41}\ \overline{\underline{\Gamma'}_F\{\mu X.A\}} \\ \mathsf{ac}\ \overline{\underline{\Gamma}_F\{\mu X.A, \mu X.A\}} \\ \underline{\Gamma}_F\{\mathsf{ctr}\}\ \overline{\underline{\Gamma}_F\{\mu X.A\}} \\ \mathsf{ac}\ \overline{\underline{\Gamma\{\mu X.A\}}_F}\end{array}$$

To obtain the claim about guarded formulas, we let the depth of the proof on the left be α and the depth of the premise be α'. Then there are finite ordinals m, n such that the proof on the right has the depth

$$\begin{aligned}&2 \cdot (\omega \cdot (\alpha' + 1) + m) + 1 + n = 2 \cdot (\omega \cdot (\alpha' + 1)) + 2m + 1 + n \\ &= (2 \cdot \omega) \cdot (\alpha' + 1) + 2m + 1 + n = \omega \cdot (\alpha' + 1) + 2m + 1 + n \\ &\leq \omega \cdot (\alpha' + 1) + \omega \leq \omega \cdot (\alpha + 1).\end{aligned}$$

Remember that ordinal multiplication is associative and that $2 \cdot \omega = \omega$, see, for instance, [17]. □

Combining the results about embedding the shallow system into the deep system (with cut), cut-elimination for the deep system, and embedding the deep system into the shallow system (without cut), we obtain the following corollary about syntactic cut-elimination for the shallow system. Note that by Lemma 2, we know that the rank of any formula is smaller than ω^2. Hence if $\mathsf{G}_\mu \vdash^{\alpha}_{\beta,\gamma} \Gamma$, then there is a natural number n such that $\beta, \gamma \leq \omega \cdot n$.

Corollary 43. *Let Γ be a sequent of restricted formulas. Assume we have $\mathsf{G}_\mu \vdash^{\alpha}_{\omega \cdot n, \omega \cdot n} \Gamma$. Then we have $\mathsf{G}_\mu \vdash_{0} \Gamma$. If we consider guarded formulas only, then we have $\mathsf{G}_\mu \vdash^{\omega \cdot (\varphi_1^n(\omega^2 \cdot \alpha) + 1)}_{0} \Gamma$.*

5 Conclusion

We looked at syntactic cut-elimination for modal fixed point logics from the general perspective of the modal μ-calculus. We introduced a deep system D_μ for a fragment of the μ-calculus that includes the logic of common knowledge and PDL (modulo a multi-modal language, see Remark 21). We then showed that D_μ enjoys syntactic cut-elimination. We also showed that D_μ is not complete for the modal μ-calculus, which provides some evidence that our method cannot be extended in a straightforward way to larger fragments of the modal μ-calculus. Via embedding a traditional shallow system into D_μ and vice versa, we obtain cut-elimination for a given traditional shallow system. Thus our results subsume and extend previous cut-elimination results for particular logics like PDL and the logic of common knowledge.

The main technical contribution of this paper are the embeddings of shallow into deep and vice versa. For PDL no such embeddings were available so far and the embeddings for common knowledge are much simpler than those for the general case. In particular, the embedding of shallow into deep is for free in the case of common knowledge since the operator form for common knowledge corresponds almost directly to the structural connective $[\cdot]$ on the level of nested sequents.

If we restrict ourselves to guarded formulas, we also obtain upper bounds on the growth of the proof depth during cut-elimination. So we get an upper bound on the depth of a cut-free proof given the depth of an original proof with cut. Results of this kind were previously only known for common knowledge. Our bounds match those results but apply also to other logics like PDL.

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