About the strength of operational regularity

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Dedicated to Helmut Schwichtenberg on his retirement

We analyze the consistency strength of regularity on the basis of Feferman's operational set theory OST. Our main result shows that regularity over OST for operations corresponds to regularity with respect to set-theoretic functions in frameworks like Kripke-Platek set theory. As a consequence, we obtain that OST plus the operational axiom (Inac) is consistency-equivalent to Kripke-Platek set theory with infinity extended by the strong limit axiom (SLim) stating that any ordinal is majorized by a functionally regular ordinal. Thus OST + (Inac) is significantly stronger than originally expected.

1 Introduction

Operational set theory OST has been introduced in Feferman [9] and further discussed, from various perspectives, in Cantini [5], Cantini and Corosilla [6, 7], Feferman [10], and Jäger [13, 14, 15]. The basic theory OST is proof-theoretically equivalent to Kripke-Platek set theory KP with infinity (cf. Feferman [9, 10] and Jäger [13]). On the other hand, OST plus unbounded existential quantification and power set has the same consistency strength as von Neumann-Bernays-Gödel set theory NBG plus $\in$-induction for arbitrary formulas and a class version of Σ^1_1 choice (cf. Jäger [14] and Jäger and Krähenbühl [16]). Also, there exists a natural subsystem of OST with unbounded existential quantification and power set which is conservative over ZFC (cf. Jäger [13]).

The program of operational set theory is described in detail in Feferman [10] and summarized in the abstract of this article as follows: *A new axiomatic system* OST *of operational set theory is introduced in which the usual language of set theory is expanded to allow us to talk about (possibly partial) operations applicable both to sets and to operations.* OST *is equivalent in strength to admissible set theory, and a natural extension of* OST *is equivalent in strength to* ZFC. *The language of* OST *provides a framework in which to express "small" large cardinal notions – such as those of being an inaccessible cardinal, a Mahlo cardinal, and a weakly* *compact cardinal – in terms of operational closure conditions that specialize to the analogue notions on admissible sets. This illustrates a wider program whose aim is to provide a common framework for analogues of large cardinal notions that have appeared in admissible set theory, admissible recursion theory, constructive set theory, constructive type theory, explicit mathematics, and systems of recursive ordinal notations that have been used in proof theory.*

In this paper we concentrate on the simplest form of small large cardinals axioms, confine ourselves to operational regularity and the operational axiom (Inac) which claims that any ordinal is majorized by an operationally regular ordinal, prove some conjectures made in the literature, and disprove others. Our main result states that operational regularity corresponds in strength to regularity with respect to set-theoretic functions. As it turns out, sets L_κ, with κ a functionally regular ordinal, have strong closure properties and satisfy, for example, separation for arbitrary first order formulas.

From this analysis of operational regularity we deduce that $\mathsf{OST} + (\mathsf{Inac})$ is equiconsistent with KP plus the axiom (SLim) which claims that any ordinal is majorized by a functionally regular ordinal. Hence we also know that $\mathsf{OST} + (\mathsf{Inac})$ is very strong, certainly going (in strength) beyond full second order arithmetic.

2 OST and operational regularity

In this section we briefly recapitulate the syntax of Feferman's OST and some of its properties needed below. Then we turn to the formulation of operational regularity, as proposed in Feferman [9, 10]. We follow Jäger [13, 14] very closely and even use the same formulations whenever it seems adequate.

Let $\mathcal{L}$ be a typical language of first order set theory with countably many set variables $a, b, c, f, g, u, v, w, x, y, z, \dots$ (possibly with subscripts) and a symbol for the element relation as its only relation symbol. In addition, we have the constant ω for the first infinite ordinal. The formulas of $\mathcal{L}$ are defined as usual.

$\mathcal{L}^\circ$, the language of OST, augments $\mathcal{L}$ by the binary function symbol $\circ$ for partial term application, the unary relation symbol $\downarrow$ (defined), and the following constants: (i) the combinators $\mathbf{k}$ and $\mathbf{s}$; (ii) $\top$, $\bot$, $\mathbf{el}$, $\mathbf{non}$, $\mathbf{dis}$, and $\mathbf{e}$ for logical operations; (iii) $\mathbb{S}$, $\mathbb{R}$, and $\mathbb{C}$ for set-theoretic operations. The meaning of these constants follows from the axioms below.

The *terms* $(r, s, t, r_1, s_1, t_1, \dots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. The variables and constants of $\mathcal{L}^\circ$ are terms of $\mathcal{L}^\circ$.

2. If s and t are terms of $\mathcal{L}^\circ$, then so is $\circ(s, t)$.

In the following we often abbreviate $\circ(s,t)$ as $(s \circ t)$, (st), or simply as st. We also adopt the convention of association to the left so that $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2)\ldots s_n)$. In addition, we often write $s(t_1, \ldots, t_n)$ for $st_1 \ldots t_n$ if this seems more intuitive. Moreover, we frequently make use of the vector notation $\vec{s}$ as shorthand for a finite string $s_1, \ldots, s_n$ of $\mathcal{L}^\circ$ terms whose length is either not important or evident from the context.

Self-application is possible and meaningful, but not necessarily total, and there may be terms not denoting an object. We make use of the definedness predicate $\downarrow$ to single out those which do, and $(t\downarrow)$ is read "t is defined" or "t has a value".

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. All expressions of the form $(s \in t)$ and $(t\downarrow)$ are formulas of $\mathcal{L}^\circ$, the so-called *atomic* formulas.

2. If A and B are formulas of $\mathcal{L}^\circ$, then so are $\neg A$, $(A \vee B)$ and $(A \wedge B)$.

3. If A is a formula of $\mathcal{L}^\circ$ and if t is a term of $\mathcal{L}^\circ$ which does not contain x, then $(\exists x \in t)A$, $(\forall x \in t)A$, $\exists x A$ and $\forall x A$ are formulas of $\mathcal{L}^\circ$.

We will be working within classical logic so that the remaining logical connectives can be defined as usual. Parentheses and brackets are often omitted whenever there is no danger of confusion. The free variables of t and A are defined in the conventional way; the closed $\mathcal{L}^\circ$ terms and closed $\mathcal{L}^\circ$ formulas, also called $\mathcal{L}^\circ$ sentences, are those which do not contain free variables. Equality of sets is introduced by

$$(s = t) \ := \ (s\downarrow) \ \wedge \ (t\downarrow) \ \wedge \ \forall x(x \in s \ \leftrightarrow \ x \in t).$$

Given an $\mathcal{L}^\circ$ formula A and a variable u not occurring in A, we write A^u for the result of replacing each unbounded set quantifier $\exists x(\ldots)$ and $\forall x(\ldots)$ in A by $(\exists x \in u)(\ldots)$ and $(\forall x \in u)(\ldots)$, respectively. Suppose now that $\vec{u} = u_1, \ldots, u_n$ and $\vec{s} = s_1, \ldots, s_n$. Then $A[\vec{s}/\vec{u}]$ is the $\mathcal{L}^\circ$ formula which is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the $\mathcal{L}^\circ$ terms $\vec{s}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. If the $\mathcal{L}^\circ$ formula A is written as $B[\vec{u}]$, we often simply write $B[\vec{s}]$ instead of $B[\vec{s}/\vec{u}]$. Further variants of this notation will be obvious.

The logic of OST is the classical *logic of partial terms* due to Beeson and Feferman with the usual strictness axioms (cf. Beeson [2, 3]), including the common equality axioms. Partial equality of terms is introduced by

$$(s \simeq t) \ := \ (s\downarrow \vee\, t\downarrow \ \rightarrow \ s = t)$$

and says that if either s or t denotes anything, then they both denote the same object.

The non-logical axioms of OST comprise axioms about the applicative structure of the universe, some basic set-theoretic properties, the representation of elementary logical connectives as operations, and operational set existence axioms. They divide into four groups.

I. Applicative axioms.

(A1) $\mathbf{k} \neq \mathbf{s}$,

(A2) $\mathbf{k}xy = x$,

(A3) $\mathbf{s}xy\downarrow \wedge \mathbf{s}xyz \simeq (xz)(yz)$.

Thus the universe is a partial combinatory algebra. We have λ-abstraction and thus can introduce for each $\mathcal{L}^\circ$ term t a term $\lambda x.t$ whose variables are those of t other than x such that

$$\lambda x.t\downarrow \ \wedge \ (\lambda x.t)y \simeq t[y/x].$$

Furthermore, there exists a closed $\mathcal{L}^\circ$ term **fix**, a so-called fixed point operator, with

$$\mathbf{fix}(f)\downarrow \ \wedge \ (\mathbf{fix}(f) = g \ \rightarrow \ gx \simeq f(g, x)).$$

II. Basic set-theoretic axioms. They comprise: (i) the existence of the empty set; (ii) pair, union and infinity; (iii) $\in$-induction is available for arbitrary formulas $A[x]$ of $\mathcal{L}^\circ$,

$$\forall x((\forall y \in x)A[y] \rightarrow A[x]) \ \rightarrow \ \forall x A[x]. \qquad (\mathcal{L}^\circ\text{-}\mathsf{I}_\in)$$

To increase readability, we will freely use standard set-theoretic terminology, for example,

$$\mathit{Tran}[a] \ := \ (\forall x \in a)(x \subseteq a) \quad \text{and} \quad \mathit{Ord}[a] \ := \ \mathit{Tran}[a] \wedge (\forall x \in a)\,\mathit{Tran}[x].$$

Also, if $A[x]$ is an $\mathcal{L}^\circ$ formula, then $\{x : A[x]\}$ denotes the collection of all sets satisfying A; it may be (extensionally equal to) a set, but this is not necessarily the case. In particular, we set

$$\mathbf{V} \ := \ \{x : x\downarrow\} \quad \text{and} \quad \mathbf{B} \ := \ \{x : x = \top \vee x = \bot\}$$

so that $\mathbf{V}$ denotes the collection of all sets, but is not a set itself, and $\mathbf{B}$ stands for the unordered pair consisting of the truth values $\top$ and $\bot$, which is a set by the previous axioms. The following shorthand notations, for n an arbitrary natural number,

$$
\begin{aligned}
(f : a \to b) \quad &:= \quad (\forall x \in a)(fx \in b), \\
(f : a^{n+1} \to b) \quad &:= \quad (\forall x_1, \ldots, x_{n+1} \in a)(f(x_1, \ldots, x_{n+1}) \in b)
\end{aligned}
$$

express that f, in the operational sense, is a unary and $(n{+}1)$-ary mapping from a to b, respectively. They do not say, however, that f is a unary or $(n{+}1)$-ary function in the set-theoretic sense (see below). In this definition the set variables a and/or b may be replaced by $\mathbf{V}$ and/or $\mathbf{B}$. So, for example, $(f : a \to \mathbf{V})$ means that f is total on a, and $(f : \mathbf{V} \to b)$ means that f maps all sets into b.

III. Logical operations axioms.

(L1) $\top \neq \bot$,

(L2) $(\mathbf{el} : \mathbf{V}^2 \to \mathbf{B}) \wedge \forall x \forall y(\mathbf{el}(x, y) = \top \leftrightarrow x \in y)$,

(L3) $(\mathbf{non} : \mathbf{B} \to \mathbf{B}) \wedge (\forall x \in \mathbf{B})(\mathbf{non}(x) = \top \leftrightarrow x = \bot)$,

(L4) $(\mathbf{dis} : \mathbf{B}^2 \to \mathbf{B}) \wedge (\forall x, y \in \mathbf{B})(\mathbf{dis}(x, y) = \top \leftrightarrow (x = \top \vee y = \top))$,

(L5) $(f : a \to \mathbf{B}) \to (\mathbf{e}(f, a) \in \mathbf{B} \wedge (\mathbf{e}(f, a) = \top \leftrightarrow (\exists x \in a)(fx = \top)))$.

The Δ_0 *formulas* of $\mathcal{L}^\circ$ are those $\mathcal{L}^\circ$ formulas which do not contain the function symbol $\circ$, the relation symbol $\downarrow$ or unbounded quantifiers. Hence they are the usual Δ_0 formulas of set theory, possibly containing additional constants. The logical operations make it possible to represent all Δ_0 formulas by constant $\mathcal{L}^\circ$ terms.

Lemma 1. *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every Δ_0 formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term t_A such that the axioms introduced so far yield*

$$t_A\downarrow \wedge (t_A : \mathbf{V}^n \to \mathbf{B}) \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

For a proof of this lemma see Feferman [9, 10]. Now we turn to the operational versions of separation, replacement, and choice.

IV. Set-theoretic operations axioms.

(S1) Separation for definite operations:

$$(f : a \to \mathbf{B}) \to (\mathbb{S}(f, a)\downarrow \wedge \forall x(x \in \mathbb{S}(f, a) \leftrightarrow (x \in a \wedge fx = \top))).$$

(S2) Replacement:

$$(f : a \to \mathbf{V}) \;\to\; (\mathbb{R}(f,a){\downarrow} \,\land\, \forall x(x \in \mathbb{R}(f,a) \;\leftrightarrow\; (\exists y \in a)(x = fy))).$$

(S3) Choice:

$$\exists x(fx = \top) \;\to\; (\mathbb{C}f{\downarrow} \,\land\, f(\mathbb{C}f) = \top).$$

This finishes our description of the non-logical axioms of OST. From Feferman [9] and Jäger [13] we know that, provably in OST, there exist closed $\mathcal{L}^\circ$ terms $\emptyset$ for the empty set, $\mathbf{uopa}$ for forming unordered pairs, $\mathbf{un}$ for forming unions, $\mathbf{p}$ for forming ordered pairs (Kuratowski pairs) and $\mathbf{prod}$ for forming Cartesian products. In addition, there are closed $\mathcal{L}^\circ$ terms $\mathbf{p}_L$ and $\mathbf{p}_R$ which act as projections with respect to $\mathbf{p}$, i.e.

$$\mathbf{p}_L(\mathbf{p}(a,b)) = a \quad \text{and} \quad \mathbf{p}_R(\mathbf{p}(a,b)) = b.$$

To comply with the set-theoretic conventions, we generally write $\{a,b\}$ instead of $\mathbf{uopa}(a,b)$, $\cup a$ instead of $\mathbf{un}(a)$, $\langle a,b\rangle$ instead of $\mathbf{p}(a,b)$ and $a \times b$ instead of $\mathbf{prod}(a,b)$. Remember that ω is a constant for the first infinite ordinal and belongs to the base language $\mathcal{L}$.

We end this section with a few remarks concerning the relationship between functions in the set-theoretic sense and operations in the sense of our form of term application. Similar questions for similar operational set theories are discussed in Beeson [4] and in Cantini and Crosilla [6].

It is well-known (see, for example, Barwise [1]) that there are Δ_0 formulas $\mathit{Rel}[a]$ and $\mathit{Fun}[a]$ of our basic language $\mathcal{L}$, stating that the set a is a binary relation and function, respectively, in the typical set-theoretic sense. It can also be expressed in Δ_0 form that a is a relation with domain b, abbreviated as $\mathit{Dom}[a] = b$, and that a is a relation with range b, abbreviated as $\mathit{Ran}[a] = b$. If $\mathit{Fun}[a]$ holds and u belongs to the domain of a we often write $a\,'u$ for the unique v such that $\langle u,v\rangle \in a$.

Lemma 2. *There exist closed $\mathcal{L}^\circ$ terms* $\mathbf{dom}$, $\mathbf{ran}$, $\mathbf{op}$, *and* $\mathbf{fun}$ *so that* OST *proves the following assertions:*

1. $\mathbf{dom}(f){\downarrow} \,\land\, \mathbf{ran}(f){\downarrow} \,\land\, \mathbf{op}(f){\downarrow}$.
2. $\mathit{Rel}[a] \;\to\; (\mathit{Dom}[a] = \mathbf{dom}(a) \,\land\, \mathit{Ran}[a] = \mathbf{ran}(a))$.
3. $(\mathit{Fun}[f] \,\land\, a \in \mathbf{dom}(f)) \;\to\; f\,'a = \mathbf{op}(f,a)$.
4. $(f : a \to \mathbf{V}) \;\to\; (\mathit{Fun}[\mathbf{fun}(f,a)] \,\land\, \mathit{Dom}[\mathbf{fun}(f,a)] = a)$.
5. $(f : a \to \mathbf{V}) \;\to\; (\forall x \in a)(\mathbf{fun}(f,a)\,'x = fx)$.

This lemma, whose proof can be found in Feferman [9, 10] as well, implies that: (i) each set-theoretic function can be translated into an operation acting on the same domain and yielding the same values; (ii) to each operation total on a set a corresponds a set-theoretic function with domain a so that the values of this operation and of this function on a agree.

We use the lower case Greek letters $\alpha, \beta, \gamma, \kappa, \lambda, \zeta, \eta, \xi, \ldots$ (possibly with subscripts) to range over the ordinals; also, as customary in the context of ordinals, we write 0 instead of $\emptyset$. Following the usual set-theoretic definition of regularity as closely as possible, Feferman [9, 10] suggests the following formulation of operational regularity.

Definition 3. *Ordinal κ is called* operationally regular*, in symbols $Org[\kappa]$, if and only if $\omega < \kappa$ and*

$$\forall f (\forall \alpha < \kappa)((f : \alpha \to \kappa) \;\to\; (\exists \beta < \kappa)(f : \alpha \to \beta)).$$

This definition also is meaningful in ZFC if we let the variable f range over ordinary set-theoretic functions. Under this interpretation $Org[\kappa]$ implies that κ is identical to its cofinality and thus regular in the usual set-theoretic sense.

Clearly, the existence of operationally regular ordinals cannot be proved in OST. However, Feferman [10] suggests to consider the axiom

$$\forall \alpha \exists \beta (\alpha < \beta \;\wedge\; Org[\beta]), \tag{Inac}$$

stating that any ordinal is majorized by an operationally regular ordinal. It is conjectured in Feferman [10] that $\mathsf{OST} + (\mathsf{Inac})$ is consistency equivalent to the theory KPi of iterated admissible sets – see (for example) Jäger [12] for a detailed description of this system – which describes a recursively inaccessible universe.

It will turn out, however, that $\mathsf{OST} + (\mathsf{Inac})$ is dramatically stronger than KPi. To give a precise characterization of $\mathsf{OST} + (\mathsf{Inac})$ we show it to be equivalent to Kripke-Platek set theory with infinity plus an axiom stating that any ordinal is smaller than a so-called functionally regular ordinal.

3 The theories KP and KP + (SLim)

We begin this section with briefly recalling the system KP of Kripke-Platek set theory with infinity. Then we turn to the notion of a functionally regular ordinal and say something about the closure properties of the constructible sets L_κ for κ being functionally regular. We end this section with a theorem about specific inductive definitions, which later will be used to model OST. For further reading about KP, its proof-theoretic analysis and some interesting subsystems and extensions consult, for example, Jäger [11, 12] and Rathjen [18].

KP is formulated in our basic language $\mathcal{L}$ with $\in$ as its only relation symbol and equality of sets defined by

$$(a = b) \; := \; (\forall x \in a)(x \in b) \; \wedge \; (\forall x \in b)(x \in a).$$

The collections of Δ_0, Σ, and Π formulas are introduced as usual. If T is a theory in $\mathcal{L}$ containing KP and A a formula of $\mathcal{L}$, then A is Δ over T if there is a Σ formula B and a Π formula C, both with the same free variables as A, such that T proves the equivalence of A and B plus that of A and C. Also, as in the case of OST, we make use of other standard set-theoretic terminology.

The underlying logic of KP is classical first order logic with equality, its non-logical axioms are: pair, union, infinity (i.e. the assertion that ω is the least infinite ordinal), Δ_0 separation and Δ_0 collection, i.e.

$$\exists x(x = \{y \in a : B[y]\}), \qquad (\Delta_0\text{-Sep})$$

$$(\forall x \in a)\exists y C[x, y] \; \rightarrow \; \exists z(\forall x \in a)(\exists y \in z)C[x, y] \qquad (\Delta_0\text{-Col})$$

for arbitrary Δ_0 formulas $B[u]$ and $C[u, v]$ of $\mathcal{L}$, as well as $\in$-induction for arbitrary formulas $A[x]$ of $\mathcal{L}$,

$$\forall x((\forall y \in x)A[y] \rightarrow A[x]) \; \rightarrow \; \forall x A[x]. \qquad (\mathcal{L}\text{-I}_\in)$$

Clearly, the formula $Ord[a]$, which says that a is an ordinal, is a Δ_0 formula of $\mathcal{L}$, and we use the lower case Greek letters $\alpha, \beta, \gamma, \kappa, \lambda, \zeta, \eta, \xi$ (possibly with subscripts) to range over the ordinals, as we do in OST. In the following we will be working within the constructible universe, but cannot introduce it here. Most relvant details can be found, for example, in Barwise [1] or Kunen [17].

Very briefly, $(a \in L_\alpha)$ states that the set a is an element of the αth level L_α of the constructible hierarchy and $a \in \mathbf{L}$ is short for $\exists \alpha(a \in L_\alpha)$. Given a set $a \in \mathbf{L}$, we write $rk_{\mathbf{L}}[a]$ for the least ordinal α such that $a \in L_{\alpha+1}$ and $(a <_{\mathbf{L}} b)$ means that a is smaller than b according to the well-ordering $<_{\mathbf{L}}$ on the constructible universe $\mathbf{L}$. The *axiom of constructibility* is the statement $(\mathbf{V}{=}\mathbf{L})$, i.e. $\forall x \exists \alpha(x \in L_\alpha)$. It is well-known that the assertions $(a \in L_\alpha)$ and $(a <_{\mathbf{L}} b)$ are Δ over KP and that the systems KP and KP + $(\mathbf{V}{=}\mathbf{L})$ are of the same consistency strength; both systems prove the same absolute sentences.

Now we turn to the more common set-theoretic variant of operational regularity obtained by working with set-theoretic functions rather than operations. To make this distinction also notationally clear, we speak of functional regularity in this case.

Definition 4. *Ordinal κ is called* functionally regular*, in symbols $Frg[\kappa]$, if and only if $\omega < \kappa$ and*

$$\forall f(\forall \alpha < \kappa)(Fun[f] \wedge Dom[f] = \alpha \wedge Ran[f] \subseteq \kappa \;\rightarrow\; (\exists \beta < \kappa)(Ran[f] \subseteq \beta)).$$

Functionally regular ordinals κ and the corresponding sets L_κ have fairly strong closure properties. We will discuss some of those, and in doing so, frequently make use of notions and results of the theory of admissible sets as presented, for example, in Barwise [1]. We begin with an immediate observation.

Lemma 5. KP *proves that any functionally regular ordinal is a limit ordinal.*

The next remark is about a property of the constructible hierarchy which is convenient for proving Lemma 7 below. For a proof we refer to Kunen [17], verifying that all arguments work for the restricted framework of $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$.

Lemma 6. $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves for any limit ordinal $\alpha \geq \omega$ that there exists a bijective set-theoretic function from α to L_α.*

We are now ready lift the regularity property from functionally regular ordinals κ to the corresponding constructible sets L_κ.

Lemma 7. *Provable in $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$, we have for all functionally regular ordinals κ that*

$$a \in L_\kappa \wedge Fun[f] \wedge Dom[f] = a \wedge Ran[f] \subseteq L_\kappa \;\rightarrow\; (\exists b \in L_\kappa)(Ran[f] \subseteq b).$$

Proof. Assume that $a \in L_\kappa$ and f is a set-theoretic function from a to L_κ. Then there exists an α such that $\omega < \alpha < \kappa$ and $a \subseteq L_\alpha$. We apply the previous lemma and let g be a set-theoretic bijection from α to L_α. Now we introduce the set-theoretic function h from α to κ with

$$h'\xi \;=\; \begin{cases} rk_{\mathbf{L}}[f'(g'\xi)] & \text{if } g'\xi \in a, \\ 0 & \text{if } g'\xi \notin a \end{cases}$$

for all $\xi < \alpha$. We see immediately that h is a set-theoretic function from α to κ. Hence the functional regularity of κ provides us with an ordinal $\beta < \kappa$ for which $Ran[h] \subseteq \beta$. Hence $Ran[f] \subseteq L_\beta$. □

Lemma 8. $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves that for any functionally regular κ the set L_κ is admissible.*

Proof. Let κ be functionally regular. Then κ is a limit ordinal greater than ω, and so, apart from Δ_0 collection, all axioms of KP are trivially satisfied by L_κ. To treat Δ_0 collection, let $A[v,w]$ be a Δ_0 formula of $\mathcal{L}$, posssibly with additional parameters, and assume

$$a \in L_\kappa \ \wedge \ (\forall x \in a)(\exists y \in L_\kappa) A[x,y].$$

Now let f be the set-theoretic function from a to L_κ with

$$f\,'x \ = \ \min(\{\xi < \kappa : (\exists y \in L_\xi) A[x,y]\})$$

for all $x \in a$. The previous lemma tells us that there is a $b \in L_\kappa$ for which $Ran[f] \subseteq b$. For $\beta := \sup(\{\xi : \xi \in b\})$ we thus have $\beta < \kappa$, thus $L_\beta \in L_\kappa$, and

$$(\forall x \in a)(\exists y \in L_\beta) A[x,y].$$

Hence L_β is a possible witness as required for satisfying Δ_0 collection. □

The sets L_κ with κ being functionally regular have even much stronger closure properties. In particular, as we will see below, that they satisfy full separation. Before turning to this theorem, we prove a useful lemma.

Lemma 9. *For all $\mathcal{L}$ formulas $A[\vec{u},v,w]$ with all its free variables in $\vec{u},v,w$, the theory* KP + $(\mathbf{V}{=}\mathbf{L})$ *proves*

$$\begin{aligned} &Frg[\kappa] \ \wedge \ a, \vec{u} \in L_\kappa \ \rightarrow \\ &\qquad (\exists b \in L_\kappa)(\forall x \in a)((\exists y \in L_\kappa) A^{L_\kappa}[\vec{u},x,y] \ \leftrightarrow \ (\exists y \in b) A^{L_\kappa}[\vec{u},x,y]). \end{aligned}$$

Proof. If z is a non-empty set, we write $\min_{\mathbf{L}}[z]$ for the with respect to the well-ordering $<_{\mathbf{L}}$ least element of z. Given a functionally regular κ and $a, \vec{u} \in L_\kappa$, we consider the function f from a to L_κ defined by, for $x \in a$,

$$f\,'x \ := \ \begin{cases} \min_{\mathbf{L}}[\{z \in L_\kappa : A^{L_\kappa}[\vec{u},x,z]\}] & \text{if } (\exists y \in L_\kappa) A^{L_\kappa}[\vec{u},x,y], \\ \emptyset & \text{otherwise.} \end{cases}$$

Since κ is functionally regular, Lemma 7 implies the existence of a set $b \in L_\kappa$ such that $Ran[f] \subseteq b$. Consequently,

$$(\forall x \in a)((\exists y \in L_\kappa) A^{L_\kappa}[\vec{u},x,y] \rightarrow (\exists y \in b) A^{L_\kappa}[\vec{u},x,y]).$$

Since the converse of this implication is obvious, our lemma is proved. □

If $\vec{x} = x_1, \ldots, x_n$ and $\vec{a} = a_1, \ldots, a_n$, we often write $\vec{x} \in \vec{a}$ and $\langle \vec{x} \rangle \in \times(\vec{a})$ instead of $(x_1 \in a_1 \wedge \ldots \wedge x_n \in a_n)$ and $\langle x_1, \ldots, x_n \rangle \in a_1 \times \ldots \times a_n$, respectively.

Theorem 10. *For all $\mathcal{L}$ formulas $A[\vec{u}, \vec{v}]$ with all its free variables in $\vec{u}, \vec{v}$, the theory* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves*

$$Frg[\kappa] \ \rightarrow \ (\forall \vec{a} \in L_\kappa)(\forall \vec{x} \in L_\kappa)(\exists b \in L_\kappa)(b = \{\langle \vec{y} \rangle \in \times(\vec{a}) : A^{L_\kappa}[\vec{x}, \vec{y}]\}).$$

Proof. We show this assertion by induction on the formula $A[\vec{u}, \vec{v}]$. If $A[\vec{u}, \vec{v}]$ is an atomic formula, a negation, a disjuction, a conjunction, or a formula beginning with a bounded quantifier, then the assertion is obvious or an immediate consequence of the induction hypothesis. This leaves the two interesting cases of $A[\vec{u}, \vec{v}]$ beginning with an unbounded quantifier.

Assume that $A[\vec{u}, \vec{v}]$ is of the form $\exists z B[\vec{u}, \vec{v}, z]$. For any functionally regular κ and $\vec{a}, \vec{x} \in L_\kappa$ the previous lemma gives us a set $c \in L_\kappa$ such that

$$(\forall \vec{y} \in \vec{a})((\exists z \in L_\kappa) B^{L_\kappa}[\vec{x}, \vec{y}, z] \leftrightarrow (\exists z \in c) B^{L_\kappa}[\vec{x}, \vec{y}, z]). \qquad (*)$$

By the induction hypothesis there exists a set $b_0 \in L_\kappa$ such that

$$b_0 \ = \ \{\langle \vec{y}, z \rangle \in \times(\vec{a}, c) : B^{L_\kappa}[\vec{x}, \vec{y}, z]\},$$

and with this b_0 as parameter we define by Δ_0 separation in L_κ the set

$$b \ := \ \{\langle \vec{y} \rangle \in \times(\vec{a}) : (\exists z \in c)(\langle \vec{y}, z \rangle \in b_0)\}.$$

Clearly, $b \in L_\kappa$, and for all $\vec{y}$ we have because of (*) that

$$\begin{aligned} \langle \vec{y} \rangle \in b \quad &\leftrightarrow \quad \vec{y} \in \vec{a} \wedge (\exists z \in c) B^{L_\kappa}[\vec{x}, \vec{y}, z] \\ &\leftrightarrow \quad \vec{y} \in \vec{a} \wedge (\exists z \in L_\kappa) B^{L_\kappa}[\vec{x}, \vec{y}, z] \\ &\leftrightarrow \quad \vec{y} \in \vec{a} \wedge A^{L_\kappa}[\vec{x}, \vec{y}]. \end{aligned}$$

Therefore b is a set in L_κ as required. The remaining case of $A[\vec{u}, \vec{v}]$ beginning with an unbounded universal quantifier is left to the reader. □

In view of this theorem we can deduce that, within $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$, for any functionally regular κ the set L_κ is a standard model of the theory ZFC^-, the subsystem of ZFC without the power set axiom.

The referee has pointed out that Lemma 6 can be proved in KP alone, making use of Lemma II.6.7 of Devlin [8]. Clearly, $(\mathbf{V}{=}\mathbf{L})$ is not used elsewhere in the proofs of the following Lemmas 7–9 and in the proof of Theorem 10. Hence the extra hypothesis $(\mathbf{V}{=}\mathbf{L})$ can be eliminated there and, as a consequence, KP is sufficient to prove that for any functionally regular κ, L_κ is a model of ZFC^-.

Now we add to Kripke-Platek set theory the strong limit axiom which states that any ordinal is majorized be a functionally regular ordinal,

$$\forall \alpha \exists \beta (\alpha < \beta \wedge Frg[\beta]) \qquad (\mathsf{SLim})$$

and write KPS for the theory KP + (SLim). Observe that the ordinals of KPS form an admissible limit of functionally regular ordinals, but are not necessarily functionally regular.

For any $\mathcal{L}$ formula A we write $A^{\mathbf{L}}$ for the $\mathcal{L}$ formula obtained from A by replacing all unbounded quantifiers $\exists x(\ldots)$ and $\forall x(\ldots)$ by $\exists x(x \in \mathbf{L} \land \ldots)$ and $\forall x(x \in \mathbf{L} \to \ldots)$, respectively. As the following theorem states, $\mathbf{L}$ is an inner model of KPS.

Theorem 11. *For the universal closure A of any axiom of* KPS *we have that*

$$\mathsf{KPS} \vdash A^{\mathbf{L}}.$$

Proof. It is a folklore result in admissible set theory that KP proves $A^{\mathbf{L}}$ for the universal closure of any axiom of KP. Hence we can confine ourselves to verifying $(\mathsf{SLim})^{\mathbf{L}}$ in KPS.

To do so, let α be an arbitrary ordinal. Then (SLim) implies the existence of a functionally regular ordinal β which contains α. Trivially, this implies that β is functionally regular in $\mathbf{L}$, i.e. $\mathit{Frg}^{\mathbf{L}}[\beta]$. Thus we have $(\mathsf{SLim})^{\mathbf{L}}$. □

Hence the usual inner model considerations yield the following result about the equiconsistency of KPS and KPS + $(\mathbf{V}{=}\mathbf{L})$.

Corollary 12. *The theory* KPS *and its extension* KPS$+(\mathbf{V}{=}\mathbf{L})$ *are equiconsistent; both systems prove the same absolute formulas.*

For establishing the equiconsistency of OST + (Inac) and KPS it is thus sufficient to embed KPS into OST + (Inac) and to reduce OST + (Inac) to the theory KPS + $(\mathbf{V}{=}\mathbf{L})$.

We end this section with a specific form of Σ recursion over the universe which helps us in modeling OST+(Inac) within KPS+$(\mathbf{V}{=}\mathbf{L})$. Let R be a fresh $(n{+}1)$-ary relation symbol and extend $\mathcal{L}$ to the language $\mathcal{L}(R)$ with expressions $R(\alpha, \vec{a})$ as additional atomic formulas. Given a formula $A[R]$ of $\mathcal{L}(R)$ and another formula $B[\vec{a}]$ of $\mathcal{L}$ with distinguished free variables $\vec{a}$, we write $A[B[.]]$ for the result of substituting $B[\vec{s}]$ for each occurrence of the form $R(\vec{s})$ in $A[R]$, renaming bound variables as necessary to avoid collision.

Theorem 13 (Σ recursion)**.** *Let R be a fresh n-ary relation symbol and $A[R, \alpha, \vec{a}]$ a formula of $\mathcal{L}(R)$ with distinguished free variables $\vec{a} = a_1, \ldots, a_n$ which is Δ over* KP. *Then there exists a Σ formula $B[\alpha, \vec{a}]$ of $\mathcal{L}$ such that* KP *proves*

$$B[\alpha, \vec{a}] \;\leftrightarrow\; (\vec{a} \in L_\alpha \land A[(\exists \xi < \alpha)B[\xi, .], \alpha, \vec{a}]).$$

Proof. We proceed similarly to the proof of the theorem about definition by Σ recursion in Barwise [1]. Let $n = 1$ to simplify notation and let $C[f, \alpha, b]$ be the $\mathcal{L}$ formula given by the conjunction of

(i) $Fun[f] \land Dom[f] = \alpha$,

(ii) $(\forall \beta < \alpha)(f{'}\beta = \{x \in L_\beta : A[\bigcup_{\xi<\beta} f{'}\xi, \beta, x]\})$,

(iii) $b = \{x \in L_\alpha : A[\bigcup_{\xi<\alpha} f{'}\xi, \alpha, x]\}$.

Clearly, $C[f, \alpha, a]$ is Δ over KP, and we let $D[f, \alpha, a]$ be the Σ formula provably equivalent to $C[f, \alpha, a]$ in KP. By transfinite induction on α we can then show that

$$D[f, \alpha, b] \land D[g, \alpha, c] \rightarrow f = g \land b = c, \tag{1}$$

$$\exists f \exists b D[f, \alpha, b]. \tag{2}$$

The proof of (1) is straightforward, the proof of (2) requires Σ reflection or Σ replacement, which both follow (in KP) from Δ_0 collection. Now set

$$B[\alpha, a] := \exists f \exists b(D[f, \alpha, b] \land a \in b).$$

Obviously, $B[\alpha, a]$ is a Σ formula, and it is now easy to check that it has the property required in our theorem. □

4 Modeling OST + (Inac) within KPS + ($\mathbf{V}$=$\mathbf{L}$)

The following model construction is a modification of that in Jäger [13]. We begin with the similar notational preliminaries:

- For any natural number n greater than 0 and any natural number i we select a Δ_0 formulas $Tup_n(a)$ and $(a)_i = b$ formalizing that a is an ordered n-tuple and b the projection of a on its ith component; hence $Tup_n(\langle a_0, \dots, a_{n-1}\rangle)$ and $(\langle a_0, \dots, a_{n-1}\rangle)_i = a_i$ for $0 \le i \le n-1$.

- Then we fix pairwise different constructible sets $\widehat{\mathbf{k}}$, $\widehat{\mathbf{s}}$, $\widehat{\top}$, $\widehat{\bot}$, $\widehat{\mathbf{el}}$, $\widehat{\mathbf{non}}$, $\widehat{\mathbf{dis}}$, $\widehat{\mathbf{e}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$, and $\widehat{\mathbb{C}}$ making sure that they all do not belong to the collection of ordered pairs and triples; they will later act as the codes of the corresponding constants of $\mathcal{L}^\circ$.

We are going to code the $\mathcal{L}^\circ$ terms $\mathbf{k}x$, $\mathbf{s}x$, $\mathbf{s}xy$, ... by the ordered tuples $\langle \widehat{\mathbf{k}}, x\rangle$, $\langle \widehat{\mathbf{s}}, x\rangle$, $\langle \widehat{\mathbf{s}}, x, y\rangle$, ... of the corresponding form. For example, to satisfy $\mathbf{k}xy = x$ we interpret $\mathbf{k}x$ as $\langle \widehat{\mathbf{k}}, x\rangle$, and "$\langle \widehat{\mathbf{k}}, x\rangle$ applied to y" is taken to be x.

Next let R be a fresh 3-place relation symbol and extend $\mathcal{L}$ to the language $\mathcal{L}(R)$ as above. The following definition introduces the $\mathcal{L}(R)$ formula $\mathfrak{A}[R, \alpha, a, b, c]$ which will lead to the interpretation of the application relation $(ab = c)$.

Definition 14. *We choose $\mathfrak{A}[R, \alpha, a, b, c]$ to be the $\mathcal{L}(R)$ formula defined as the disjunction of the following formulas (1) – (22):*

(1) $a = \widehat{\mathbf{k}} \ \wedge \ c = \langle \widehat{\mathbf{k}}, b \rangle$,

(2) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{k}} \ \wedge \ (a)_1 = c$,

(3) $a = \widehat{\mathbf{s}} \ \wedge \ c = \langle \widehat{\mathbf{s}}, b \rangle$,

(4) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{s}} \ \wedge \ c = \langle \widehat{\mathbf{s}}, (a)_1, b \rangle$,

(5) $\mathit{Tup}_3(a) \ \wedge \ (a)_0 = \widehat{\mathbf{s}} \ \wedge$
$(\exists x, y \in L_\alpha)(R((a)_1, b, x) \ \wedge \ R((a)_2, b, y) \ \wedge \ R(x, y, c))$,

(6) $a = \widehat{\mathbf{el}} \ \wedge \ c = \langle \widehat{\mathbf{el}}, b \rangle$,

(7) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{el}} \ \wedge \ (a)_1 \in b \ \wedge \ c = \widehat{\top}$,

(8) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{el}} \ \wedge \ (a)_1 \notin b \ \wedge \ c = \widehat{\bot}$,

(9) $a = \widehat{\mathbf{non}} \ \wedge \ b = \widehat{\top} \ \wedge \ c = \widehat{\bot}$,

(10) $a = \widehat{\mathbf{non}} \ \wedge \ b = \widehat{\bot} \ \wedge \ c = \widehat{\top}$,

(11) $a = \widehat{\mathbf{dis}} \ \wedge \ c = \langle \widehat{\mathbf{dis}}, b \rangle$,

(12) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{dis}} \ \wedge \ (a)_1 = \widehat{\top} \ \wedge \ c = \widehat{\top}$,

(13) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{dis}} \ \wedge \ (a)_1 = \widehat{\bot} \ \wedge \ b = \widehat{\top} \ \wedge \ c = \widehat{\top}$,

(14) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{dis}} \ \wedge \ (a)_1 = \widehat{\bot} \ \wedge \ b = \widehat{\bot} \ \wedge \ c = \widehat{\bot}$,

(15) $a = \widehat{\mathbf{e}} \ \wedge \ c = \langle \widehat{\mathbf{e}}, b \rangle$,

(16) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{e}} \ \wedge \ (\exists x \in b) R((a)_1, x, \widehat{\top}) \ \wedge \ c = \widehat{\top}$,

(17) $\mathit{Tup}_2(a) \ \wedge \ (a)_0 = \widehat{\mathbf{e}} \ \wedge \ (\forall x \in b) R((a)_1, x, \widehat{\bot}) \ \wedge \ c = \widehat{\bot}$,

(18) $a = \widehat{\mathbb{S}} \ \wedge \ c = \langle \widehat{\mathbb{S}}, b \rangle$,

(19) $Tup_2(a) \ \wedge \ (a)_0 = \widehat{\mathbb{S}} \ \wedge \ (\forall x \in b)(R((a)_1, x, \widehat{\top}) \vee R((a)_1, x, \widehat{\bot})) \ \wedge$
$(\forall x \in c)(x \in b \ \wedge \ R((a)_1, x, \widehat{\top})) \ \wedge \ (\forall x \in b)(R((a)_1, x, \widehat{\top}) \ \rightarrow \ x \in c)$,

(20) $a = \widehat{\mathbb{R}} \ \wedge \ c = \langle \widehat{\mathbb{R}}, b \rangle$,

(21) $Tup_2(a) \ \wedge \ (a)_0 = \widehat{\mathbb{R}} \ \wedge \ (\forall x \in b)(\exists y \in c)R((a)_1, x, y) \ \wedge$
$(\forall y \in c)(\exists x \in b)R((a)_1, x, y)$,

(22) $a = \widehat{\mathbb{C}} \ \wedge \ R(b, c, \widehat{\top}) \ \wedge \ (\forall x \in L_\alpha)(x <_{\mathbf{L}} c \ \rightarrow \ \neg R(b, x, \widehat{\top})) \ \wedge$
$(\forall x \in L_\alpha)\neg R(\widehat{\mathbb{C}}, b, x)$.

It is a matter of routine to check that $\mathfrak{A}[R, \alpha, a, b, c]$ is Δ over KP. It is also easy ro verify that $\mathfrak{A}[R, \alpha, a, b, c]$ is deterministic in the following sense: from $\mathfrak{A}[R, \alpha, a, b, c]$ we can conclude that exactly one of the clauses (1)–(22) of the previous definition is satisfied for these α, a, b and c.

We continue with applying Theorem 13 to this formula $\mathfrak{A}[R, \alpha, a, b, c]$: any formula $B[\alpha, a, b, c]$ provided by this theorem may be used to describe the αth level of the interpretation of the OST application ($ab = c$). Accordingly, we proceed as follows.

Definition 15. *Let $B_{\mathfrak{A}}[\alpha, a, b, c]$ be a Σ formula of $\mathcal{L}$ associated to the formula $\mathfrak{A}[R, \alpha, a, b, c]$ according to Theorem 13 such that* KP *proves*

$$B_{\mathfrak{A}}[\alpha, a, b, c] \ \leftrightarrow \ (a, b, c \in L_\alpha \ \wedge \ \mathfrak{A}[(\exists \xi < \alpha)B_{\mathfrak{A}}[\xi, .], \alpha, a, b, c]). \qquad (\mathfrak{A})$$

Then we define

$$B_{\mathfrak{A}}^{<\alpha}[a, b, c] \ := \ (\exists \beta < \alpha)B_{\mathfrak{A}}[\beta, a, b, c],$$

$$Ap_{\mathfrak{A}}[a, b, c] \ := \ \exists \alpha B_{\mathfrak{A}}[\alpha, a, b, c].$$

As we will see, $Ap_{\mathfrak{A}}[a, b, c]$ is functional in its third argument. The next lemma takes care of the only critical case in the proof of this property and motivates the rather complicated clause (22) of Definition 14 above.

Lemma 16. *We can prove in* KP *that*

$$B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a] \ \wedge \ B_{\mathfrak{A}}[\beta, \widehat{\mathbb{C}}, f, b] \ \rightarrow \ \alpha = \beta \ \wedge \ a = b.$$

Proof. Working informally in KP, we assume $B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a]$, $B_{\mathfrak{A}}[\beta, \widehat{\mathbb{C}}, f, b]$, and, without loss of generality, $\alpha \leq \beta$. Then $\widehat{\mathbb{C}}, f, a \in L_\alpha$ and $b \in L_\beta$. In view of $(\mathfrak{A})$ and clause (22) the assumption $B_{\mathfrak{A}}[\beta, \widehat{\mathbb{C}}, f, b]$ also implies

$$(\forall x \in L_\beta)\neg B_{\mathfrak{A}}^{<\beta}[\widehat{\mathbb{C}}, f, x].$$

We also have $B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a]$, hence $\alpha = \beta$ is an immediate consequence. Moreover, ($\mathfrak{A}$) and clause (22) plus the two assumptions $B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a]$ and $B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, b]$ also give us

$$B_{\mathfrak{A}}^{<\alpha}[f, a, \widehat{\top}] \wedge (\forall x \in L_\alpha)(x <_{\mathbf{L}} a \to \neg B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]),$$

$$B_{\mathfrak{A}}^{<\alpha}[f, b, \widehat{\top}] \wedge (\forall x \in L_\alpha)(x <_{\mathbf{L}} b \to \neg B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]).$$

Consequently, we also have $a = b$, as desired. □

Lemma 17. *We can prove in* KP*:*

1. $B_{\mathfrak{A}}^{<\alpha}[a, b, u] \wedge B_{\mathfrak{A}}^{<\alpha}[a, b, v] \;\to\; u = v.$
2. $Ap_{\mathfrak{A}}[a, b, u] \wedge Ap_{\mathfrak{A}}[a, b, v] \;\to\; u = v.$

Proof. Since the previous lemma is at our disposal, the first assertion is easily proved by induction on α. The second assertion is a straightforward consequence of the first. □

Now we proceed as in Jäger [13] and associate to each term t of $\mathcal{L}^\circ$ a formula $[\![t]\!]_{\mathfrak{A}}(u)$ of $\mathcal{L}$ expressing that u is the value of t under the interpretation of the operational application via the formula $Ap_{\mathfrak{A}}$.

Definition 18. *For each $\mathcal{L}^\circ$ term t we introduce an $\mathcal{L}$ formula $[\![t]\!]_{\mathfrak{A}}(u)$, with u not occurring in t, which is inductively defined as follows:*

1. *If t is a variable or the constant ω, then $[\![t]\!]_{\mathfrak{A}}(u)$ is the formula $(t = u)$.*
2. *If t is another constant, then $[\![t]\!]_{\mathfrak{A}}(u)$ is the formula $(\widehat{t} = u)$.*
3. *If t is the term (rs), then we set*

$$[\![t]\!]_{\mathfrak{A}}(u) \;:=\; \exists x \exists y([\![r]\!]_{\mathfrak{A}}(x) \wedge [\![s]\!]_{\mathfrak{A}}(y) \wedge Ap_{\mathfrak{A}}[x, y, u]).$$

For every term t of $\mathcal{L}^\circ$ its translation $[\![t]\!]_{\mathfrak{A}}(u)$ is a Σ formula of $\mathcal{L}$. This treatment of terms leads to a canonical translation of formulas of $\mathcal{L}^\circ$ into formulas of $\mathcal{L}$.

Definition 19. *The translation of an $\mathcal{L}^\circ$ formula A into the $\mathcal{L}$ formula $A^\star$ is inductively defined as follows:*

1. *For the atomic formulas of $\mathcal{L}^\circ$ we stipulate*

$$(t{\downarrow})^\star \;:=\; \exists x [\![t]\!]_{\mathfrak{A}}(x),$$

$$(s \in t)^\star \;:=\; \exists x \exists y([\![s]\!]_{\mathfrak{A}}(x) \wedge [\![t]\!]_{\mathfrak{A}}(y) \wedge x \in y).$$

2. *If A is a formula $\neg B$, then $A^\star$ is $\neg B^\star$.*

3. *If A is a formula $(B \diamond C)$ for $\diamond$ being the binary junctor $\vee$ or $\wedge$, then $A^\star$ is $(B^\star \diamond C^\star)$.*

4. *If A is a formula $(\exists x \in t)B[x]$, then*

$$A^\star \;:=\; \exists y(\llbracket t \rrbracket_{\mathfrak{A}}(y) \;\wedge\; (\exists x \in y)B^\star[x]).$$

5. *If A is a formula $(\forall x \in t)B[x]$, then*

$$A^\star \;:=\; \forall y(\llbracket t \rrbracket_{\mathfrak{A}}(y) \;\rightarrow\; (\forall x \in y)B^\star[x]).$$

6. *If A is a formula $QxB[x]$ for a quantifier Q, then $A^\star$ is $QxB^\star[x]$.*

We notice immediately that the translation $A^\star$ of any $\mathcal{L}^\circ$ formula A which does not contain the application operation (i.e. all terms occuring in A are constants of variables) is equivalent to A.

Based on this interpretation of $\mathcal{L}^\circ$ in $\mathcal{L}$ we can now turn to the desired embedding of $\mathsf{OST} + (\mathsf{Inac})$ into $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$. A substantial part of the work has been done in Jäger [13] already, where a corresponding translation has been used to embed OST into $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$. The strong limit axiom together with the following obsrvation will take care of the axiom (Inac).

Lemma 20. *In $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ we can prove that*

$$\mathit{Frg}[\kappa] \;\rightarrow\; \mathit{Org}^\star[\kappa].$$

Proof. Working in KP, let κ be a functionally regular ordinal. For any f and $\alpha < \kappa$ we have to show that

$$(f : \alpha \rightarrow \kappa)^\star \;\rightarrow\; (\exists \beta < \kappa)(f : \alpha \rightarrow \beta)^\star.$$

Hence assume $(f : \alpha \rightarrow \kappa)^\star$, and thus

$$(\forall \eta < \alpha)(\exists \xi < \kappa)Ap_{\mathfrak{A}}[f, \eta, \xi].$$

This is a Σ formula, and therefore Σ reflection implies

$$(\forall \eta < \alpha)(\exists \xi < \kappa)Ap^a_{\mathfrak{A}}[f, \eta, \xi].$$

for a suitable set a. Let g be the set-theoretic function from α to κ which maps any $\eta < \kappa$ to the uniquely determined $\xi < \kappa$ for which $Ap^a_{\mathfrak{A}}[f, \eta, \xi]$. Since κ is functionally regular there exists a $\beta < \kappa$ such that $\mathit{Ran}[g] \subseteq \beta$. This yields

$$(\forall \eta < \alpha)(\exists \xi < \beta)Ap^a_{\mathfrak{A}}[f, \eta, \xi].$$

Thus $(f : \alpha \rightarrow \beta)^\star$ by Σ persistency, as desired. □

Theorem 21. *The theory* $\mathsf{OST} + (\mathsf{Inac})$ *is* $\star$*-interpreted in* $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$*; i.e. for all* $\mathcal{L}^\circ$ *formulas* A *we have*

$$\mathsf{OST} + (\mathsf{Inac}) \vdash A \quad \Longrightarrow \quad \mathsf{KPS} + (\mathbf{V}{=}\mathbf{L}) \vdash A^\star.$$

Proof. Our treatment of operational application is so that the axioms of the logic of partial terms fall off directly. For verifying the interpretations of the mathematical axioms of OST simply follow Jäger [13]. So it remains to prove $(\mathsf{Inac})^*$ in $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$.

Following our $\star$-translation this means that we have to show that the theory $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ proves $\forall\alpha\exists\beta(\alpha < \beta \wedge \mathit{Org}^\star[\beta])$. But given any α, the strong limit axiom (SLim) guarantees the existence of a functionally regular ordinal β which contains α and, by the previous lemma, satisfies $\mathit{Org}^\star[\beta]$.

So we know that the $\star$-translations of all axioms of $\mathsf{OST} + (\mathsf{Inac})$ are provable in $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$. Since $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ is closed under (the translations of) all rules of inference of OST, our theorem is established. □

5 Reducing KPS to OST + (Inac)

Since $\mathsf{KP} \subseteq \mathsf{OST}$ has been established in Feferman [9, 10] and Jäger [13] not much is left to be done for establishing the reduction of KPS to $\mathsf{OST} + (\mathsf{Inac})$. We can immediately turn to the desired theorem.

Theorem 22. *The theory* $\mathsf{OST} + (\mathsf{Inac})$ *contains the theory* KPS*; i.e. for all* $\mathcal{L}$ *formulas* A *we have*

$$\mathsf{KPS} \vdash A \quad \Longrightarrow \quad \mathsf{OST} + (\mathsf{Inac}) \vdash A.$$

Proof. As just mentioned, we have $\mathsf{KP} \subseteq \mathsf{OST}$, and thus only the axiom (SLim) remains to be proved in $\mathsf{OST} + (\mathsf{Inac})$. Hence pick an arbitrary ordinal α. We have to show that there exists a functionally regular ordinal β which contains α.

From (Inac) we conclude that there exists an ordinal β which contains α and is operationally regular, and thus

$$\omega < \beta \;\wedge\; \forall f(\forall\eta < \beta)((f : \eta \to \beta) \;\to\; (\exists\xi < \beta)(f : \eta \to \xi)). \tag{*}$$

To prove that this β is functionally regular, let η be an arbitrary ordinal less than β and g be an arbitrary set-theoretic function from η to β. Now we apply the closed term $\mathbf{op}$ to g and conclude with Lemma 2 that $\mathbf{op}(g)\downarrow$ and, for all $x \in \mathit{Dom}[g] = \eta$,

$$\mathbf{op}(g, x) \;\simeq\; g'x. \tag{**}$$

Hence $(\mathbf{op}(g) : \eta \to \beta)$, and because of (*) there exists a $\xi < \beta$ such that $(\mathbf{op}(g) : \eta \to \xi)$. Together with (**) this implies $Ran[g] \subseteq \xi$, as needed for establishing the functional regularity of β. □

Corollary 23. *The three theories* $\mathsf{OST} + (\mathsf{Inac})$*,* KPS*, and* $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ *are equiconsistent and prove the same absolute formulas of the language* $\mathcal{L}$*.*

This corollary is an immediate consequence of Theorem 21, Theorem 22, and Theorem 11.

The purpose of this article was to clarify the proof-theoretic strength of operational regularity as introduced in Feferman [9, 10]. The main question in this context is now to find out whether there exist a natural variant $(\mathsf{Inac})'$ of the axiom (Inac) complying with the ideology of operational set theory:

(i) $\mathsf{OST} + (\mathsf{Inac})'$ is proof-theoretically equivalent to the theory KPi and thus "describes" a recursively inaccessible universe,

(ii) if interpreted in the sense of classical set theory, $(\mathsf{Inac})'$ nevertheless provides for a weakly inaccessible universe.

References

[1] K. J. Barwise. *Admissible Sets and Structures*, volume 7 of *Perspectives in Mathematical Logic*. Springer, 1975.

[2] M. J. Beeson. *Foundations of Constructive Mathematics: Metamathematical Studies*, volume 3/6 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1985.

[3] M. J. Beeson. Proving programs and programming proofs. In R. Barcan Marcus, G. J. W. Dorn, and P. Weingartner, editors, *Logic, Methodology, and Philosophy of Science VII*, volume 114 of *Studies in Logic and the Foundations of Mathematics*, pages 51–82. North-Holland, 1986.

[4] M. J. Beeson. Towards a computation system based on set theory. *Theoretical Computer Science*, 60(3):297–340, 1988.

[5] A. Cantini. Extending constructive operational set theory by impredicative principles. *Mathematical logic Quarterly*, 57(3):299–322, 2011.

[6] A. Cantini and L. Crosilla. Constructive set theory with operations. In A. Andretta, K. Kearnes, and D. Zambella, editors, *Logic Colloquium 2004*, volume 29 of *Lecture Notes in Logic*, pages 47–83. Association for Symbolic Logic and Cambridge University Press, 2008.

[7] A. Cantini and L. Crosilla. Elementary constructive operational set theory. In R. Schindler, editor, *Ways of Proof Theory*, pages 199–240. Ontos Verlag, 2010.

[8] K. J. Devlin. *Constructibility*, volume 6 of *Perspectives in Mathematical Logic*. Springer, 1984.

[9] S. Feferman. Notes on Operational Set Theory, I. Generalization of “small” large cardinals in classical and admissible set theory. Technical Notes, 2001.

[10] S. Feferman. Operational set theory and small large cardinals. *Information and Computation*, 207:971–979, 2009.

[11] G. Jäger. Zur Beweistheorie der Kripke-Platek-Mengenlehre über den natürlichen Zahlen. *Archiv für Mathematische Logik und Grundlagenforschung*, 22:121–139, 1982.

[12] G. Jäger. *Theories for Admissible Sets: A Unifying Approach to Proof Theory*, volume 2 of *Studies in Proof Theory, Lecture Notes*. Bibliopolis, 1986.

[13] G. Jäger. On Feferman’s operational set theory OST. *Annals of Pure and Applied Logic*, 150(1–3):19–39, 2007.

[14] G. Jäger. Full operational set theory with unbounded existential quantification and power set. *Annals of Pure and Applied Logic*, 160(1):33–52, 2009.

[15] G. Jäger. Operations, sets and classes. In C. Glymour, W. Wei, and D. Westerståhl, editors, *Logic, Methodology and Philosophy of Science - Proceedings of the Thirteenth International Congress*, pages 74–96. College Publications, 2009.

[16] G. Jäger and J. Krähenbühl. Σ^1_1 Choice in a Theory of Sets and Classes. In R. Schindler, editor, *Ways of Proof Theory*, pages 283–314. Ontos Verlag, 2010.

[17] K. Kunen. *Set Theory. An Introduction to Independence Proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1980.

[18] M. Rathjen. Fragments of Kripke-Platek set theory. In P. Aczel, H. Simmons, and S. Wainer, editors, *Proof Theory*, pages 251–273. Cambridge University Press, 1992.