# Canonical proof nets for classical logic

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**Abstract.** Proof nets provide abstract counterparts to sequent proofs modulo rule permutations; the idea being that if two proofs have the same underlying proof-net, they are in essence the same proof. Providing a convincing proof-net counterpart to proofs in the classical sequent calculus is thus an important step in understanding classical sequent calculus proofs. By convincing, we mean that (a) there should be a canonical function from sequent proofs to proof nets, (b) it should be possible to check the correctness of a net in polynomial time, (c) every correct net should be obtainable from a sequent calculus proof, and (d) there should be a cut-elimination procedure which preserves correctness. Previous attempts to give proof-net-like objects for propositional classical logic have failed at least one of the above conditions. In [23], the author presented a calculus of proof nets (expansion nets) satisfying (a) and (b); the paper defined a sequent calculus corresponding to expansion nets but gave no explicit demonstration of (c). That sequent calculus, called $\mathbf{LK}^*$ in this paper, is a novel one-sided sequent calculus with both additively and multiplicatively formulated disjunction rules. In this paper (a self-contained extended version of [23]) , we give a full proof of (c) for expansion nets with respect to $\mathbf{LK}^*$, and in addition give a cut-elimination procedure internal to expansion nets – this makes expansion nets the first notion of proof-net for classical logic satisfying all four criteria.

## 1 Introduction

*Proof theory*, the study of formal proofs, was invented as a tool to study the consistency of mathematical theories, one of Hilbert's famous 23 problems. However, Hilbert had originally considered presenting at his Paris lecture a 24th problem [26] which concerned proofs directly: he proposed "develop(ing) a theory of mathematical proof in general". Central to this question is the idea that usual proofs, as written down by mathematicians, or formalized in, for example, Gentzen's sequent calculus [11], are syntactic representations of much more abstract proof objects. Given that, we should be able to tell when two syntactic proofs represent the same abstract proof.

It is striking how difficult this question seems to be, even for propositional classical logic. In contrast to the well-developed theory of proof-identity for intuitionistic natural deduction (given by interpretation of proofs in a cartesian-closed category), the theory of identity for proofs in classical logic is very poorly understood. Investigations by several researchers over the last ten years [25, 10, 19, 20, 2, 17] have only served to underline the difficulty of the problem. Many of these difficulties concern proofs with cuts. The identity of non-analytic proofs is not problematic for intuitionistic logic; since each proof has a unique normal form, he problem reduces to that of the identity of normal proofs. Reduction to normal form in the classical sequent calculus is in general neither confluent nor strongly normalizing, and so the identity of proofs containing cuts must also be considered.

Yet even for cut-free proofs, opinions on the "right notion" of proof-identity differ. It is not reasonable, as it is for natural deduction proofs, to declare two cut-free sequent proofs equal only if they are syntactically identical; a good minimum notion of equality is that proofs differing by *commuting conversions* of non-interfering sequent rules should be equal. Proof-nets [14] are a tool for providing canonical representants of such equivalence classes of proofs in linear logic [12]. A proposal by Robinson [25], following ideas from Girard [13], gives proof-nets for propositional classical logic, and these nets do indeed identify proofs differing by commutative conversions. However, they fail to provide canonical representants for sequent proofs owing to the presence of *weakening attachments*; explicit information about the context of a weakening not present in sequent proofs. As a result one sequent proof corresponds to many different nets, the exact opposite of the situation one expects. In addition, the proof-identities induced by Robinson's nets do not include, among other desirable equations, commutativity/associativity of contraction, a key assumption in the development of abstract models of proofs (such identities are assumed in [10], in [2] and also in [20]). Other notions of abstract proof for classical logic (Combinatorial proofs [16] and $\mathbb{B}/\mathbb{N}$-nets [19]) make such identifications, but at the cost of losing sequentialization into a sequent calculus.

The current paper concerns *expansion-nets*: a calculus of proof-nets for classical logic first presented in [23] which, unlike Robinson's nets, provide canonical representants of equivalence classes of classical sequent proofs. To avoid the problems inherent in weakening, we restrict attention to proofs in a new sequent calculus, $\mathbf{LK}^*$ (see Figure 1). This calculus has no weakening rule, nor does it have implicit weakening at the axioms: instead, it has both the multiplicative and additive forms of disjunction rule. This new calculus has all the properties one might hope of a sequent calculus for classical logic (except, perhaps, terminating proof search): it has the subformula property, is cut-free complete, and even has syntactic cut-elimination (although this is perhaps easier to see via the proof nets than directly in the sequent calculus, owing to the curious nature of the cut-elimination theorem: if $\Gamma$ is provable in $\mathbf{LK}^*$ with cut, then some subsequent $\Delta \subseteq \Gamma$ is provable without cut). Treating the introduction of weak formulae in this way allows us to define a canonical function mapping sequent proofs in $\mathbf{LK}^*$ to expansion nets. Correctness for expansion nets (whether a net really corresponds to a sequent proof) can be checked in polynomial time, using small adaptations of standard methods from the theory of proof nets for $\mathbf{MLL}^- + \textsc{Mix}$ (multiplicative linear logic, plus the mix rule, without units, as studied in $[1, 8, 9]$) – meaning that expansion-nets form a *propositional proof system* [6]. Translating from sequent proofs to expansion nets identifies, in addition to nets differing by commuting conversions, nets differing by the order in which contractions are performed. The current paper (a self-contained extension of [23]) gives a detailed account of the connection between expansion-nets and their associated sequent calculus: in particular, an explicit proof of sequentialization for expansion nets as(Theorem 5), which was missing in [23]. In addition, we present a cut-elimination procedure for expansion nets (proof transformations which we prove, in Propositions 11 – 14 to preserve correctness) which are weakly normalizing (Lemma 6 and Theorem 6 detail a strategy for reducing any net with cuts to a cut-free net). This result was absent from [23]: with it, we can see that expansion nets have polynomial-time proof checking, sequentialization into a sequent calculus and cut-elimination preserving sequent-calculus correctness – the first notion of abstract proof for propositional classical logic to satisfy all of these properties.

### 1.1 Structure of the paper

Section 2 gives some preliminaries, and then Section 3 introduces the variant sequent calculus $\mathbf{LK}^*$, showing completeness and some other key properties. Section 4 surveys the existing notions of abstract proof in propositional classical logic. Section 5 defines expansion nets, and compares them with the existing notions of abstract proof in the literature.

The next two chapters contain most of the novel technical material in the paper. Section 6 deals with the notion of *subnet*, a key analogue of the notion of subproof in sequent calculus which we will need to define cut-reduction. This technology (including the new notion of *contiguous empire*) also affords a proof of sequentialization of expansion-nets into $\mathbf{LK}^*$. Section 7 then provides the cut-reduction steps themselves, and a proof of cut-elimination for expansion nets.

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## 2 Preliminaries

### 2.1 Formulae of propositional classical logic

Let $\mathcal{P}$ be a countable set of *proposition symbols*. An *atom* is a pair $(a, i)$, where $a \in \mathcal{P}$ and $i \in \{+, -\}$. By an abuse of notation, but in line with common use, we will simply write $a$ for $(a, +)$, and write $\bar{a}$ for $(a, -)$. Two atoms are *dual* if they differ only in their second component.

$$\dfrac{}{a,\ \bar{a}}\ \textsc{Ax} \qquad \dfrac{}{\top}\ \textsc{Ax}_\top$$

$$\dfrac{\Gamma,\ A}{\Gamma, A \vee B}\ \vee_0 \qquad \dfrac{\Gamma,\ A, B}{\Gamma, A \vee B}\ \vee \qquad \dfrac{\Gamma,\ B}{\Gamma, A \vee B}\ \vee_1 \qquad \dfrac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta,\ A \wedge B}\ \wedge$$

$$\dfrac{\Gamma \qquad \Delta}{\Gamma, \Delta}\ \textsc{Mix} \qquad \dfrac{\Gamma,\ a, a}{\Gamma, a}\ \mathrm{C} \qquad \dfrac{\Gamma,\ \bar{a}, \bar{a}}{\Gamma, \bar{a}}\ \mathrm{C} \qquad \dfrac{\Gamma,\ A \wedge B, A \wedge B}{\Gamma, A \wedge B}\ \mathrm{C}$$

**Fig. 1. LK$^*$**: A variant sequent calculus without weakening

The classical formulae over $\mathcal{P}$ are given by the following grammar

$$A ::= a \mid \bar{a} \mid \top \mid \bot \mid A \wedge A \mid A \vee A.$$

Negation is not a connective in our systems, but is defined by De Morgan duality. We will use the notation $\bar{A}$ to denote the De Morgan dual of the formula $A$. The *rank* $\mathrm{rk}(A)$ of a formula $A$ is defined as follows:

$$\mathrm{rk}(\top) = \mathrm{rk}(\bot) = \mathrm{rk}(a) = \mathrm{rk}(\bar{a}) = 1$$

$$\mathrm{rk}(A \wedge B) = \mathrm{rk}(A \vee B) = 1 + \max(\mathrm{rk}(A), \mathrm{rk}(B))$$

### 2.2 Forests and sequents

A *forest* (in this paper) is a pair $(A, \mathrm{pr})$ consisting of a set $A$ of nodes and a partial endofunction pr (predecessor) on $A$ (the elements of $A$ on which pr is undefined being the *roots*) such that, for each element $x$ of $A$, there is an $n \geq 0$ such that $\mathrm{pr}^n(x)$ is a root. Clearly, a forest with one root is a tree. Given a $y$ such that $\mathrm{pr}(x) = y$, we will say that $x$ is a *successor* of $y$. A node with no successors is a *leaf.* A node $x$ in a forest is *ordered* if it comes equipped with an injective function from its set of successors to $\mathbb{N}$ — otherwise it is *unordered.*

A forest defines a natural partial order $\leq$ on its nodes derived from predecessor: $x \leq y$ if there exists $n \geq 0$ with $x = \mathrm{pr}^n(y)$. A forest also gives rise to a directed graph (the graph of the forest) with nodes the same as the nodes of the forest, and a directed edge from every node to its predecessor.

A *subforest* of $F$ is a nonempty set $G$ of nodes of $F$ such that if $g_1$ is a member of $G$ and $g_1 \leq g_2$ then $g_2$ is a member of $G$.

Given that a formula is a tree, it is natural to consider a sequent to be a forest: a *classical sequent* will be, for us, a finite forest whose trees are classical propositional formulae.

*Remark 1.* Sequents are typically defined either as sets, multisets or sequences of formulae: why then have we chosen to define sequents as forests? For an fine-grained analysis of proofs, sets are a bad representation, as they throw away all explicit information about contraction. Sequences, on the other hand, distinguish too much; what we need is a representation which allows us to distinguish individual occurrences of the same formula in a sequent without caring in which order they appear. The problem with the multiset representation of sequents lies in confusion over the meaning of "multiset", which is different depending on context, and in essential ways. In particular, problems arise for structural proof theory if the intended meaning of multiset is "set with multiplicities". Suppose that from $\bar{A}, A$ we derive $\bar{A}, A, A$ by weakening. If we wish to form a cut against $A$, we must choose which copy of $A$ to cut against: the choice will have drastic consequences during cut-elimination. But in the "set with multiplicities" understanding of multisets, there is no notion of an individual copy of $A$ in the sequent.

By defining a sequent to be a forest, we avoid this conceptual hurdle: each formula in the sequent corresponds to a distinct root of the forest. When we want to think about sequents as multisets to make sense, for example, of the expression $\varDelta \subseteq \varGamma$ ("$\varDelta$ is a *subsequent* of $\varGamma$"), we can use the *set* of roots of the sequent (the above expression is interpreted as "$\varDelta$ is a subforest of $\varGamma$, each of whose roots is a root of $\varGamma$").

We write sequent proofs without turnstiles: if **L** is a sequent system, we write $\mathbf{L} \vdash \Gamma$ to mean "there is a sequent derivation in **L** with $\Gamma$ at the root and axioms at the leaves.

$$\frac{}{a,\ \bar{a}}\,\mathrm{Ax} \qquad \frac{}{\top}\,\mathrm{Ax}_\top$$

$$\frac{\Gamma,\ A, B}{\Gamma, A \vee B}\,\vee \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta,\ A \wedge B}\,\wedge$$

$$\frac{\Gamma,\ A, A}{\Gamma, A}\,\mathsf{C} \qquad \frac{\Gamma}{\Gamma, B}\,\mathsf{W}$$

**Fig. 2.** Cut-free multiplicative **LK** (one-sided)

## 3 A variant sequent calculus for classical logic

The completeness of expansion nets relies on the completeness of a variant sequent calculus $\mathbf{LK}^*$ (shown in Figure 1). This sequent calculus was introduced, along with expansion-nets, in [23]. The calculus bears some similarities to Hughes's "minimal calculus" **Mp** [18], in that it has both multiplicatively and additively formulated disjunction rules. However, while **Mp** has a mixed additive/multiplicative conjunction rule, $\mathbf{LK}^*$ has the standard multiplicative conjunction rule. Given these logical rules, we need the contraction rule (which is absent from **Mp**) to be complete with respect to classical logic. This would ordinarily make the multiplicative disjunction rule redundant, as it is derivable from the two additive rules plus contraction; however, in $\mathbf{LK}^*$ contraction is forbidden on disjunctions. Contraction is, however, admissible in $\mathbf{LK}^*$; we will prove this using the following two easy lemmata:

**Lemma 1 (Pseudo-invertibility of $\vee$).** *If $\mathbf{LK}^* \vdash \Gamma, A \vee B$, then one of the following holds:*

- $\mathbf{LK}^* \vdash \Gamma, A, B$
- $\mathbf{LK}^* \vdash \Gamma, A$
- $\mathbf{LK}^* \vdash \Gamma, B$

**Lemma 2.** *If $\Gamma$ is nonempty and $\mathbf{LK}^* \vdash \Gamma, \top$, then $\mathbf{LK}^* \vdash \Gamma$.*

**Proposition 1.** *Contraction is admissible in $\mathbf{LK}^*$.*

*Proof.* Contraction is admissible for $\top$ by Lemma 2, and for atoms/conjunctions by the contraction rule. Now suppose that contraction is admissible for all formulae of rank $< n$, and let $A \vee B$ have rank $n$. Given a proof of $\Gamma, A \vee B, A \vee B$, apply pseudo invertibility (Lemma 1) to obtain a proof of $\Gamma, A^{(n)}, B^{(m)}$, (here $C^{(n)}$ denotes $n$ copies of the formula $C$) where $0 \leq m, n \leq 2$ and $n + m \geq 2$. Using a combination of the induction hypothesis and one of the disjunction rules of $\mathbf{LK}^*$ we obtain a proof of $\Gamma, A \vee B$. $\square$

In common with **Mp**, $\mathbf{LK}^*$ has the curious property of being sound and complete for formulae ($\vdash A$ iff $\vDash A$) but not "sequent complete": that is, there are sequents provable in **LK** which cannot be proved in the variant system. For example, if $a$ and $b$ are distinct propositional letters, then $a, \bar{a}, b$ does not have a proof in $\mathbf{LK}^*$. For this reason, our proof of completeness proceeds by showing that each **LK**-provable sequent has an $\mathbf{LK}^*$-provable subsequent:

**Proposition 2.** *Let $\Gamma$ be provable in* ***LK*** *(we take as* ***LK*** *the system in Figure 2). Then we may partition the formulae in $\Gamma$ (in terms of forests, the roots of $\Gamma$) into $\Gamma_s$ (the* strong *formulae of $\Gamma$) and $\Gamma_w$ (the* weak *formulae of $\Gamma$), such that* ***LK***$^*$ *proves $\Gamma_s$.*

*Proof.* By induction on the length of an **LK** derivation. Clearly, the proposition is true for consequences of the **LK** axiom. We proceed by case analysis on the last rule $\rho$ used in the **LK** derivation:

$[\rho = \mathsf{W}]$ The induction hypothesis gives us the strong formulae $\Gamma_s$ of the premiss $\Gamma$ of $\rho$, such that $\mathbf{LK}^* \vdash \Gamma_s$. The sequent $\Gamma_s$ is also a subsequent of the conclusion $\Gamma, B$ of $\rho$, and so we may take it as the strong formulae of the conclusion (i.e. $B$ is a weak formula in the conclusion).

$[\rho = \vee]$ Let $\Gamma, A, B$ be the premiss of $\rho$, and $\Gamma, A \vee B$ the conclusion. Apply the induction hypothesis to $\Gamma, A, B$, yielding a sequent $\Gamma_s$ of strong formulae provable in $\mathbf{LK}^*$:

- If $A$ and $B$ are both strong, then $\Gamma_s = \Delta, A, B$ is provable in $\mathbf{LK}^*$, and $\Delta, A \vee B$ is an $\mathbf{LK}^*$ provable subsequent of the conclusion of $\rho$.
- If $A$ and $B$ are both weak, then $\Gamma_s$ is also a subsequent of the conclusion of $\rho$, and so we may take $\Gamma_s$ as the strong formulae of the concluion of $\rho$.
- If $A$ is weak and $B$ is strong, then $\Gamma_s = \Delta, B$, and thus, using $\vee_1$, $\Delta, A \vee B$ is an $\mathbf{LK}^*$ provable subsequent of the conclusion of $\rho$. Symmetrically if $A$ strong and $B$ weak.

$[\rho = \mathrm{C}]$ This is similar to the case for disjunction, with the added twist that we must use admissible contraction where a contraction rule is not available in $\mathbf{LK}^*$. Let $\Gamma, A, A$ be the premiss of $\rho$, and $\Gamma, A$ the conclusion. Apply the induction hypothesis to $\Gamma, A, A$, yielding a sequent $\Gamma_s$ of strong formulae provable in $\mathbf{LK}^*$:

- If both copies of $A$ are strong, then $\Gamma_s = \Delta, A, A$ is provable in $\mathbf{LK}^*$, and $\Delta, A$ is an $\mathbf{LK}^*$ provable subsequent of the conclusion of $\rho$ by contraction admissibility.
- If both copies of $A$ are weak, then $\Gamma_s$ is also a subsequent of the conclusion of $\rho$, and so we may take $\Gamma_s$ as the strong formulae of the conclusion of $\rho$.
- If one copy of $A$ is weak, then $\Gamma_s = \Delta, A$ is also an $\mathbf{LK}^*$ provable subsequent of the conclusion of $\rho$.

$[\rho = \wedge]$ This is the most interesting case. Let $\Gamma, A,$ be one premiss of $\rho$ and $\Delta, B$ the other. The induction hypothesis applied to both premisses gives us a subsequents $\Gamma_s$ and $\Delta_s$ of strong formulae respectively for each premiss.

- If $A$ and $B$ are both strong in their respective sequents, then $\Gamma_s = \Gamma', A$ and $\Delta_s = \Delta', B$, and so $\Gamma', \Delta', A \wedge B$, a subsequent of the conclusion of $\rho$, is provable in $\mathbf{LK}^*$.
- If $A$ and $B$ are both weak, then $\Gamma_s, \Delta_s$ is a subsequent of the conclusion of $\rho$, provable in $\mathbf{LK}^*$ using the Mix rule.
- If $A$ is weak and $B$ is strong, then $\Gamma_s$ does not contain $A$, and is therefore a subsequent of $\Gamma, \Delta, A \wedge B$ provable in $\mathbf{LK}^*$. Symmetrically if $A$ strong and $B$ weak.

*Remark 2.* $\mathbf{LK}^*$ is also formula complete without the Mix rule; we only use Mix in the completeness argument once, where a conjunction is applied to two weak formulae; the Mix rule allows us to translate this derivation into $\mathbf{LK}^*$ in a symmetric manner. Without Mix, we would be forced to choose one or other of the premisses as the strong formulae of the conclusion.

## 4 Existing notions of proof-net for classical logic

To underline the need for a new notion of proof-net, we consider the existing notions of proof-net for classical logic, and underline their strengths and weaknesses as canonical representatives of equivalence classes of proofs.

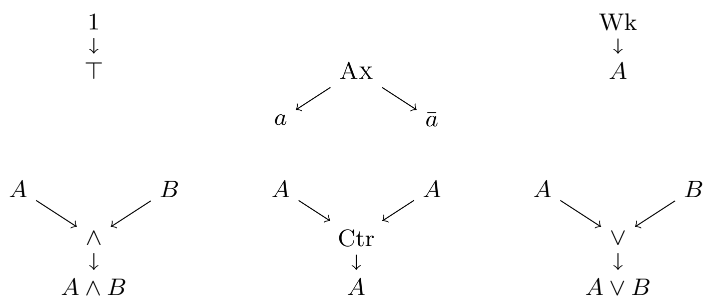



**Fig. 3.** Naïve classical nets: graph figures

### 4.1 Naïve classical nets

The basic idea for a rudimentary form of classical proof-net comes from Girard [13], and the details were first worked out by Robinson in [25]: the underlying structure of the nets is identical to that for **MLL** nets, and correctness is given by treating the conjunctions and axioms of classical logic in the same way as the linear logic axiom and tensor,treating both contraction and disjunction in the same way as the linear logic "par" connective, and treating weakenings as $\perp$ is treated in **MLL** nets.

*Remark 3.* The following presentation of classical nets differs from that of Robinson, in that we work with one-sided proofs, and we use weakening attachments for correctness rather than explicit weakening nodes. Since these nets represent the most basic idea for developing **MLL** nets into nets for classical logic, and since they lack many of the properties we would desire of proof-objects for classical logic, we call them *naïve classical nets.*

A graph-like presentation of naïve classical nets can be found in Figure 3: a naïve classical proof-structure is a graph built from the individual graph elements by matching types, such that the resulting graph has no sources (nodes with no incoming edges) labelled with formulae. There is an inductive definition mapping sequent-proofs in **LK** to proof-structures, which can be very easily obtained by considering proof-structures not as graphs, but as forests of trees:

**Definition 1.** *Let $\mathcal{X}$ be a countable set of* wire symbols*. A* wire variable *is an atom over $\mathcal{X}$, as defined in Section 2.1: a pair of a member $x$ of $\mathcal{X}$ and a polarity ( + or −). Thus wire variables occur in dual pairs, for example $x$ and $\bar{x}$. A* contraction-weakening tree *(or* cw-tree*) over $\mathcal{X}$ is a member of the following grammar.*

$$s ::= 1 \mid \mathrm{Wk} \mid x \mid \bar{x} \mid (s \vee s) \mid (s \wedge s) \mid \mathrm{Ctr}(s, s)$$

*where $x$ and $\bar{x}$ are wire variables over $\mathcal{X}$.*

We use these cw-trees to define the mapping from sequent proofs to proof structures, by *annotating* formulae appearing in **LK** derivations with cw-trees. The system in Figure 4 derives sequents of "annotated formulae", in which each formula has an associated *cw*-tree: the tree attached to a formula provides a history of how it was proved.

We can recover the more usual graph-like presentation of proof structures by considering the *graph* of an annotated sequent, given by adding axiom links to the forest of cw-terms as suggested by the dual wire variables.

*Example 1.* The following annotated sequent represents a proof of Pierce's law

$$((\bar{x} \vee \mathrm{Wk}) \wedge \bar{y}) : (\bar{p} \vee q) \wedge \bar{p}, \quad \mathrm{Ctr}(x, y) : p \tag{1}$$

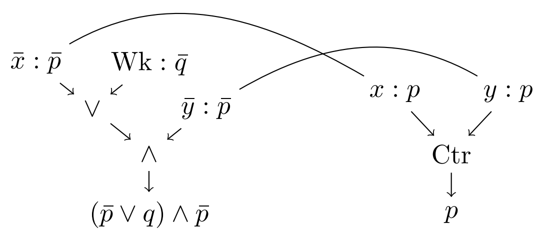
$$\frac{}{x : a,\ \bar{x} : \bar{a}}\ \mathrm{Ax} \qquad \frac{F \qquad G}{F, G}\ \textsc{Mix} \qquad \frac{}{1 : \top}\ \mathrm{Ax}_\top$$

$$\frac{G,\ t : A,\ s : B}{G, s \vee t : A \vee B}\ \vee \qquad \frac{G,\ s : A \qquad F,\ t : B}{G, F,\ s \wedge t : A \wedge B}\ \wedge$$

$$\frac{G,\ s : A,\ t : A}{G,\ \mathrm{Ctr}(s,t) : A}\ \mathsf{C} \qquad \frac{G}{G,\ \mathrm{Wk} : B}\ \mathsf{W}$$

**Fig. 4.** $\mathbf{LK}_{\mathrm{net}}$: annotating **LK** plus Mix with a naïve form of classical proof-net

The graph of this annotated sequent is

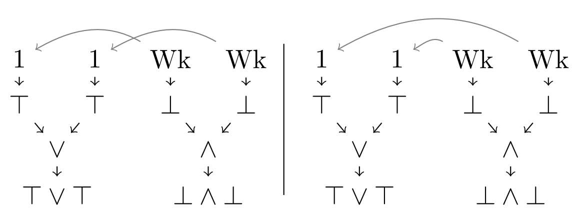



(2)

To obtain a correctness criterion, it is necessary to anchor each weakening to some other node of the proof. In [25] this anchoring is part of the structure of the weakening node: we instead use the more usual notion of an *attachment*

**Definition 2.** *An* attachment $f$ *for a naïve classical proof structure* $F$ *is a function mapping each rule node labelled with Wk to some other rule-node of the proof-structure. By an attached proof structure, we mean a pair* $(F, f)$ *of a proof structure* $F$ *and an attachment* $f$ *for* $F$*.*

*Example 2.* Below we see two different attachments of the same proof structure, represented by the grey arrows:




(3)

The annotated sequent calculus in Figure 4 provides a function from LK proofs to proof structures. To extend this to *attached* proof structures, we must give an attachment for each weakening in the sequent proof. We may choose any one of the formulae present in the context of the weakening rule; this arbitrary choice means that attached proof-nets themselves cannot be the canonical proof objects we seek. For **MLL**, the right notion of canonical proof object is a *quotient* of attached proof-nets by so-called *Trimble rewiring* [27], whereby two proof-nets are equivalent if they can be transformed into one another by several steps of "rewiring" a single unit: a rewiring is a change of attachment for the unit which yields a correct net. According to Trimble rewiring, the two attached nets in (3) are different, as rewiring any one unit would result in a structure which is not a net; this is important, as the corresponding morphisms are distinguished in some $*$-autonomous categories.

The standard problem in the theory of proof-nets is to give a global *correctness criterion* for identifying, among the proof-structures, those which can be obtained from desequentializing a sequent proof. This then leads to a *sequentialization theorem*, allowing one to reconstruct a sequent proof out of a correct proof-net. Naïve classical nets are very closely modelled on **MLL** nets; this means we may adapt any of the many equivalent formulations of correctness for **MLL** nets to provide a correctness criterion for them. For example, the following is the switching graph criterion [7], suitably altered for our setting:

**Definition 3.** *Let $F$ be a naïve classical proof-structure.*

*(a) A rule-node of $F$ is* switched *if it is a Ctr or $\vee$ node. A* switching *of a naïve classical proof-structure is a choice, for each switched node, of one of its successors.*
*(b) Given an attachment $f$ for $F$, and a switching $\sigma$ for $F$, the* switching graph $\sigma(F, f)$ *is the graph obtained by deleting from $F$ all edges from a switched node to its successor not chosen by $\sigma$, forgetting directedness of edges, and adding an edge from each Wk node to its image under $f$.*
*(c) $(F, f)$ is* ACC-correct *if, for each switching $\sigma$, $\sigma(F, f)$ is acyclic and connected.*
*(d) $F$ is a* naïve classical net *if, for some $f$, $(F, f)$ is ACC-correct.*

**Theorem 1 (Robinson).**

*(a) Every proof-structure arising from an **LK** proof is a naïve classical net.*
*(b) Every naïve classical net can be obtained by desequentializing an **LK** proof.*

Using the techniques developed in [8, 9], we can capture classical reasoning in the presence of the MIX rule (which does not allow us to prove any new theorems, but extends the space of cut-free proofs):

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\ \textsc{Mix}$$

**Definition 4.** *Let $F$ be a Robinson proof-structure, and $f$ an attachment for $F$*

*(a) $(F, f)$ is* AC-correct *if, for each switching $\sigma$, $\sigma(F, f)$ is acyclic.*
*(b) $F$ is a* MIX-net *if there is an attachment $f$ such that $(F, f)$ is AC-correct.*

**Theorem 2.** *(a) Every proof-structure arising from a sequent proof in the system in Figure 2 plus* MIX *is a* MIX*-net.*
*(b) Every* MIX*-net can be obtained by desequentializing a sequent proof with* MIX*.*

Correctness for naïve classical nets, and sequentialization, can be developed easily by analogy with **MLL** nets; for details see [25].

As intrinsic representations of proofs, naïve classical nets have a number of drawbacks:

**Either correctness is NP, or weakening introduces noncanonicity** Correctness for naïve classical proof structures is **NP**-complete; it is in **NP**, since the correctness criterion goes via guessing an attachment for each Wk: without an attachment it is not possible to adapt the correctness criterion from **MLL**. Correctness for unattached naïve nets is **NP** hard since there is an evident surjective map from cw-annotated sequents to unattached **MLL** nets, for which correctness is known to be **NP** hard [21]. We could, instead, take attached naïve nets as our abstract proof objects, having, as in Robinson's original formulation, an explicit attachent for each weakening. Then correctness would be checkable in polynomial time (so we would have a propositional proof system) but there would no longer be a canonical function mapping sequent proofs to proof-nets – that is, we would not have a calculus of abstract proofs.

**Contraction is not associative, commutative** Given a *cw*-annotated sequent $F,\ t : A,\ s : A,\ u : A$, there are twelve distinct ways to contract the three displayed terms in the sequent calculus, each leading to a different naïve net. For example, the net

$$F,\ \mathrm{Ctr}(\mathrm{Ctr}(t, s), u) : A$$

is syntactically distinct from the net

$$F,\ \mathrm{Ctr}(t, \mathrm{Ctr}(s, u)) : A$$

Naïve classical nets satisfy neither the identity $\mathrm{Ctr}(\mathrm{Ctr}(t, s), u) = \mathrm{Ctr}(t, \mathrm{Ctr}(s, u))$, nor the identity $\mathrm{Ctr}(s, t) = \mathrm{Ctr}(t, s)$; taken together, these equations ensure a canonical way to contract multiple instances of the same formula.

**Weakening is not a unit for contraction** Given a net $G = F, t : A$, we can weaken to arrive at a net $F, t : A, \mathrm{Wk} : A$, and then contract to form a net $F, \mathrm{Ctr}(t, \mathrm{Wk}) : A$. This net differs from $G$, but we would prefer it to be identified with $G$: that is, Wk should be a unit for the contraction operation.

**Contraction on disjunctions is not pointwise** Given a cw-annotated sequent

$$F,\ t_1 : A,\ t_2 : A,\ s_1 : B,\ s_2 : B,$$

we can apply the rules of **LK** to obtain a single term of type $A \vee B$ in five distinct ways, which once again we would prefer were identified. Two of them are displayed below

$$F,\ \mathrm{Ctr}((t_1 \vee s_1), (t_2 \vee s_2)) : A \vee B \mid F,\ (\mathrm{Ctr}(t_1, t_2) \vee \mathrm{Ctr}(s_1, s_2)) : A \vee B.$$

If these two derivations, are identified, we will say that contraction on disjunctions is *constructed pointwise*: in naïve classical nets this is clearly not the case.

Two further proposals for proof-net-like objects exist in the literature. They do not suffer from the above problems but pay a heavy price for doing so, lacking as they do a strong connection with the sequent calculus. We will not discuss these proposals in as great a depth as naïve classical nets, as there is not such a close connection between them and expansion-nets.

### 4.2 Lamarche-Strassburger nets

The Lamarche-Strassburger approach to classical proof-nets [19] (hereafter LS-nets) are a generalization of $\mathbf{MLL}^-$ proof nets which allow classical logic to be captured: instead of changing the underlying forests, as with naïve proof structures, this approach changes the behaviour of the *links*. Specifically, while in $\mathbf{MLL}^-$ nets each leaf takes part in precisely one axiom link, in LS-nets a leaf may take part in several links, or indeed none – it is this liberalized notion of axiom link that allows LS-nets to capture classical logic. Depending on the particular flavour of net, there may even be more than one link between a pair of dual atoms. The "proof-structures" of these calculi of nets are the following:

- A $\mathbb{B}$-prenet over $\Gamma$ is a set $\mathcal{L}$ of pairs of leaves of $\Gamma$, such that the first member of each pair is labelled with a positive atom $a$, and the second member of the pair is labelled with the dual $\bar{a}$ of that atom.
- A $\mathbb{N}$-prenet over $\Gamma$ is a multiset $\mathcal{L}$ of pairs of leaves of $\Gamma$, such that the first member of each pair is labelled with a positive atom $a$, and the second member of the pair is labelled with the dual $\bar{a}$ of that atom.

The difference between $\mathbb{B}$-prenets, $\mathbb{N}$-prenets and naïve classical nets can be readily seen in Figure 5: while contraction is explicit in naïve nets, in a $\mathbb{B}$-prenet it is represented by an atom's participation in multiple axiom links. In a $\mathbb{N}$-prenet, there can, in addition, be multiple links between the same pair of atoms: thus

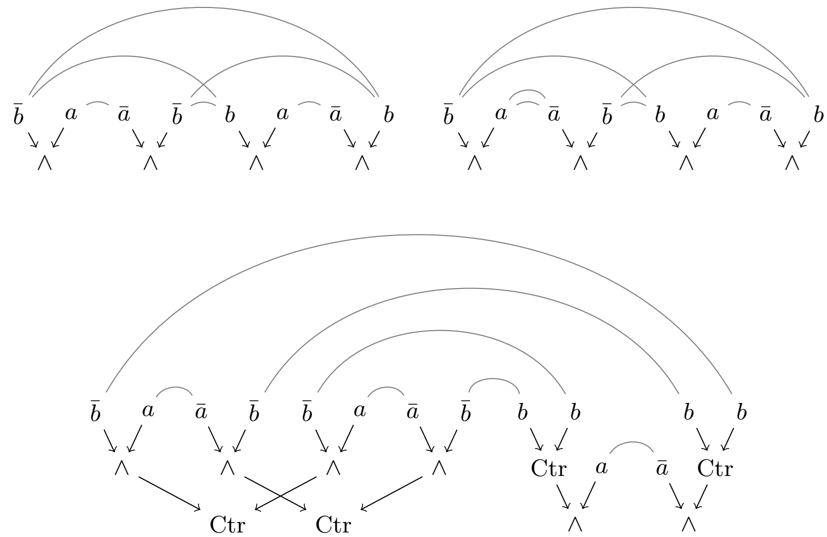


**Fig. 5.** The same $\mathbf{LK}^*$ proof, rendered as a $\mathbb{B}$-net, $\mathbb{N}$-net, and naïve net

more information about contraction is present in naive nets than in $\mathbb{N}$-nets, and more in $\mathbb{N}$-nets than in $\mathbb{B}$-nets.

The translation from sequent proofs to pre-nets is almost immediate: it arises simply by tracing the occurrences of atoms through the sequent proof (for full details see [19]). If we are interested in extracting a $\mathbb{B}$-prenet, we only care if there is a path between two atoms: in the case of $\mathbb{N}$-prenets we are also interested in *how many* paths there are. Neither flavour of LS-net suffers from the non-canonicity problems of Robinson-style nets, but they introduce new problems:

**No polynomial-time correctness algorithm for $\mathbb{B}$-nets** Strassburger and Lamarche give in [19] an exponential-time criterion singling out those $\mathbb{B}$-prenets which correspond to sequent proofs; since the size of a $\mathbb{B}$-net is polynomially bounded by the size of its conclusion, we cannot reasonably hope to do better. The condition given for $\mathbb{N}$-nets in [19] simply collapses a $\mathbb{N}$-net to a $\mathbb{B}$-net and checks correctness of the $\mathbb{B}$-net, the result being that there are "correct" $\mathbb{N}$-nets that are not the translation of any sequent proof. There is some hope that a different polynomial-time correctness criterion might be found for these nets, or for the similar *atomic flows* [15], but none has been found so far, despite substantial effort. Consequently, there is currently no notion of sequentializing $\mathbb{N}$-nets, either into a sequent system or some other calculus.

**Cut-elimination does not preserve correctness** Cut-elimination is easy to define on LS-nets: as shown in [19], it suffices, when opposing atomic contractions in a cut, to simply count the number of paths through the cut between each pair of atoms. This procedure is proved in [19] to be strongly normalizing, confluent, and correctness preserving on $\mathbb{B}$-nets. However, applying this procedure to $\mathbb{N}$-nets, there is a $\mathbb{N}$-net which is the image of a sequent proof, but whose cut reduct is not the image of a sequent proof; cut-reduction does not preserve correctness with respect to the sequent-calculus.

### 4.3 Hughes's Combinatorial proofs

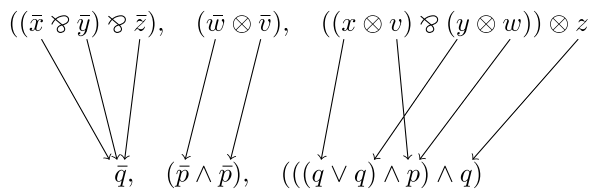
The combinatorial proofs of Hughes [17, 16] are a more radical departure from the standard notions of proof net than Lamarche-Strassburger or Robinson-style nets. Broadly, combinatorial proofs represent classical proofs as "fibered" linear proofs, with the fibring representing the structural rules. The "semi-combinatorial" presentation of combinatorial proofs given in [16] is the most immediately graspable for a proof-theorist: a combinatorial proof of a sequent $\Gamma$ of classical propositional logic is a function $f$ from the leaves of an **MLL** + MIX proof net $\pi$ (which we can represent as a *binary* **MLL** formula, in which atoms occur in dual pairs) to the leaves of $\Gamma$ preserving

- Duality (if leaves $X$ and $Y$ are dual, then so are $f(X)$ and $f(Y)$)
- Conjunctive relationships (If the topmost connective between $X$ and $Y$ is a $\otimes$, then the topmost connective between $f(X)$ and $f(Y)$ is a $\wedge$.

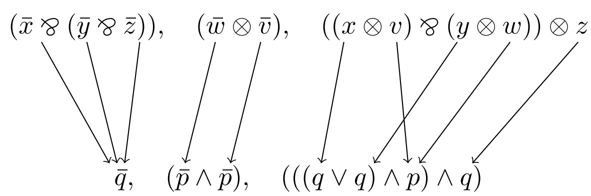
and such that $f$ is a *contraction-weakening*:

- $f$ is built from pure contraction ($c : A \wedge A \to A$), weakening ($w : A \wedge B \to A$), and associativity/commutativity of the connectives, using function composition and "horizontal" composition (if $f : A \to B$ and $g : A' \to B'$ are contraction-weakenings, then so are the evident functions $f \wedge g : A \wedge A' \to B \wedge B'$ and $f \vee g : A \vee A' \to B \vee B'$).

*Example 3.* An example of a semi-combinatorial proof is the following:




Semi-combinatorial proofs suffer from the same problems as naïve nets with regard to associativity of contraction: differences in the association of contractions manifest in the **MLL** + MIX formula: for example, the following is also a semi-combinatorial proof, differing from the one above only by the association of the left-hand $\parr$:




Combinatorial proofs themselves avoid this problem by representing the binary **MLL** + MIX theorem not as a formula, but as its *co-graph*: two **MLL** formulae have the same co-graph if and only if they differ by associativity and commutativity of connectives. Thus, combinatorial proofs provide a sufficiently abstract notion of proof for our purposes.

The contraction-weakening requirement is equivalent to two other requirements, as proved by Hughes: the *skew fibration* condition and the fact that $f$ preserves maximal cliques of conjunctively related leaves. The surprising result of [16] is that these conditions can be checked in polynomial time: thus Combinatorial proofs, unlike unattached naïve classical nets or LS-nets, form a propositional proof system.

Combinatorial proofs fail to satisfy our other two specifications for a good notion of abstract classical proof:

**Sequentialization into a nonstandard calculus** There are combinatorial proofs which are not the image of any sequent-calculus proof, as shown in [16]; Hughes introduces in that paper an extended calculus (the *Homomorphism calculus* for which the map from proofs to invariants is surjective. This calculus can be seen as a generalization of the sequent calculus which replaces the usual structural rules with a homomorphism rule

$$\frac{\Gamma, A}{\Gamma, B}\, f : A \to B \text{ is a contraction-weakening}$$

but is less well understood than the sequent calculus: in addition, it lacks certain desirable properties, such as the subformula property.

**Cut-reduction does not preserve sequent correctness** We might hope that some other, more sophisticated correctness condition might identify the combinatorial proofs arising from sequent calculus derivations. This may be so, but such a correctness criterion would be incompatible with the dynamic aspects of combinatorial proofs shown in [16]. In that paper Hughes defines a notion of combinatorial proof with cut, gives a strongly normalizing cut-elimination procedure for combinatorial proofs which preserves his correctness criterion. However, this procedure does not stay within this subclass of sequent-correct combinatorial proofs.

## 5 Expansion nets

As we saw in the previous section, weakening causes substantial problems in naïve classical proof-nets, but the alternatives ($\mathbb{N}$-nets and combinatorial proofs) lack correctness/sequentialization with respect to a sequent calculus. In this section we give a calculus of nets which retains a connection to the sequent calculus while also having a polynomial-time correctness criterion, without the need for weakening attachment and its attendant noncanonicity.

The basic idea can be seen already in naïve classical nets: if weakening only happens within a disjunction, then attachment is redundant. Let $F$ be a naïve proof-structure. If a weakening subterm Wk of $F$ is the successor of a disjunction, and if the other successor $t$ of that disjunction is not an instance of Wk, we will say that the weakening subterm has a *default attachment*, namely $t$. If every weakening subterm of $F$ has a default attachment, we will say it is *default-attached*. If $F$ is default-attached, the *default attachment* of $F$ is the function from instances of Wk to nodes of $F$ assigning each instance of Wk to its default attachment.

*Example 4.* The net (3) for Pierce's formula is default attached: the only weakening in that net appears as an immediate subtree of a disjunction, and the setting the other disjunct $\bar{x} : \bar{p}$ as the attachment for it yields an ACC correct attached net.

The cw-annotated sequent $x : p,\ \bar{x} : \bar{p},\ \mathrm{Wk} : q$ is not default-attached, as the weakening appears outside of a disjunction. The following is also not default-attached:

$$1 : \top,\ \mathrm{Wk} \wedge \mathrm{Wk} : \bot \wedge \bot,\ 1 : \top,\ \mathrm{Wk} \wedge \mathrm{Wk} : \bot \wedge \bot,\ 1 : \top$$

Since the difficult part of correctness for naïve nets is guessing the attachment, correctness for default-attached nets is easy:

**Proposition 3.** *Correctness of default-attached naïve proof-structures can be checked in polynomial time.*

*Proof.* Correctness for naïve structures is **NP** because the attachment of the weakenings must be guessed. For a default-attached structure, the default attachment can be computed in linear time, and the polynomial correctness algorithm for attached nets may then be applied. □

Default-attached nets improve on general naïve nets by having a polynomial-time verifiable correctness criterion, without the need for an explicit weakening attachment (which compromises the canonicity of naïve nets). However, we still have the problem that contraction is neither associative, commutative, nor pointwise on disjunctions. The first two of these problems were noticed by Girard at the same time he proposed nets for classical logic, and there is an evident solution: make contraction n-ary, while at the same time forbidding either weakenings or contractions from being the successors of a contraction. The last of these problems (pointwise contraction) can be solved by forbidding contraction on disjunctions. We enforce those conditions by moving to a new kind of proof-net, which we call *expansion nets*: these nets were introduced in [23]. The terminology is inspired by Miller's *expansion-tree proofs* [24], which are a representation of proofs in first- and higher-order logic. Expansion-tree proofs represent n-ary contraction in a similar fashion to expansion nets; in expansion trees contraction happens only on existentially quantified subformulae (not on universally quantified formula), and is represented by formal sums (expansions) of witnessing terms rather than binary contractions. Expansion-tree proofs provide a compact, bureaucracy-free representation of proofs for first- and higher-order classical logic; expansion-nets provide a similar technology for propositional classical logic.

Expansion-nets are built from trees we call *propositional expansion trees* (to distinguish from Miller's expansion trees):

**Definition 5 (Propositional Expansion trees).** *Let $\mathcal{X}$ be a set of wire symbols, with $= x, y, \bar{x}, \bar{y} \dots$ the corresponding* wire variables *– atoms over $\mathcal{X}$. An* propositional expansion tree *over $\mathcal{X}$ is of the form $t$ below:*

$$t ::= 1 \mid (w + \cdots + w) \mid (t \vee t) \mid (t \vee *) \mid (* \vee t) \qquad w ::= x \mid \bar{x} \mid t \otimes t$$

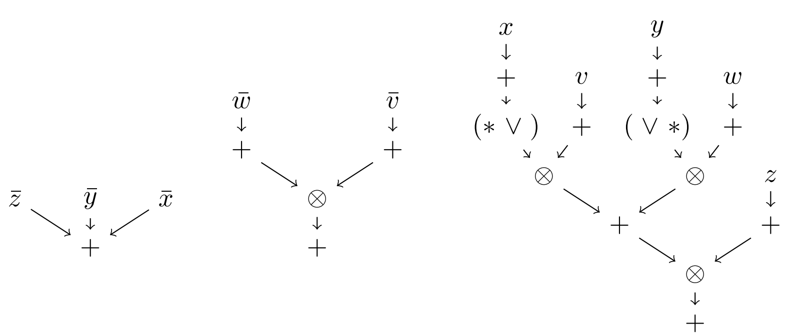
*where $(w + \cdots + w)$ denotes a nonempty finite formal sum, which we call an* expansion*. We call the members of the grammar $w$ "witnesses". In line with the previous section we will call trees of the form $(t \vee *)$ and $(* \vee t)$* default weakenings*.*

Just as cw-trees gave us a succinct way to write down and reason about naïve nets, so propositional expansion trees will give us a nice way to present expansion nets. However, it will be just as important to think of expansion-nets as a graphical proof calculus, in particular when we want to talk about paths in a net. For this purpose, we will need to consider the tree (in the sense of Section 2.2) defined by a propositional expansion tree: that is, a set of nodes and a predecessor function. We should also consider which of the nodes of this tree are ordered.

The parse tree for an expansion-tree/witness (given by the grammars in Definition 5) gives us an immediate reading of a propositional expansion tree (or witness) as a tree: for example, the propositional expansion trees

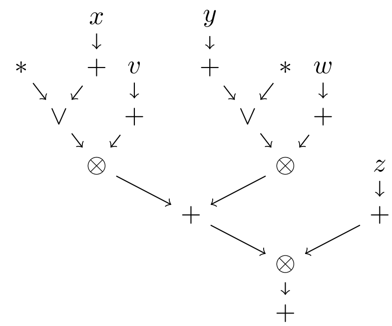
$$(\bar{x} + \bar{y} + \bar{z}) \quad ((\bar{w}) \otimes (\bar{v})) \quad ((((* \vee (x)) \otimes (v)) + (((y) \vee *) \otimes (w))) \otimes (z))$$

can be seen as trees




(4)

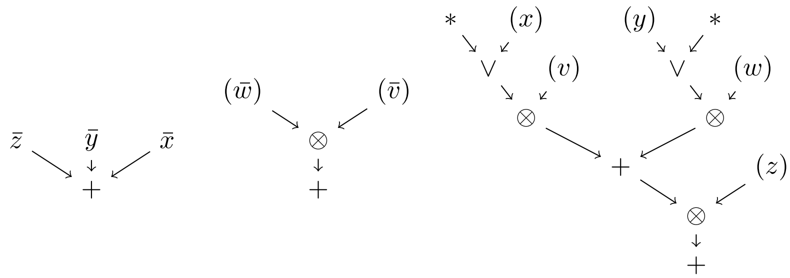
However, this tree-reading of an expansion-tree treats the subtrees $(t \vee *)$ and $(* \vee t)$ as having only one successor. It will be useful at certain points to regard $*$ as a subtree of $(t \vee *)$ (resp $(* \vee t)$) even though the symbol $*$ never appears outside of a default weakening. Treating the occurrences of $*$ as nodes, we obtain the tree




(5)

We will call the nodes of a propositional expansion tree which are not instances of $*$ *proper nodes*.

When showing examples of expansion-nets, we will sometimes not show the expansion structure on trivial expansions of atomic type: this improves readability and makes some diagrams smaller. For example, using this shorthand the three expansion trees above are:




(6)

The successors of a node $t \vee s$ or $t \otimes s$ are the nodes $t$ and $s$: these nodes are ordered, as they correspond to the sequent-calculus introduction rules for the connectives. The successors of a node $(w_1 + \cdots + w_n)$ are the nodes $w_1$ to $w_n$. Since $+$ denotes a formal sum, the successors of an expansion are *unordered*: this corresponds to the fact that contraction is a symmetric operation.

The successors of a node $(t \vee *)$ are the node $t$ and a node labelled $*$, ordered such that the order of $t$ is 0 and the order of the $*$ is 1. Similarly for $(* \vee t)$, but with the orders reversed. The nodes labelled with $*$, $x$, $\bar{x}$ and 1 have no successors: they are the leaves of the tree.

Our proof structures will be *typed* forests of propositional expansion trees: we type propositional expansion trees with formulae of classical propositional logic. To maintain the associativity and commutativity of the formal sum (which interprets contraction), we make a distinction at the level of types between witnesses and expansions: the expansions recieve a special "witness types", while the expansion is typed with a formula. This enforces that contractions are n-ary and of maximum size.

**Definition 6.** *A* type *is either*

*(a) A formula of classical propositional logic;*
*(b) A* witness type*: one of the three following forms:*
- *A* positive witness type*, written $[a]$, where $a$ is a positive atom;*
- *A* negative witness type*, written $[\bar{a}]$, where $\bar{a}$ is a negative atom; or*
- *A conjunctive witness type, written $A \otimes B$, where $A$ and $B$ are formulae of propositional classical logic.*

*Each witness type has an underlying classical formula: for $A \otimes B$ this is $A \wedge B$, for $[a]$ this is $a$ and for $[\bar{a}]$ this is $\bar{a}$.*

*A* typed tree/typed witness *is a pair of a propositional expansion tree/witness and a type, derivable in the typing system shown in Figure 6. This typing system should be thought of as an analogue of Figure 3 for expansion-nets: it specifies the shape of the "proof-structures" we consider.*

$$\frac{}{\bar{x} : [\bar{p}]} \qquad \frac{}{1 : \top} \qquad \frac{t : B}{(* \vee t) : A \vee B} \qquad \frac{t : A}{(t \vee *) : A \vee B} \qquad \frac{}{x : [p]}$$

$$\frac{t : A \quad s : B}{(t \vee s) : A \vee B} \qquad \frac{t : A \quad s : B}{t \otimes s : A \otimes B}$$

$$\frac{w_1 : [p] \ \cdots \ w_n : [p]}{(w_1 + \cdots + w_n) : p} \qquad \frac{w_1 : [\bar{p}] \ \cdots \ w_n : [\bar{p}]}{(w_1 + \cdots + w_n) : \bar{p}} \qquad \frac{w_1 : A \otimes B \ \cdots \ w_n : A \otimes B}{(w_1 + \cdots + w_n) : A \wedge B}$$

**Fig. 6.** Typing derivations for propositional expansion trees

*Example 5.* The wire variable $x$ can be assigned the witness type $[p]$, while the expansions $(x)$ (a *trivial* expansion) and $(x + y)$ can be assigned as a type the propositional formula $p$.

*Example 6.* The following are correctly typed propositional expansion trees:

$$(\bar{x} + \bar{y} + \bar{z}) : \bar{q} \quad ((\bar{w}) \otimes (\bar{v})) : (\bar{p} \wedge \bar{p}) \quad ((((* \vee (x)) \otimes (v)) + (((y) \vee *) \otimes (w))) \otimes (z)) : (((q \vee q) \wedge p) \wedge q)$$

**Definition 7.** *A* typed forest *is a finite forest $F$ of typed propositional expansion trees and witnesses, in which axiom variables occur in dual pairs: that is*

*(a) each axiom variable $x$, and each negated variable $\bar{y}$, occurs at most once in $F$, and*
*(b) there is an occurrence of $\bar{x}$ in $F$ if and only if there is an occurrence of $x$.*

*The type of a typed forest $F$ is the forest of types of the terms in $F$. We will say that $F$ is an e-annotated sequent if all the terms in $F$ are expansion-trees: equivalently, if the type of $F$ is a sequent of classical propositional logic (that is, it contains no witness types).*

*Example 7.* The forest consisting of the three typed propositional expansion trees shown in Example 6 is an e-annotated sequent.

*Example 8.* The following is a typed forest:

$$((\bar{w}) \otimes (\bar{v})) : \bar{p} \wedge \bar{p},\ w : [p],\ v : [p]$$

It is not an e-annotated sequent, since some of its roots are witnesses.

The e-annotated sequents are our notion of *proof-structure*; the more general notion of typed forests is needed to study subproofs and cut-elimination.

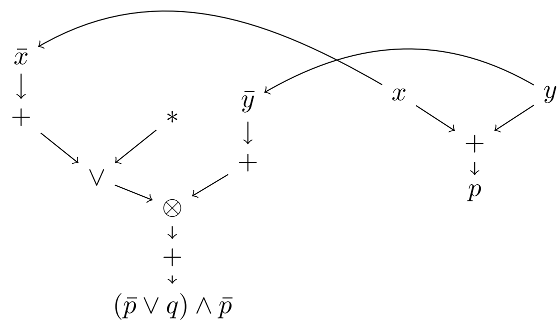
*Example 9.* The following e-annotated sequent arises by annotating the standard proof of Pierce's law

$$(((\bar{x}) \vee *) \otimes (\bar{y})) : (\bar{p} \vee q) \wedge \bar{p}, \quad (x + y) : p \tag{7}$$

As with cw-annotated sequents, we can consider the *graph* of this annotated sequent by adding in the axiom wires, giving a representation of our proof-structures closer to that usually seen for proof-nets:

**Definition 8.** *The* graph *of an e-annotated sequent $F$ is a directed graph with vertices identical to the nodes of the forest of $F$. The edges of the graph are given by the forest structure (with edges directed toward the root), plus an edge from $x$ to $\bar{x}$ for each wire variable $x$ appearing in $F$.*

For example, this graph represents the proof of Pierce's formula given above:



(8)

The $e$-annotated sequents are our notion of proof structure: the expansion nets are those $e$-annotated sequents which arise from sequent proofs in $\mathbf{LK}^*$. The procedure of inductively constructing a proof-net from a sequent proof is given via the annotated sequent calculus shown in Figure 7.

**Definition 9.** *An* expansion-net *is an e-annotated sequent derivable in the system shown in Figure 7.*

*Remark 4.* Notice that the order in which contractions occur in the sequent proof is no longer relevant to the net derived, as it was in naïve classical nets, since we represent contractions by the formal sum of witnesses. This can be seen in the following two examples of annotated derivations:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{(\bar{x}):\bar{a},\ (x):a}\,\mathrm{Ax} \qquad \cfrac{}{(\bar{y}):\bar{a},\ (y):a}\,\mathrm{Ax}}{(\bar{x}):\bar{a},\ (\bar{y}):\bar{a}, (x\otimes y):a\wedge a}\,\wedge \qquad \cfrac{}{(\bar{z}):\bar{a},\ (z):a}\,\mathrm{Ax}}{(\bar{x}):\bar{a},\ (\bar{y}):\bar{a},\ \bar{z}:\bar{a},\ ((x\otimes y)\otimes z):(a\wedge a)\wedge a}\,\wedge}{(\bar{x}):\bar{a},\ (\bar{y}+\bar{z}):\bar{a},\ ((x\otimes y)\otimes z):(a\wedge a)\wedge a}\,\mathrm{C}}{(\bar{x}+\bar{y}+\bar{z}):\bar{a},\ ((x\otimes y)\otimes z):(a\wedge a)\wedge a}\,\mathrm{C}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{(\bar{x}):\bar{a},\ (x):a}\,\mathrm{Ax} \qquad \cfrac{}{(\bar{y}):\bar{a},\ (y):a}\,\mathrm{Ax}}{(\bar{x}):\bar{a},\ (\bar{y}):\bar{a}, (x\otimes y):a\wedge a}\,\wedge \qquad \cfrac{}{(\bar{z}):\bar{a},\ (z):a}\,\mathrm{Ax}}{(\bar{x}):\bar{a},\ (\bar{y}):\bar{a},\ \bar{z}:\bar{a},\ ((x\otimes y)\otimes z):(a\wedge a)\wedge a}\,\wedge}{(\bar{x}+\bar{z}):\bar{a},\ (\bar{y}):\bar{a},\ ((x\otimes y)\otimes z):(a\wedge a)\wedge a}\,\mathrm{C}}{(\bar{x}+\bar{y}+\bar{z}):\bar{a},\ ((x\otimes y)\otimes z):(a\wedge a)\wedge a}\,\mathrm{C}$$

*Example 10.* The e-annotated sequent

$$(\bar{x}+\bar{y}+\bar{z}):\bar{q} \quad ((\bar{w})\otimes(\bar{v})):(\bar{p}\wedge\bar{p}) \quad ((((*\vee(x))\otimes(v))+(((y)\vee *)\otimes(w)))\otimes(z)):(((q\vee q)\wedge p)\wedge q)$$

is an expansion-net: if we let $t = (((y)\vee *)\otimes(w))$, and $s = ((*\vee(x))\otimes(v))$, then we have the following derivation:

$$\frac{}{1:\top}\,\mathrm{Ax}_\top \qquad \frac{F \quad G}{F,\ G}\,\textsc{Mix} \qquad \frac{}{(\bar{x}):\bar{p},\ (x):p}\,\mathrm{Ax}$$

$$\frac{F,\ t:A,\ s:B}{F,\ t\vee s:A\vee B}\,\vee \qquad \frac{F,\ t:A}{F,\ t\vee *:A\vee B}\,\vee_0 \qquad \frac{F,\ s:B}{F,\ *\vee s:A\vee B}\,\vee_1 \qquad \frac{F,\ t:A \qquad G,\ s:B}{F,G,\ (t\otimes s):A\wedge B}\,\wedge$$

$$\frac{F,\ t:A\wedge B,\ s:A\wedge B}{F,\ t+s:A\wedge B}\,\mathrm{C}_\wedge \qquad \frac{F,\ s:p,\ t:p}{F,\ s+t:p}\,\mathrm{C}_p \qquad \frac{F,\ s:\bar{p},\ t:\bar{p}}{F,\ s+t:\bar{p}}\,\mathrm{C}_{\bar{p}}$$

**Fig. 7.** $\mathbf{LK}^*_e$: an annotated version of $\mathbf{LK}^*$ deriving expansion nets

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\bar{x}):\bar{q},(x):q}\,\mathrm{Ax}}{(\bar{x}):\bar{q},(*\vee(x)):(q\vee q)}\,\vee_1 \quad \dfrac{}{(w):p,(\bar{w}):\bar{p}}\,\mathrm{Ax}}{(\bar{x}):\bar{q},t:((q\vee q)\wedge p),(\bar{w}):\bar{p}}\,\wedge \quad \dfrac{\dfrac{\dfrac{}{(\bar{y}):\bar{q},(y):q}\,\mathrm{Ax}}{(\bar{y}):\bar{q},((y)\vee *):(q\vee q)}\,\vee_0 \quad \dfrac{}{(v):p,(\bar{v}):\bar{p}}\,\mathrm{Ax}}{(\bar{y}):\bar{q},s:((q\vee q)\wedge p),(\bar{v}):\bar{p}}\,\wedge}{((\bar{w})\otimes(\bar{v})):(\bar{p}\wedge\bar{p}),(\bar{x}):\bar{q},(\bar{y}):\bar{q},t:((q\vee q)\wedge p),s:((q\vee q)\wedge p)}\,\wedge}{((\bar{w})\otimes(\bar{v})):(\bar{p}\wedge\bar{p}),\ (\bar{x}+\bar{y}):\bar{q},\ s+t:((q\vee q)\wedge p)}\,\mathrm{C}^2 \quad \dfrac{}{(z):q,(\bar{z}):\bar{q}}\,\mathrm{Ax}}{(\bar{x}+\bar{y}):\bar{q},\ (\bar{z}):\bar{q},((\bar{w})\otimes(\bar{v})):(\bar{p}\wedge\bar{p}),\ s+t:(((q\vee q)\wedge p)\wedge q)}\,\wedge}{(\bar{x}+\bar{y}+\bar{z}):\bar{q},\ ((\bar{w})\otimes(\bar{v})):(\bar{p}\wedge\bar{p}),\ ((s+t)\otimes(z)):(((q\vee q)\wedge p)\wedge q)}\,\mathrm{C}$$

Cut-free formula-completeness of $\mathbf{LK}^*$ gives us the following:

**Theorem 3.** *A formula $A$ of classical propositional logic is valid if and only if there is an expansion net $t:A$.*

Each e-annotated sequent corresponds to an equivalence-class of default-attached cw-annotated sequents, modulo the associativity and commutativity of contraction and the pointwise construction of contractions. Furthermore, it is easy to verify that, given an equivalence class of cw-annotated sequents induced by an e-annotated sequent, either all or none of them are correct. Thus, correctness of a member of the equivalence class can be used to define a notion of correctness for expansion-nets. However, it will be useful later to consider the idea of a switching path in an expansion-net, and for this reason we give now an independent definition of correctness for expansion-nets – actually, for all typed forests. We give here the notion of AC-correctness (AC for ACyclic, as distinct from ACC, ACyclic and Connected, the usual criterion for $\mathbf{MLL}^-$ nets) for typed forests:

**Definition 10.** *Let $F$ be a typed forest*

*(a) A node $X$ of $F$ is a* switched node *if it is an expansion, or if it is a $\vee$ node $t\vee s$ where neither $t$ nor $s$ is an instance of $*$.*
*(b) A switching $\sigma$ for $F$ is a choice of successor for each switched node.*
*(c) The switching graph $\sigma(F)$ is obtained from the graph of $F$ by:*
  *1: deleting all incoming edges to each switched node other than those coming from the nodes chosen by the switching, and*
  *2: forgetting the directedness of edges.*
*(d) $F$ is AC-correct if, for every switching $\sigma$ of $F$, $\sigma(F)$ is acyclic.*

*Remark 5.* Notice that nodes of the form $(t\vee *)$ and $(*\vee t)$ are *unswitched*; we can see this as implicitly adding to the switching graph an attachment from $*$ to $t$.

While the switching graph definition of correctness suggests an exponential-time correctness algorithm, it is essentially the same as the **MLL** + Mix switching criterion, and can therefore be checked in polynomial time; such a polynomial time algorithm is given, for example, by attempting to sequentialize by searching for *splitting pars*, a technique first described in [8], and available in English translation in the Linear Logic Primer [9].

A useful notion arising from the switching graphs is that of a *switching path*: a nonempty sequence $P = X_1, \dots X_n$ of nodes of $F$ which defines a path in some switching graph $F\sigma$ of $F$. The AC correctness criterion can, using this notion, be stated as follows: an annotated sequent is correct if it all its switching paths are acyclic. We will refer to a switching path as "entering a node $X$ through its successor $Y$" (or "entering $X$ from above") on a switching path $P$ if the node $Y$ is immediately followed by the node $X$ in $P$, and "entering a node $X$ through its predecessor $Z$" (or "entering $X$ from below") if $X$ is immediately preceded by $Z$ in $P$. Terminology related to a path leaving a node through predecessors/successor is defined analogously.

The *AC* correctness criterion characterizes, of course, those e-annotated sequents derivable in $\mathbf{LK}_e^*$:

**Theorem 4.** *An e-annotated sequent $F$ is an expansion-net (i.e. $\mathbf{LK}_e^* \vdash F$) if and only if $F$ is AC-correct.*

This result can be proved via a number of techniques, including the aforementioned "splitting pars" technique, or the earlier "splitting tensors" technique. The latter was adapted for **MLL** + Mix by Bellin in [1]. In Section 6 below, we give a proof directly for expansion-nets which uses the new notion of a contiguous subnet.

### 5.1 Comparison with other notions of invariant

It should be clear that expansion nets identify more proofs of $\mathbf{LK}^*$ than naïve classical nets: two proofs differing only by the order in which contractions are performed will have different naïve nets but the same expansion net. We take some time now to compare the equivalence classes of proofs defined by expansion nets and the other existing notions of abstract proof – $\mathbb{N}$-nets and combinatorial proofs (since $\mathbb{B}$-nets identify strictly more proofs than $\mathbb{N}$-nets, we will not consider them further).

**$\mathbb{N}$-nets identify more $\mathbf{LK}^*$ derivations than expansion nets** To obtain an $\mathbb{N}$-net from a given derivation, one simply traces paths from positive to negative atoms in the conclusion of the proof. For both $\mathbf{LK}^*$ derivations and expansion-nets, there is an obvious way to this: and it is not difficult to establish he following:

**Proposition 4.** *Let $\Phi$ be an $\mathbf{LK}^*$ derivation, and let $F$ be its corresponding expansion net. The $\mathbb{N}$-nets of $\Phi$ and $F$ coincide.*

This means that expansion-nets cannot distinguish two proofs identified by their $\mathbb{N}$-nets; said differently, expansion nets contain at least as much information as $\mathbb{N}$-nets. In fact, they contain strictly more information. Consider the following two sequent derivations proving the same sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bar a, a}{\bar a \vee \bar b, a}\vee_0 \qquad \dfrac{\bar c, c}{\bar c \vee \bar d, c}\vee_0}{(\bar a \vee \bar b)\wedge(\bar c \vee \bar d), a \vee c}\wedge,\vee \qquad \dfrac{\dfrac{\bar b, b}{\bar a \vee \bar b, b}\vee_1 \qquad \dfrac{\bar d . d}{\bar c \vee \bar d, d}\vee_1}{(\bar a \vee \bar b)\wedge(\bar c \vee \bar d), b \vee d}\wedge,\vee}{(\bar a \vee \bar b)\wedge(\bar c \vee \bar d),\ (\bar a \vee \bar b)\wedge(\bar c \vee \bar d),\ (a \vee c)\wedge(b \vee d)}\wedge}{(\bar a \vee \bar b)\wedge(\bar c \vee \bar d), (a \vee c)\wedge(b \vee d)}\mathrm{C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{a, \bar a}{a \vee c, \bar a}\vee_0 \qquad \dfrac{b, \bar b}{b \vee d, \bar b}\vee_0}{(a \vee c)\wedge(b \vee d), \bar a \vee \bar b}\wedge,\vee \qquad \dfrac{\dfrac{\bar c, c}{a \vee c, \bar c}\vee_1 \qquad \dfrac{d, \bar d}{b \vee d, \bar d}\vee_1}{(a \vee c)\wedge(b \vee d), (\bar c \vee \bar d)}\wedge,\vee}{(\bar a \vee \bar b)\wedge(\bar c \vee \bar d),\ (a \vee c)\wedge(b \vee d), (a \vee c)\wedge(b \vee d)}\wedge}{(\bar a \vee \bar b)\wedge(\bar c \vee \bar d), (a \vee c)\wedge(b \vee d)}\mathrm{C}$$

These two proofs have the same $\mathbb{N}$-net, with one link between each pair of dual atoms, but different expansion nets:

$$\begin{tikzcd}[column sep=tiny]
\bar{x} \arrow[dr] & & * \arrow[dl] & & \bar{y} \arrow[dr] & & * \arrow[dl] & & * \arrow[dr] & & \bar{z} \arrow[dl] & & * \arrow[dr] & & \bar{w} \arrow[dl] \\
& \vee \arrow[drr] & & & & \vee \arrow[dll] & & & & \vee \arrow[drr] & & & & \vee \arrow[dll] & \\
& & & \otimes \arrow[drrrr] & & & & & & & & \otimes \arrow[dllll] & & & \\
& & & & & & & + \arrow[d] & & & & & & & \\
& & & & & & & (\bar{a} \vee \bar{b}) \wedge (\bar{c} \vee \bar{d}) & & & & & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
x \arrow[dr] & & y \arrow[dl] & & z \arrow[dr] & & w \arrow[dl] \\
& \vee \arrow[drr] & & & & \vee \arrow[dll] & \\
& & & \otimes \arrow[d] & & & \\
& & & + \arrow[d] & & & \\
& & & (a \vee c) \wedge (b \vee d) & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
\bar{x} \arrow[dr] & & \bar{y} \arrow[dl] & & \bar{z} \arrow[dr] & & \bar{w} \arrow[dl] \\
& \vee \arrow[drr] & & & & \vee \arrow[dll] & \\
& & & \otimes \arrow[d] & & & \\
& & & + \arrow[d] & & & \\
& & & (\bar{a} \vee \bar{b}) \wedge (\bar{c} \vee \bar{d}) & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
x \arrow[dr] & & * \arrow[dl] & & y \arrow[dr] & & * \arrow[dl] & & * \arrow[dr] & & z \arrow[dl] & & * \arrow[dr] & & w \arrow[dl] \\
& \vee \arrow[drr] & & & & \vee \arrow[dll] & & & & \vee \arrow[drr] & & & & \vee \arrow[dll] & \\
& & & \otimes \arrow[drrrr] & & & & & & & & \otimes \arrow[dllll] & & & \\
& & & & & & & + \arrow[d] & & & & & & & \\
& & & & & & & (a \vee c) \wedge (b \vee d) & & & & & & &
\end{tikzcd}$$

Identifying these two proofs, as suggested by their $\mathbb{N}$-nets, does not seem at all natural in the multiplicatively formulated sequent calculus (it arises very naturally, however, in the *deep inference* proof theory of classical logic [5, 4], which provided inspiration for the design of $\mathbb{N}$-nets.) In light of this, and the sequentialization theorem, we claim that expansion-nets provide a better notion of abstract proof for sequent proofs than $\mathbb{N}$-nets.

**Combinatorial proofs identify at least as many $\mathbf{LK}^*$ derivations as expansion nets** To see how to extract a combinatorial proof from an expansion-net, we will need the following intuitive notion: an expansion tree $F$ of type $\Gamma$ induces a function $f$ from the wire variables of $F$ to the leaves (atom occurrences) of $\Gamma$. This function arises in much the same way as the $\mathbb{N}$-net of an expansion-net: by tracing the atoms through the tree. Given an expansion net $F$, extract a co-graph from $F$ as follows: the vertices of the co-graph are the wire variables of $F$, and there is an edge between two wire variables if and only if smallest subtree of $F$ containing both variables is an $\otimes$ tree. The function $f$ from wire variables to atoms is a contraction-weakening, by the structure of propositional expansion trees, and so the pair of co-graph and function given by an expansion-net defines a correct combinatorial proof. For example, the expansion-net in Example 6 yields the (semi-)combinatorial proof in Example 3.

This combinatorial proof is the same proof as would be extracted directly from an $\mathbf{LK}^*$ derivation giving rise to $F$: thus combinatorial proofs identify, at the very least, all the proofs identified by expansion-nets. It is likely that, in fact, combinatorial proofs identify the same $\mathbf{LK}^*$ derivations as expansion-nets; if so, this would provide a criterion identifying just those combinatorial proofs arising from $\mathbf{LK}^*$ derivations.

## 6 Subnets of expansion nets

In the sequent calculus, we have a clear notion of "subproof of a sequent proof", given by subtrees. In proof-nets, it is harder to see, intuitively, the correct notion of subproof, and this causes a number of conceptual problems when manipulating proofs. The notion of "subnet" captures, in proof nets, the concept of subproof.

Subproofs play two important roles in the proof theory of classical logic. The first is that proving cut-elimination often relies on a principal lemma in which it is shown that a single cut can be eliminated from an otherwise cut-free proof: in this case the cut is always the final rule in the proof. Full cut-elimination then follows by considering uppermost cuts: the subproof introducing an uppermost cut contains no other cuts. In proof nets, there is no clear notion of uppermost cut, or of the subproof containing a cut. It is with a view to obtaining such a notion that we define the subnets of a net.

The second role that subproofs play is in the definition of cut-reduction steps, where one of the cut-formulae is the result of a structural rule. For example, the usual way to reduce a cut against contraction, such as

$$\dfrac{\begin{array}{c}\vdots\,\Phi\\ F, A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\Psi\\ F', \bar{A}, \bar{A}\end{array}}{F, \bar{A}}\,C}{F, F'}\,\textsc{Cut} \tag{9}$$

is to duplicate the subproof $\Phi$, and then contract the resulting duplicated conclusions:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\Phi\\ F, A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\Phi\\ F, A\end{array} \qquad \begin{array}{c}\vdots\,\Psi\\ F'\bar{A}, \bar{A}\end{array}}{F, F', \bar{A}}\,\textsc{Cut}}{F, F, F'}\,\textsc{Cut}}{F, F'}\,\mathrm{C}^* \tag{10}$$

To perform such an operation in proof-nets requires that we know what a subproof is, and can find them. In linear logic proof nets with exponentials, duplication and deletion are typically mediated by *boxes* – that is, the subgraphs to be duplicated are explicitly marked regions of the net. Expansion-nets are box-free, and so the appropriate subgraph to delete or duplicate must be calculated; further, we must ensure that this duplication or deletion does not break correctness.

A *subnet* of an expansion-net $F$ (a concept first introduced for $\mathbf{MLL}^-$ nets in [3]) is a graph corresponding to a subproof of $F$: we define them as follows:

**Definition 11.** *Let $F$ be a typed forest: a substructure of $F$ is a subforest $G$ of $F$ which is*

- *closed under axiom links: that is, if the leaf annotated with $x$ is in $G$, then so is the leaf annotated with $\bar{x}$.*
- *closed under default attachment: that is, if an instance of $*$ occurs in $G$, then its predecessor $(t \vee *)$ or $(* \vee t)$ is in $G$.*

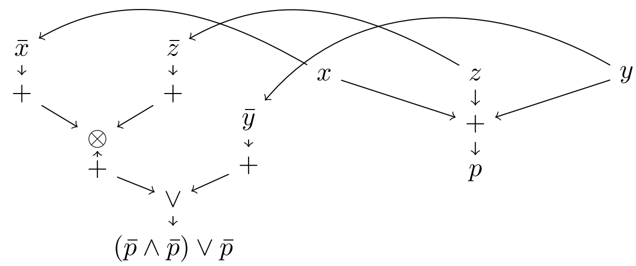
*If $X$ is a node of $F$, let $\mathrm{str}(X)$ be the smallest substructure of $F$ containing $X$.*

**Definition 12.** *Let $F$ be an AC typed forest: a* subnet *of $F$ is a substructure $G$ of $F$ such that, for any two roots $X$, $Y$ of $G$, every switching path between $X$ and $Y$ lies inside $G$.*

A more obvious (but incorrect) notion of subnet for expansion-nets would be, simply, a subforest which is, itself, an expansion net. This simplistic kind of definition works for $\mathbf{MLL}^-$ proof-nets, for example. Consider, however, the following sequent proof in classical logic:

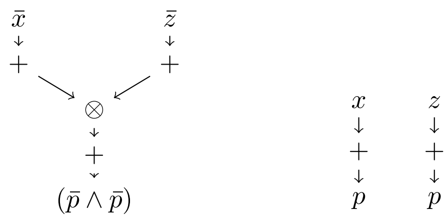
$$\pi = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bar{p}, p}\,\textsc{Ax} \qquad \dfrac{}{\bar{p}, p}\,\textsc{Ax}}{(\bar{p} \wedge \bar{p}), p, p}\,\mathrm{C}}{(\bar{p} \wedge \bar{p}), p}\,\mathrm{C} \qquad \dfrac{}{\bar{p}, \quad p}\,\textsc{Ax}}{(\bar{p} \wedge \bar{p}), \bar{p}, \quad p, p}\,\textsc{Mix}}{(\bar{p} \wedge \bar{p}) \vee \bar{p}, \quad p, p}\,\vee}{(\bar{p} \wedge \bar{p}) \vee \bar{p}, \quad p}\,\mathrm{C}$$

The expansion net $F$ corresponding to $\pi$ is:



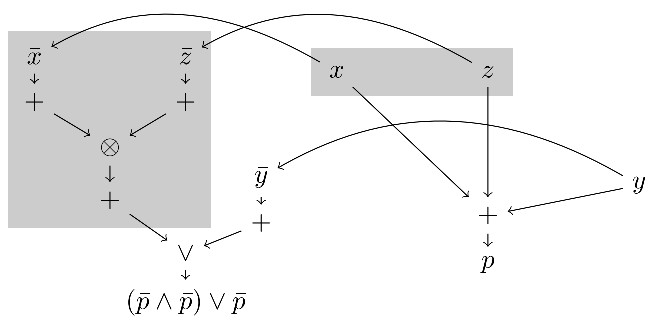

(11)

Now consider the sub-proof of the sequent proof proving $p \wedge p, p, p$. The expansion-net corresponding to that proof is the following:




(12)

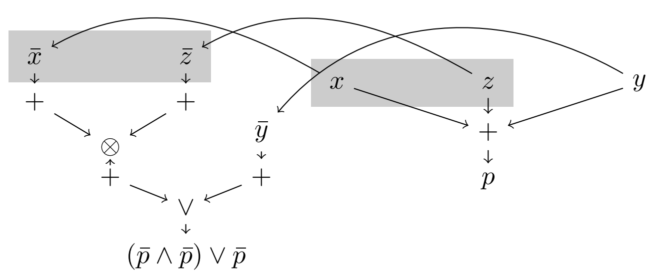
This does not appear as a subforest of $F$; in other words, the subforests of $F$ which are themselves expansion-nets do not suffice to express the sub-proofs of $F$. The subnet corresponding to the subproof is in this case *not* an expansion net: it is the shaded subgraph in the following:




(13)

or, alternatively, $(\bar{x} + \bar{y}) : \bar{p} \wedge \bar{p}, x : [p], y : [p]$, which is not an expansion net, as it has witnesses as roots.

Now consider the following shaded substructure of $F$, which is *not* a subnet of $F$:




(14)

This typed forest satisfies the AC correctness criterion: each of its switching graphs is acyclic. However, it is not a subnet of $F$, since there is a switching path from $\bar{x}$ to $\bar{z}$ which passes outside the shaded substructure. This shaded substructure does *not* correspond to any subproof of $\pi$, nor of any other sequentialization of $F$. For more discussion on subnets in the presence of the mix rule, see [1].

### 6.1 Kingdoms and Empires

Given a proper node $X$ (that is, a node which is not an instance of $*$), the subnets with $X$ as a root correspond to subproofs with $X$ in the conclusion.

Given any proper node $X$, the set of subnets with $X$ as a root are closed under intersection:

**Lemma 3.** *Let $G_1$ and $G_2$ be subnets of an AC typed forest $F$ having the node $X$ as a root. Let $G_1 \cap G_2$ denote the substructure of $F$ defined by the nodes of $F$ common to $G_1$ and $G_2$. Then $G_1 \cap G_2$ is a subnet of $F$.*

*Proof.* Suppose there is a switching path $P$ between two roots of $G_1 \cap G_2$ but outside of $G_1 \cap G_2$. If both $X$ and $Y$ are roots of $G_1$, then $G_1$ is not a subnet, similarly for $G_2$: therefore without loss of generality $X$ is only a root of $G_1$, and $Y$ only a root of $G_2$. Since $Y$ is not a root of $G_1$, $P$ passes through some root of $G_1$: but then the path from that root to $X$ is a path between two roots of $G_1$, outside of $G_1$, and so $G_1$ is not a subnet.

This means, particular, that we can consider the smallest subnet with $X$ as a root, given by taking the intersection of all such subnets : the following terminology originates in [3].

**Definition 13.** *Let $F$ be an AC typed forest, and let be $X$ a node of $F$, such that at least one subnet of $F$ has $X$ as a root. The kingdom $k(X)$ of $X$ in $F$ is the smallest subnet of $F$ which has $X$ as a root.*

Notice that, by this definition, only proper nodes can have a kingdom or empire: there is no substructure of any expansion net having a $*$ as a root.

*Example 11.* The shaded net shown in example 13 is the kingdom of its leftmost root.

Kingdoms are of interest because they allow us to see additional dependencies between nodes in an expansion-net. If in an expansion-net $F$ a node $Y$ is in the kingdom of a node $X$, then in every sequentialization of F (every $\mathbf{LK}^*_e$ derivation resulting in $F$) the rule introducing $Y$ will occur in a subproof of the rule introducing $X$. We will use the relation symbol $\ll$ to denote this *kingdom ordering*:

$$X \ll Y \text{ if and only if } X \text{ is in the kingdom of } Y.$$

This relation plays a key role in our proof of cut-elimination for expansion-nets (Theorem 6). It allows us to recover a notion of "uppermost cut" in an expansion net: a cut which is $\ll$-maximal corresponds to a cut which can be sequentialized such that no other cut lies above it.

The relation $\ll$ also plays an important role in our proof of sequentialization for expansion-nets (Theorem 5). In fact, sequentialization is nothing more than the completion of the relation $\ll$ to a tree-relation on the nodes of an expansion net. We will need, in the proof of the sequentialization theorem, the following fact: two nodes of an AC typed forest have the same kingdom if and only if they are a pair of dual wire variables.

**Proposition 5.** *$\ll$ is a preorder on the proper nodes of an AC typed forest $F$, and moreover is a partial order on the nonatomic proper nodes of $F$.*

*Proof.* The relation $\ll$ is clearly reflexive and transitive. We show that it is antisymmetric if restricted to the nonatomic nodes of $F$. Suppose that $X$ and $Y$ are distinct nodes of $F$, and that $X \in k(Y)$ and $Y \in k(X)$. Then clearly $k(X) = k(Y)$, since otherwise the intersection of $k(X)$ and $k(Y)$ would be a smaller subnet with both $X$ and $Y$ as roots. This equality holds in the case where $X$ and $Y$ are dual wire variables: the two ends of a wire arising from an axiom link: we must now show that it cannot hold if either $X$ or $Y$ is nonatomic. Suppose first that $X$ is a disjunction or nontrivial expansion; then by removing $X$ from $k(Y)$ (but keeping its successors) we find a smaller subnet with $Y$ as a root: contradiction. Now suppose that $X = (X_1 \otimes X_2)$. Then $k(X) = k(X_1) \cup k(X_2) \cup \{X\}$, and so $Y$ is a member of $k(X_i)$ for $i \in \{0, 1\}$. Since $Y \in k(X_i)$ and $k(X_i) \in k(Y)$, we have as above that $k(X_i) = k(Y)$. But $X \notin k(X_i)$; contradiction.

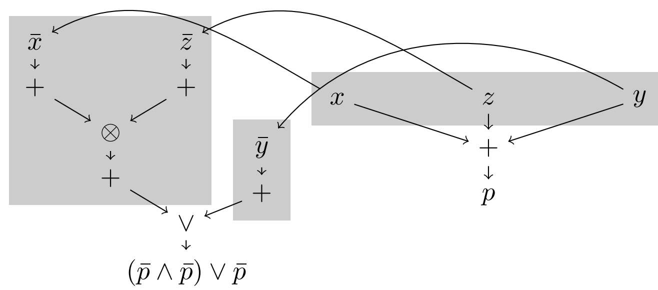
We have not yet shown that every proper node of an expansion-net has a kingdom. Bellin shows directly in [1] that every node has a kingdom, but this is a rather difficult proof: for an easier proof we turn now to the new notion of contiguousness.

### 6.2 Contiguous subnets

A natural counterpart to the notion of kingdom, the smallest subnet with a given node as root, is the notion of empire:

**Definition 14.** *Let $F$ be an AC-correct typed forest, and $X$ a proper node of $F$. The* empire $e(X)$ *of $X$ in $F$ is the largest subnet of $F$ with $X$ as a root.*

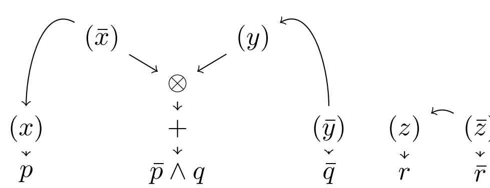
*Example 12.* Continuing our example from above, the shaded subnet in the following is the empire of its leftmost root:




(15)

In the absence of the mix rule (that is, if we assume that every switching graph is not only acyclic, but also connected), the empire is a very useful concept: it is very easy to show that every proper node of an AC-correct typed forest has an empire (indeed, there is a simple inductive definition of the empire of a node, see [3]). However, the simple proof of the existence of the empire does not carry over for proof nets with mix. In addition, the very notion of "empire" is less appealing in the presence of mix. Without mix, we have that the union of two intersecting subnets is a subnet, and therefore that the empire of a node $X$ exists if any subnet with $X$ as a root exists. Furthermore, we have the following "simultaneous empire lemma": if $X$ and $Y$ are two proper nodes, and $Y$ is not in $e(X)$, then either $e(X) \subset e(Y)$ or $e(X) \cap e(Y) = \emptyset$. The following example shows that neither of these properties hold in the presence of mix:

*Example 13.* Consider the following expansion net, which cannot be derived with the Mixrule:




The empire of $(\bar{x})$ is $(x), (\bar{x}), (z), (\bar{z})$. Similarly, the empire of $(y)$ is $(y), (\bar{y}), (z), (\bar{z})$. However, the union of those two subnets is not a subnet, since there is a switching path from $(\bar{x})$ to $(y)$ outside of it. Notice also that, while $(\bar{x})$ is not in $e((y))$, and $(y)$ is not in $e((\bar{x}))$, the two empires intersect (that is, the simultaneous empire property fails).

In this section we define a more appealing counterpart to the empire for proof-nets with mix: the "contiguous empire" of a node. It is easier to show directly that each node has a contiguous empire than to show directly that each node has a kingdom: moreover, the notion is useful in proving sequentialization of expansion-nets, and allows us to define in Section 7.1 a more pleasing notion of cut-reduction.

The new notion we introduce, to define the contiguous empire, is the property that an AC typed forest is *contiguous* with respect to one of its roots:

**Definition 15.** *(a) Let $F$ be an AC typed forest, and let $X$ be a root of $F$. We say that $F$ is* contiguous with respect to $X$ *if there is a switching path from $X$ to every other node $Y$ of $F$.*

*(b) Let $F$ be an AC typed forest and let $X$ be any proper node of $F$. The* contiguous empire *of $X$ is defined to be the largest subnet of $F$ having $X$ as a root which is contiguous with respect to $X$.*

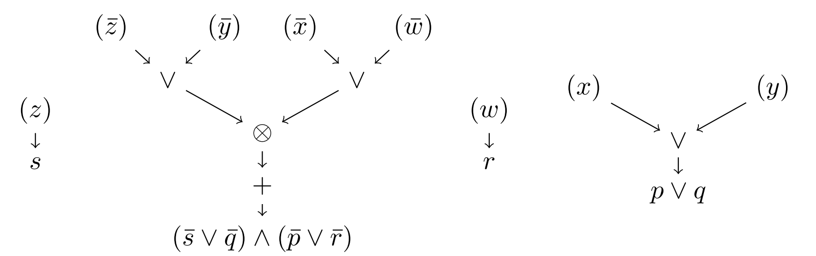
*Example 14.* The expansion net shown in Example 13 is not contiguous with respect to any of its roots. Neither is the empire of $(\bar{x})$ contiguous with respect to $(\bar{x})$. The contiguous empire of $(\bar{x})$ is $(\bar{x}), (x)$.

As we will see later, the kingdom of a node is always contiguous, and so there is no need to consider a concept of "contiguous kingdom". The advantage of the contiguous empire over the usual empire is that it admits a simple definition, which is a minor variation on the inductive definition of empires in *ACC* nets found in [3]:

**Definition 16.** *Let $F$ be an AC typed forest and let $X$ be a proper node of $F$. We define the substructure $ce(X)$ as the smallest substructure of $F$ containing $X$ and satisfying the following:*

- $(\otimes)$ *If $Y = t \otimes s$ is a node of $F$, if $t \in ce(X)$ or if $s \in ce(X)$, and if and $t, s \neq X$, then $Y$ is in $ce(X)$;*
- $(W)$ *If $Y = (t \vee *)$ (resp. $(* \vee t)$) is a node of $F$, if $t \neq X$, and if $t \in ce(X)$, then $Y$ is in $ce(X)$;*
- $(\parr\ 1)$ *If $Y$ is a switched node of $F$, and if all the successors of $Y$ are in $ce(X)$ and not equal to $X$, then $Y \in ce(X)$.*
- $(\parr\ 2)$ *If $Y$ is a switched node of $F$, if none of the successors of $Y$ are equal to $X$, and if one of the successors of $Y$ is in $ce(X)$, then $Y \in ce(X)$ if there is a switching path from $X$ to $Y$ which does not pass through any of the successors of $Y$ (that is, the path passes into $Y$ from below).*

*Remark 6.* Items $(\otimes)$, $(W)$ and $(\parr 1)$ in the above definition are derived from Girard's inductive definition of the empire in an ACC net, as described in [3]. This inductive definition fails in the presence of mix: consider, for example, the following expansion-net (based on an example from [1]):



Applying the (faulty) inductive definition of empire to the rightmost root $((x) \vee (y))$, we only obtain the substructure $((x) \vee (y)), (\bar{x}), (\bar{y})$, which is not a subnet, since there is a switching path through the $\otimes$ node from $(\bar{x})$ to $(\bar{y})$. However, by using the novel extra condition $(\parr 2)$, we can observe that since there is a switching path from $((x) \vee (y))$ to $((\bar{x}) \vee (\bar{w}))$ from below (i.e., via $(y)$, $(\bar{y})$, and $((\bar{z}) \vee (\bar{y})) \otimes ((\bar{x}) \vee (\bar{w}))$), $((\bar{x}) \vee (\bar{w}))$ is in the contiguous empire of $((x) \vee (y))$. Similarly, $((\bar{z}) \vee (\bar{y}))$ is in $ce(((x) \vee (y)))$: from which, applying the other conditions, we obtain that $ce(((x) \vee (y)))$ is the whole net.

From the definition of $ce(X)$, we can not immediately see that it is contiguous with respect to $X$: it is clear that there is a switching path from $X$ to $Y$ in $F$ for every $Y$ in $ce(X)$, but not clear that this path lies entirely within $ce(X)$. The following lemma shows that $ce(X)$ has an equivalent definition which clearly *is* contiguous:

**Lemma 4.** *Let $F$ be an AC typed forest and $X$ be a proper node of $F$. Let $ce'(X)$ be the smallest set of nodes of $F$ containing* $\mathrm{str}(X)$ *(the smallest substructure containing $X$) and closed under:*

- $(\otimes')$ *If $Y = t \otimes s$ is a node of $F$, if $t \neq X$ and $s \neq X$ and if either $t \in ce'(X)$ or $s \in ce'(X)$, then $Z$ is in $ce(X)$ for each $Z \in \mathrm{str}(Y)$;*
- $(W')$ *If $Y = (t \vee *)$ (resp. $(* \vee t)$) is a node of $F$, $t \neq X$, and either $t \in ce'(X)$, then $Y$ is in $ce'(X)$;*
- $(\parr'\ 1)$ *If $Y$ is a switched node of $F$, and all the successors of $Y$ are in $ce'(X)$ and not equal to $X$, then $Y \in ce'(X)$.*

($\invamp'$ 2) *Let $Y$ be a switched node of $F$. If none of the successors of $Y$ are equal to $X$, and if one of the successors of $Y$ is $ce'(X)$, then: if there is a switching path $P$ from $X$ to $Y$ which does not pass through any of the successors of $Y$, then $Z \in ce'(X)$ for each $Z \in \mathrm{str}(W)$, $W \in P$.*

*Proof.* We must prove that $ce'(X)$ is not larger than $ce(X)$ (it clearly contains $ce(X)$). The difficult case is to show that a structure extended by one application of ($\invamp'$ 2) can also be extended by multiple steps of ($\otimes$), ($W$),($\invamp$ 1), ($\invamp$ 2), and closing under substructure, as in the definition of $ce(X)$. We prove this by induction on the length of a switching path in the application of ( $\invamp'$ 2). Suppose that a single step of ($\invamp'$ 2) can be carried out by multiple steps of the definition of $ce(X)$ when the switching path is of length $< n$. Now suppose ($\invamp'$ 2) is applied to a structure $G$ and a path $P$ of length $n+1$, ending at a switched node $Y$. By ($\invamp$ 2), we may add $Y$ to $G$. Recall that $P$ must enter $Y$ from below. Counting from $X$, let $W$ be the penultimate switched node in $P$ entered from below on $P$ – that is, the switching path $Q$ traced from $W$ to $Y$ enters all pars in between from a successor). Seen from the opposite direction, that means that by applying ($\otimes$) and ($W$), and closing under substructure, we can add $\mathrm{str}(V)$ for every $V$ on the path $Q$ between $Y$ and $W$. In particular, at least one of the successors of $W$ is a member of $ce(X)$, since $Q$ must leave $W$ by one of those successors. The path $P$ restricted to be from $X$ to $W$ does not pass through any successor of $W$, and thus by the induction hypothesis, we may add the rest of the switching path to $ce(X)$.

**Proposition 6.** *$ce(X)$ is contiguous with respect to $X$.*

*Proof.* By the previous lemma; it is clear that each stage of construction of $ce'(X)$ yields a contiguous substructure.

**Proposition 7.** *Let $F$ be an AC-correct structure, and $X$ a proper node of $F$. $ce(X)$ is a subnet of $F$.*

*Proof.* Suppose not. Then there are roots $Y$, $Z$ of $ce(X)$ such that there is a switching path from $Y$ to $Z$ outside of $ce(X)$. There are two cases to consider

- $X$ is $Y$ ($X$ is $Z$ is symmetric). Then there is a path from $X$ to $Z$ inside $ce(X)$, and another outside $ce(X)$. By concatenating these two paths we obtain a cycle, which contradicts $AC$-correctness of $F$.
- Neither $X$ nor $Y$ is $Z$. By construction of $ce(X)$, both $Y$ and $Z$ are the successors of switched nodes in $F$. The switching path from $Y$ to $Z$ passes through both of those switched nodes. In particular, there is a switching path from $Y$ to $Z'$, the predecessor of $Z$ (a switched node), which enters $Z'$ from below. There is also a path from $X$ to $Y$ within $ce(X)$, by contiguousness. Concatenating these paths, we obtain a switching path from $X$ to $Z'$, not via a successor of $Z'$; thus $Z'$ is in $ce(X)$, contradicting the fact that $Z$ is a root of $ce(X)$.

**Corollary 1.** *Let $F$ be a AC-correct structure, and $X$ a proper node of $F$. The kingdom $k(X)$ of $X$ exists, and is contiguous with respect to $X$.*

*Proof.* For existence, note that we have demonstrated the existence of a subnet $ce(X)$ with $X$ as a root: the kingdom exists by Lemma 3. Now consider the subnet $k(X)$ as a net in its own right, with $X$ as a root. By the previous proposition, there is a subnet $ce(X)$ of $k(X)$ with $X$ as a root; by minimality of the kingdom $ce(X) = k(X)$ and so $k(X)$ is contiguous with respect to $X$.

We will not need the fact that $ce(X)$ is the contiguous empire of $X$ but we include the proof of that fact here for the sake of completeness.

**Proposition 8.** *$ce(X)$ is the contiguous empire of $X$.*

*Proof.* Suppose for a contradiction that $G$, some contiguous subnet of $F$, contains a node $W_0$ not contained in $ce(X)$. We may assume that this $W_0$ is a switched node $W_0$, which has a successor $Y_0$ which is a root of $ce(X)$, and a successor $Z_0$ not in $ce(X)$; the path from $X$ to any node outside of $ce(X)$ must leave $ce(X)$ through such a node. Since $G$ is contiguous, there is a switching path from $X$ to $W_0$: since $W_0$ is not a member of $ce(X)$, that path must come via $Z_0$, and so $Z_0$ is also a member of $G$. Applying the same logic as before, the path from $X$ to $Z_0$ in $G$ must leave $ce(X)$ at some root $Y_1$ (distinct from $Y_0$ by acyclicity). This root is also the successor of a switched node $W_1$, and $W_1$ must also have a successor $Z_1$ not in $ce(X)$. Note that we now have $Y_0$, $Y_1$, distinct roots of $ce(X)$, successors of switched nodes $W_0, W_1$; those switched nodes each have another successor $Z_0$, $Z_1$ not in $ce(X)$. There is a switching path from $W_1$ to $W_0$ via $Z_0$, leaving $W_1$ through its predecessor.

Now suppose that, repeating this line of thinking, we have found roots $Y_0 \dots Y_n$ of $ce(X)$, successors of switched nodes $W_0, \dots W_n$, such that each $W_i$ has another successor $Z_i$ not in $ce(X)$, and that there is a switching path from $X$ to each $Y_i$, for $i < n$, leaving $ce(X)$ at $Y_{i+1}$. Suppose, further, that there is a switching path from $Z_n$ to $W_1$ which, tracing from $W_n$ to $W_1$, enters each $W_i$ via $Z_i$. Since $W_n$ is in $G$, there is a switching path from $X$ to $Z_n$, leaving $ce(X)$ at $Y_{n+1}$, which has a predecessor $W_{n+1}$. There thus a switching path from $W_{n+1}$ to $Z_n$, leaving $W_{n+1}$ through its predecessor. By concatenating with the path from $Z_n$ to $W_0$, we obtain a switching path from $W_{n+1}$ to $W_1$, and consequently to each $W_i$ (to see that this concatenation is really a switching path, observe that if there is a switched node common to both paths, either there is a switching cycle or a switching path from $W_0$ to $W_{n+1}$, contradicting that $W_0$ and $W_{n+1}$ are not in $ce(X)$).

Suppose $Y_{n+1} = Y_j$ for $j \leq n$; then there is a switching path from $W_{n+1}$ to itself: a switching cycle. Thus $Y_{n+1}$ is a new root of $ce(X)$. Since $ce(X)$ has only finitely many nodes, eventually $Y_{n+1}$ will be equal to $Y_i$ for some $i$, and we obtain a contradiction.

### 6.3 Splitting and sequentialization for expansion nets

Sequentialization for expansion nets is the following:

**Theorem 5.** *Let $F$ be an e-annotated sequent. $F$ is an expansion-net (i.e. $F$ is derivable in $\boldsymbol{LK}^*_e$) if and only if $F$ is $AC$-correct.*

The proof of sequentialization for expansion nets is not so different from sequentialization for $\mathbf{MLL}^-$ plus Mix nets. The proof of sequentialization we give in this paper is *bottom-up*, and can be thought of as proof search in $\mathbf{LK}^*$, guided by the information contained in an e-annotated sequent. Given an $AC$-correct e-annotated sequent, we look for a rule of $\mathbf{LK}^*_e$ with $F$ as the conclusion and $AC$-correct e-annotated sequents as premisses. We call such a root of $F$ a *gate*.

**Definition 17.** *Let $F$ be an $AC$-correct e-annotated sequent. A* gate *of $F$ is a root $t : A$ of $F$ which is the conclusion of a rule instance $\rho$ of $\boldsymbol{LK}^*_e$, such that the premisses of $\rho$ are also $AC$-correct e-annotated sequents.*

As we will see below, disjunctions and non-trivial expansions are always gates. The major difficulty in proving sequentialization lies in showing the existence of a gate in the absence of disjunctive and non-trivial expansions. In this case, the gate will be a "splitting" instance of $\otimes$.

The proof of the existence of a splitting $\otimes$ we give here is slightly novel, in that we make use of the notion of contiguousness (the previous proof of this lemma for Mix-nets, by Bellin [1], is almost the same but less elementary).

**Lemma 5.** *Let $F$ be an $AC$-correct e-annotated sequent, and let the roots of $F$ be* trivial *expansions (of the form $(x)$, $(\bar{x})$ or $(t \otimes s)$). If at least one root of $F$ is non-atomic, then $F$ has the form*

$$F_1, F_2, (t \otimes s) : A \wedge B,$$

*where $F_1, t : A$ and $F_2, s : B$ are $AC$-correct e-annotated sequents.*

*Proof.* Since every root of $F$ is a trivial expansion, we have that

$$F = (x_1) : a_1, \dots (x_n) : a_l, (\bar{y}_1) : b_1, \dots, (\bar{y}_m) : b_m, (t_0 \otimes t'_0) : A_1 \wedge B_1, \dots (t_n \otimes t'_n) : A_n \wedge B_n.$$

Let $G$ be the typed forest consisting on the witnesses contained in the roots of $F$: that is,

$$G = x_1 : [a_1], \dots x_n : [a_l], \bar{y}_1 : [b_1], \dots, \bar{y}_m : [b_m], t_0 \otimes t_0' : A_1 \otimes B_1, \dots t_n \otimes t_n' : A_n \otimes B_n.$$

We will show that there is a root $t \otimes s$ of $G$ such that

$$G = G_1, G_2, t \otimes s : A \otimes B,$$

and such that every path from $G_1$ to $G_2$ in the graph of $G$ passes through the node $t \otimes s$. In that case, $G_1, t : A$ and $G_2, s : B$ are $AC$-correct typed forests, and $(t \otimes s) : A \wedge B$ is clearly a gate of $F$.

By Proposition 5 $G$ has a $\ll$-maximal root $X$; this node must be a tensor, without loss of generality $t_0 \otimes t_0'$. If $X$ is splitting, we are done. Suppose it is not splitting; then we know that

- $ce(X)$ is not the whole of $F$, and in addition
- there is a (non-switching) path in the graph of $G$ from a root of $ce(t_0)$ to a root of $ce(t_0')$ (if no such path exists, then $t_0 \otimes t_0'$ is splitting).

The existence of the path means that there is a root $s$ of $ce(t_0)$ whose predecessor is not in $ce(t_0)$, through which this path leaves $ce(t_0)$. This now allows us to discern the existence of another $\ll$-maximal node $Y$; if the root $Z$ below $s$ is $\ll$-maximal, then we are done; if $Z$ is not $\ll$-maximal, then there is an $\ll$-maximal node $Y$ such that $Z \ll Y$. If $Y$ is splitting we are done.

Now suppose that $t_0 \otimes t_0' \dots t_n \otimes t_n'$ are all $\ll$-maximal non-splitting roots of $G$, and that there is a switching path from $t_0$ to $t_n'$ passing through each $t_i$ and $t_i'$ in turn. As above, since $t_n \otimes t_n'$ is not splitting there is a root $s$ of $ce(t_i')$ whose predecessor is not in $ce(t_i')$. This now allows us to discern the existence of another $\ll$-maximal node. If the node is splitting we are done; otherwise it is a tensor $t_{n+1} \otimes t_{n+1}'$. Note that it must be distinct from the roots of $F$ already listed, otherwise $F$ would have a switching cycle: since $k(t_{n+1} \otimes t_{n+1}')$ is contiguous with respect to $t_{n+1} \otimes t_{n+1}'$, there is a switching path from, without loss of generality, $t_{n+1}$ to $s$. If that switching path enters $s$ from below, then we have a switching path from $t_0$ to $Y_{n+1}'$, by concatenation. Otherwise, the switching path from $t_{n+1}$ enters $s$ from above, and must pass into $ce(Y_i')$ via some other root $r$; either way, we have a switching path from $t_0$ to $t_{n+1}'$.

We have seen that, given $n$ non-splitting $\ll$-maximal roots of $G$, we can find another such root. Since $G$ has only finitely many roots, we eventually find one, $t' \otimes s'$, which is splitting. So $G = G_1, G_2, t' \otimes s'$, where $G_1, t'$, $G_2, s'$ are both $AC$-correct typed forests. ☐

*Proof (Of Theorem 5).* By induction on the number of symbols in $F$. The smallest possible numbers of symbols in an $AC$-correct annotated sequent is one ($F = 1 : \top$), which can easily seen to sequentialize.

Suppose now that $F$ contains more than one symbol. First, suppose that $F$ has a graph which is disconnected: let $F'$ be a connected component of $F$. Then $F = F', F''$, and we have

$$\frac{F' \qquad F''}{F} \textsc{Mix}$$

Both $F'$ and $F''$ have fewer symbols than $F$; by the induction hypothesis there are $\mathbf{LK}_e^*$ derivations of $F'$ and $F''$, and so also an $\mathbf{LK}_e^*$ derivation of $F$.

Suppose first that $F$ is an $AC$-correct e-annotated sequent whose graph is disconnected. Then each connected component of the graph $F$ defines an $AC$

We now show that every $AC$-correct e-annotated sequent $F$ whose graph is connected either has a gate or is of the form $(x) : a, (\bar{x}) : \bar{a}$ (i.e. a conclusion of the $\mathbf{LK}_e^*$ axiom) by induction on the number of nodes in $F$.

If $F = F', (t_1 + \dots + t_n) : A$, with $n > 2$ then $(t_1 + \dots + t_n) : A$ is a gate: for example,$F', (t_1) : A, (t_2 + \dots + t_n) : A$ is also $AC$-correct, and we have

$$\frac{F', (t_1) : A, (t_2 + \dots + t_n) : A}{F', (t_1 + \dots + t_n) : A} C$$

Similarly, if $F = F', (t \vee s) : A \vee B$, then $t \vee s$ is a gate: $F', t : A, s : B$ is $AC$-correct and

$$\frac{F', t : A, s : B}{F', (t \vee s) : A \vee B} \vee$$

A similar argument shows that roots of the form $(t \vee *)$ and $(* \vee t)$ are gates.

If $F$ is connected, and no root of $F$ is a disjunction or nontrivial expansion, then either all the roots of $F$ are of atomic type, or $F$ contains at least one root of conjunctive type. We may, therefore, apply Lemma 5 to obtain $AC$-correct e-annotated sequents $F_1, t' : A$ and $F_2, s' : B$ such that

$$\frac{F_1, t' \qquad F_2, s'}{F} \wedge$$

is a correct application of the $\wedge$-rule. Since the premisses of this rule are smaller $AC$-correct e-annotated sequent, they can be derived in $\mathbf{LK}^*_e$, and therefore so can $F$.

Finally, suppose that all the roots of $F$ are trivial expansion of atomic type, and that $F$ is connected. Then $F$ must contain at least one pair $(x) : a, (\bar{x}) : \bar{a}$, since wire variables occur in pairs. It cannot contain any more pairs, since otherwise it would not be connected, so $F = (x) : a, (\bar{x}) : \bar{a}$, and is derivable in $\mathbf{LK}^*_e$.

## 7 Cut-elimination for expansion nets

In this final technical section we define a weakly normalizing cut-elimination procedure directly on expansion-nets which preserves correctness. This result did not appear in [23], and is new to the current paper. In Propositions 11,12,13 and 14, we show that any individual cut in an expansion-net can be replaced by "smaller" cuts. Then, in Lemma 6, we show that one cut can be removed completely from an otherwise cut-free expansion-net. Finally, in Theorem 6 we show, using the kingdom ordering defined in the previous section, how to eliminate all cuts from an expansion-net.

The primary difficulty in defining cut-elimination for classical nets lies with the reductions involving weakening and contraction. In the original linear logic proof nets, deletion and duplication of subproofs is mediated by boxes. As we suggested above, in box-free settings the role of boxes is taken on by *subnets*. This means that cut-reduction in classical nets is not *local*: the subnet to be copied must be calculated, and this calculation can, in general, consider the whole net.

Cut-elimination for expansion nets is, of course, strongly related to cut-elimination for the calculus $\mathbf{LK}^*$. This calculus is cut-free complete, and so we already have a (semantic) cut-elimination result, but since this calculus is only complete for formulae, and not for sequents, the result has a somewhat nonstandard form:

**Proposition 9.** *Let $\Gamma$ be provable in $\mathbf{LK}^*$ plus cut. Then there is a sub-multiset $\Gamma'$ of $\Gamma$ provable in $\mathbf{LK}^*$.*

It is interesting to consider how one might prove this theorem syntactically, within $\mathbf{LK}^*$. A typical cut-reduction step in the sequent calculus is to identify a sub-proof ending with a cut, and replace it with a sub-proof in which no cuts appear. Applying that methodology to $\mathbf{LK}^*$, we take a subproof proving $\Gamma$ and acquire a subproof proving a subsequent $\Gamma'$. If we had access to weakening, this would be unproblematic, but in our setting we can only "weaken" within a disjunction. Thus any formula which "becomes weak" during cut-elimination must be a conclusion of the whole derivation or an immediate subformula of a disjunction: this is an unusual requirement for a cut-reduction step; it adds another place in which commutative conversions must be applied and it is not immediately clear that it can lead to cut-elimination. Fortunately, in a proof-net setting commutative conversions are not needed, and it is easy to see that such reductions can always be applied.

To begin, we need to introduce a notion of expansion-net with cut:

**Definition 18.** *(a) A* cut tree *is an unordered pair of an expansion tree $t$ of type $A$, and an expansion tree $s$ of type $\bar{A}$, where $A$ is a formula of classical propositional logic not equal to $\top$ or $\bot$. The* positive term *in the cut is the term of type $\bar{a}$ or $A \wedge B$. We write a cut tree $t \bowtie s$, where by convention the positive term is written on the left (when it is known which of $s$ and $t$ is the positive term).*

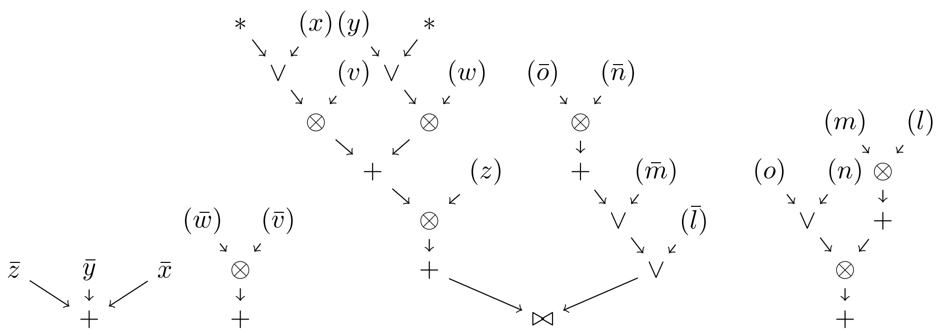
*(b) A typed forest with cut is a finite forest of typed propositional expansion trees, witnesses and cut trees, such that a wire-variable $x$ occurs at most once, and occurs if and only if its dual $\bar{x}$ occurs. The type of a typed forest with cut is the forest of types of its roots which are not cuts.*

*(c) An* expansion-net with cut *is a typed forest with cut, derivable in* $\boldsymbol{LK}^*_e$ *plus the rule*

$$\frac{F,\ t : A \qquad G,\ s : \bar{A}}{F,\ G,\ t \bowtie s}\ \textsc{Cut}$$

Extending the correctness criterion to cuts is trivial, as usual in proof-nets: we simply treat the cut $t \bowtie s$ as though it were a conjunctive witness $t \otimes s$. (i.e. an unswitched binary node). The notions of subnet, kingdom etc. carry over in an obvious manner, as does the sequentialization theorem.

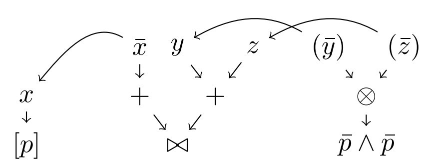
*Example 15.* The following is a correct expansion-net with cuts: it is derived by cutting the expansion net in Example 6 with the expansion-net witnessing the associativity of $\wedge$:



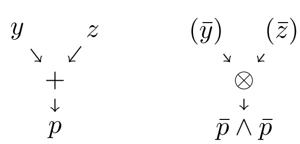

### 7.1 The basic cut-reduction operations

As in Gentzen-style cut-elimination for sequent calculi, cut-elimination in expansion nets is based on a number of basic operations. In general, the application of these rules may not terminate, but we can find a *strategy* for applying these rules such that there is a measure on proofs which decreases. We define cut-reduction, not just on expansion nets, but on $AC$ typed forests which might, in general, have witnesses as roots. This allows us to perform cut-reduction on subnets of an expansion-net, just as one eliminates cuts in a subproof in the sequent calculus.

Unlike usual cut-elimination results, the cut-elimination we define in this section does not in general preserve the type of derivations. This is for two reasons. The first has been mentioned above: namely that expansion-nets with cut are sequent-complete (can prove all sequents derivable in **LK**) while expansion-nets without cut are only formula complete. The second reason concerns cut-elimination in a general $AC$ typed forest. Consider the following example:




Cuts of this form are reduced by "yanking": the axiom link between the $x$ and $\bar{x}$, and the cut, disappear, leaving the following net:

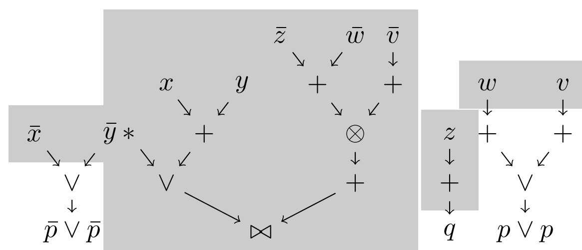


However, the type of the net has changed: we have replaced a root of type $[p]$ with a root of type $p$. We will use the term *closure* to describe a sequent resulting from deleting some formulae, and replacing others with their underlying type:

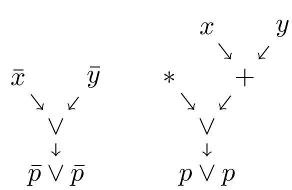
**Definition 19.** *Let $\Gamma$ be a forest of types. A* closure *of $\Gamma$ is a forest $\Gamma'$ of types, together with an injective function from the roots of $\Gamma'$ to the roots of $G$ which either preserves types or replaces a type with its underlying classical formula (see Definition 6).*

Notice that if $\Gamma$ does not contain any witness types, a closure of $\Gamma$ is just a sub-multiset of $\Gamma$.

Our cut-elimination argument relies on isolating, in an expansion net $F$, a subnet $G$ containing only one cut, and replacing it with a cut-free AC typed forest $G'$ whose type is a closure of $G$: that is, we replace each non-cut root $t$ of $G$ with $f^{-1}(G)$, where $f$ given to us by the closure of the type of $G$. If $t$ has no pre-image, and is a root, we can delete it. If $t$ has no pre-image, and is a root, we would like to replace it by $*$ (representing that the formula previously introduced by $t$ is now introduced by weakening) – however, we must be careful to ensure that the $*$ occurs within a default weakening. For example, consider the following expansion-net, with marked subnet:




The marked subnet $G$ has type $[\bar{p}], [\bar{p}], q, [p], [p]$. The AC typed forest $G' = \bar{x} : [p], \bar{y} : [q], (x + y) : p$ has type which is a completion of the marked subnet, via an injection $f$ which does not have a preimage for $q$ or the first copy of $[p]$. The trees missing a pre-image are $(z)$ (which is a root) and $w$ (which is in a disjunction), so the result of replacing $G$ by $G'$ is an expansion net:




The following proposition shows that we can replace a subnet $G$ by another $AC$ typed forest whose type is a closure of $G$, provided the deleted roots fall inside disjunctions or expansions:

**Proposition 10.**
*Let $G$ be a subnet of an AC typed forest $F$, and let $G'$ be an AC typed forest whose type is a closure $G$: that is, there is an injective function $f$ from the non-cut roots of $G'$ to those of $G$ such that $f$ either preserves types or maps a term of witness type to a term of its underlying type. Call a node $t$ of $F$ weak if it is a root of $G$ but has no $f$-preimage. Suppose further that each weak node of $F$ is either a root of $F$, or the sucessor of a switched node $Y$ (a disjunction or expansion) such that at least one other successor of $Y$ is not weak. Then, replacing $G$ by $G'$ in $F$ (by replacing $t$ by $f^{-1}(t)$ if $t$ has an $f$-preimage, deleting $t$ if it is weak and a root or a successor of an expansion, and replacing $t$ by $*$ if it is weak and the successor of a disjunctive node, and replacing the cuts of $G$ by the cuts of $G'$) yields an AC typed forest whose type is a closure of the type of $F$.*

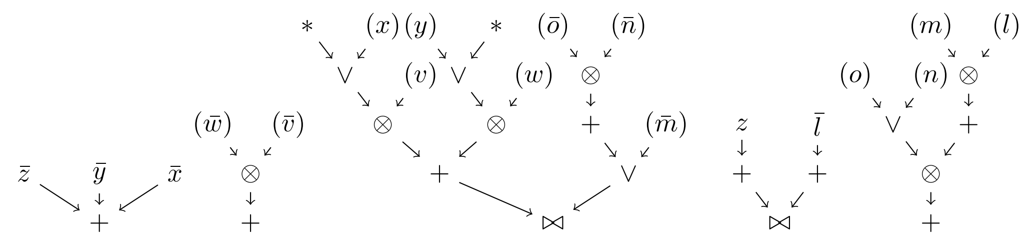
*Proof.* An easy examination of the correctness criterion.

We now introduce the basic reductions of expansion-nets, and show that they preserve AC correctness. We illustrate the reductions by showing how to reduce the net in Example 15 to cut-free form.

We will define a *logical cut* to be one in which the positive cut term is an expansion consisting of a single witness. In case the cut has non-atomic type, the definition of cut-reduction is easy:

**Proposition 11 (Logical cut – $\wedge/\vee$).** *Let $G = F, (s_1 \otimes s_2) \bowtie (t_1 \vee t_2)$ be an AC typed forest, such that $t_1, t_2 \neq *$. Then $G' = F, s_1 \bowtie t_1, s_2 \bowtie t_2$ is an AC typed forest with the same type as $G$.*

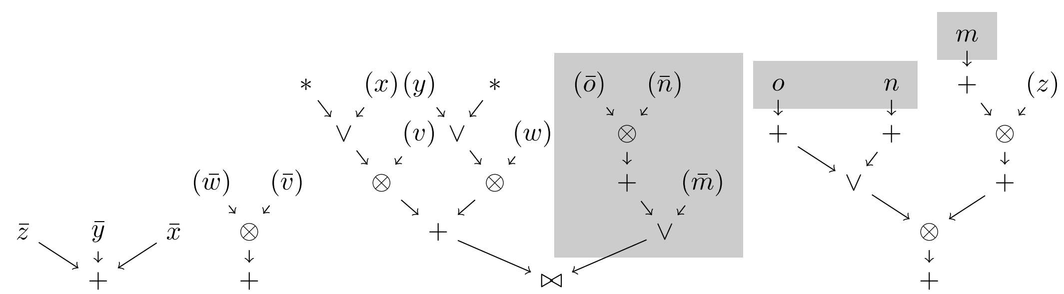
Reducing the logical cut in Example 15, we obtain the following net:



As discussed above, in case of an atomic logical cut, the whole forest has the form $G = F, (x) \bowtie t$, where $t$ is a possibly nontrivial sum of witnesses. We want to reduce this cut, as in usual proof-nets, by "yanking":

**Proposition 12 (Logical cut – atomic).** *Let $G = F, (x) \bowtie t$ be an AC typed forest. Then $F[\bar{x} := t]$ is an AC typed forest with type $G'$ a closure of $G$.*

Reducing the atomic logical cut in our example, we obtain:



Cut against contraction is dealt with by reducing the *positive width* of the cut: the number of witnesses appearing in the positive cut term. This is achieved by duplicating a subnet with the positive cut term as a root (the equivalent of duplicating a subproof with the positive cut-formula in the conclusion). In duplicating a subnet we must make sure to rename the wire-variables so that no variable occurs more than once: given a term $t$, we use the notation $t^L$, $t^R$ to denote two copies of $t$ where each wire variable $x$ has been replaced by fresh variables $x^L$, $x^R$, and each wire variable $\bar{x}$ has been replaced by fresh variables $\bar{x}^L$, $\bar{x}^R$, such that $\bar{x}^L$ is dual to $x^L$, and so on.

**Proposition 13 (Structural cut – contraction).** *Let*

$$G = F, (s_1 + \cdots + s_n) \bowtie_X t$$

*be an AC typed forest, where $s = (s_1 + \cdots + s_n)$ is nontrivial expansion. Let $s = s_1 + s_2$ (that is, $s_1$ and $s_2$ partition $s$) and let $w_1 \ldots, w_n, c_1, \ldots c_m, t$ be a subnet of $G$, whose roots other than $t$ are all either witnesses (the $w_i$), or cuts (the $c_i$). Let $F'$ be the forest defined by replacing each $w_i$ by $(w_i^L + w_i^R)$. Then*

$$G' = F', s_1 \bowtie t^L, s_2 \bowtie t^R$$

*is an AC typed forest, and the type of $G'$ is a closure of the type of $G$.*

*Proof.* Let $H, (s_1 + s_2)$ be the kingdom of $(s_1 + s_2)$: then

$$J = H, w_1, \ldots, w_n, c_1, \ldots c_m, (s_1 + s_2) \bowtie t$$

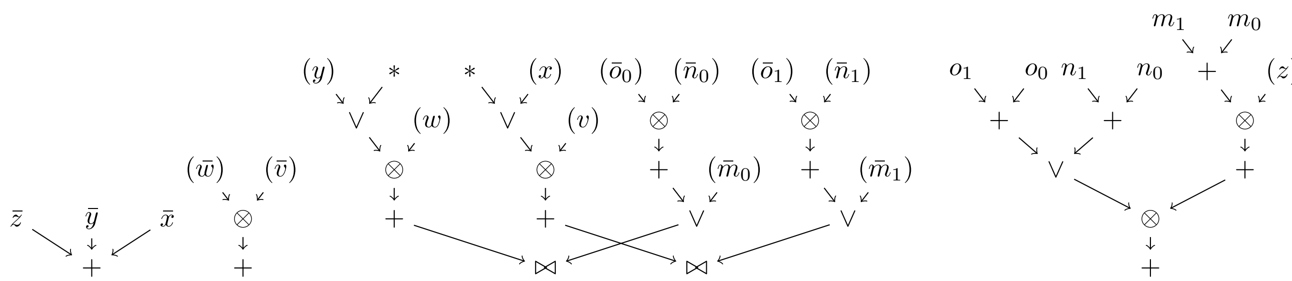
is a subnet of $G$. It is easy to see that

$$J' = H, (w_1^L + w_1^R), \ldots, (w_n^L + w_n^R), c_1^L, c_1^R \ldots c_m^L, c_m^R, s_1 \bowtie t^L, s_2 \bowtie t^R$$

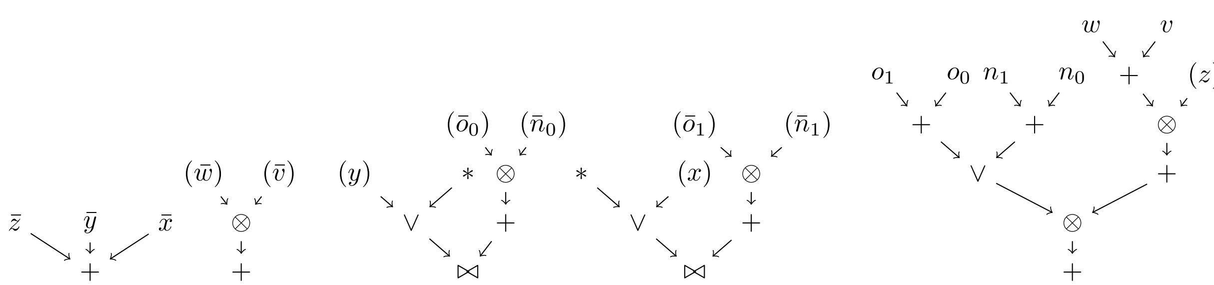
is an $AC$ typed forest with type a closure of the type of $J$. Since $J$ was a subnet, the result of replacing $J$ in $F$ by $J'$ also $AC$; the result follows.

*Remark 7.* The restriction that the duplicated subnet have only witnesses and cuts as roots ensures that we can "contract" the duplicated conclusions by adding expansions. The *kingdom* of the positive cut-term always yields such a subnet; if a root $s$ of the kingdom of $t$ were a disjunction or expansion, we could remove that node to yield a smaller subnet with $t$ as a root.

In the last step of our running example, the kingdom of the negative branch of the cut is shaded. Notice that, apart from the root taking part in the cut, all the roots of the kingdom are witnesses. Thus, we can duplicate the kingdom, cutting each copy against one of the witnesses on the positive branch of the cut:




After some logical cuts, we arrive at the following net:




This net contains two examples of our final kind of cut: a cut against default weakening. This situation superficially resembles the a cut between the *additive* propositional connectives in sequent calculus. In common with the reduction for such a cut, we delete a subproof (here subnet) of the proof. Unlike the additive reduction, we must replace the conclusions of the deleted subnet by weakenings: the catch here is that, to ensure that each weakening thus created is a default weakening, each weakened subtree must either be a component of a nontrivial expansion or of a disjunctive term which is not already a default weakening.

**Proposition 14 (Structural cut – default weakening).** *Let*

$$G = F, (s_1 \otimes s_2) \bowtie (t \vee *)$$

*be an AC typed forest. Let $E = u_1, \ldots, u_n, s_2$ be a subnet of $G$, such that each tree $u_i$ is either a root of $G$, a successor of an expansion containing at least one term not in $E$, or is the successor of a disjunction node* *the other successor of which is neither an instance of $*$ nor in $E$. Let $F'$ be the forest derived from $F$ as follows: if $u_i$ is a root of $F$, delete it: otherwise, replace it by $*$. Then*

$$G' = F', s_1 \bowtie t$$

*is a default attached AC forest, and the type of $G'$ is a subsequent of the type of $G$.*

*Proof.* Let $L, s_1$ and $M, t$ be the kingdoms of $s_1$ and $t$ respectively. Then

$$L, M, u_1, \ldots, u_n, (s_1 \otimes s_2) \bowtie (t \vee *)$$

is a subnet of $G$, and $L, M, s_2 \bowtie t$ is an $AC$ forest. The forest $F', s_1 \bowtie t$ is therefore $AC$ correct: it is default-attached, since every non-root term replaced by $*$ is either in a non-trivial expansion or forms an attached weakening.

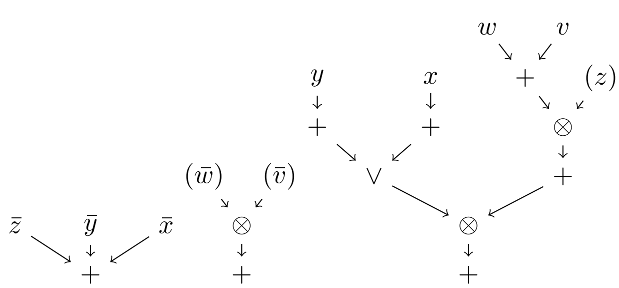
*Remark 8.* There are two nets we can canonically choose to delete which satisfy the conditions on $E$ above: namely the empire and the contiguous empire of $s_2$. This follows immediately from the definitions of (contiguous) empire. Our strategy for cut-elimination will always delete the contiguous empire, for the following reason. Consider the following rule instance in $\mathbf{LK}_e^*$:

$$\frac{F, (s_1 \otimes s_2) \bowtie (t \vee *) \qquad G}{F, G, (s_1 \otimes s_2) \bowtie (t \vee *)} \text{Mix}$$

The empire of $s_2$ changes after application of the rule, while the contiguous empire stays the same. If we delete the empire of $s_2$, then it matters in which subproof we perform the reduction, while deleting the contiguous empire is independent of that choice. Thus, deleting $ce(s_2)$ is more *compositional* than deleting $e(s_2)$, since the result depends less on the context in which the reduction takes place.

The reduction thus defined is not, however, entirely compositional: If $u : A$ is in the contiguous empire of $s_2$ in $F, u : A, (s_1 \otimes s_2) \bowtie (t \vee *)$, then before cut reduction we can form a conjunction on $A$, and afterwards we cannot. This problem is, however, not so drastic; if instead we reduce the cut after introducing the conjunction, in addition to what was deleted before, we also delete the contiguous empire of the other conjunct, which becomes part of the contiguous empire of $s_2$. In other words, the part of the proof which could not be introduced via Mix will in any case be deleted after cut-reduction.

In our running example, the contiguous empire of $(\bar{n}_0)$ is the forest $(\bar{n}_0), n_0$, and the contiguous empire of $(\bar{o}_1)$ is the forest $(\bar{o}_1), o_1$. The resulting cut-free net, after the structural reductions and a number of logical reductions, is




### 7.2 Cut-elimination theorem for expansion nets

The core of cut-elimination is the following lemma, which states that a single "topmost" cut can be removed from an expansion-net. Topmost is here defined using the relation $\ll$: given two cuts $X$ and $Y$, if $X \ll Y$ then $X$ is in the kingdom of $Y$: thus, a cut $Z$ which is minimal among the cuts of $F$ with respect to $\ll$ is not in the kingdom of any other cut, and so there is at least one sequentialization of the net such that the proof above $Z$ is cut-free. Furthermore, the lemma states that this topmost cut can be removed in such a way that duplications happen only within the kingdom of the cut: that is, the cut is eliminated by replacing the kingdom of the cut with a cut free *AC* forest, plus some supplementary deletions.

**Lemma 6 (Principal lemma for default-attached nets).** *Let $G = F, t \bowtie_X s$ be an AC forest containing $n+1$ cuts, and let the cut $\bowtie_X$ be $\ll$-minimal among the cuts in $G$. By applying the transformations in Propositions 12, 11, 13 and 14 to $G$, we can obtain an AC forest $G'$, containing $n$ cuts, such that*

*(a) $G$ and $G'$ only differ on the part of $G$ disjoint from the contiguous empire of $X$ in $G$.*
*(b) Outside of the kingdom of $X$ in $G$, $G$ and $G'$ only differ by the deletion of subtrees or their replacement with $*$.*

*The type of $G'$ is a closure of the type of $G$.*

*Proof.* By induction on the rank of the cut-formula, with a sub-induction on the positive width of the cut. Suppose first that the cut-formula is atomic, and that the cut has the form $x \bowtie_X t$. The kingdom of $X$ is $H, \bar{x}, x \bowtie_X t$, and the atomic cut reduction replaces this subnet by $H, t$. Nothing outside the kingdom of $X$ is changed.

Now assume, as an induction hypothesis, that the lemma holds for a $\ll$-minimal cut of rank $n$ and positive width $m$. Suppose first that the cut has the form $(t \vee *) \bowtie (s_1 \otimes s_2)$. To reduce this cut, we delete $ce(s_2)$, the contiguous empire of $s_2$. After one step of reduction, we obtain an *AC* forest $F' = G', t \bowtie_Y s_1$; the roots of $k(Y)/ce(Y)$ are contained within the roots of $k(X)/ce(X)$. Apply the induction hypothesis to $F'$ to obtain a cut-free expansion net with the required properties.

Now, suppose that the cut has the form $(t_1 \vee t_2) \bowtie_X (s_1 \otimes s_2)$. After one step of cut-reduction, we obtain the *AC* forest $F, t_1 \bowtie_Y s_1, t_2 \bowtie_Z s_2$. Note that $Z \notin k(Y)$ and $Y \notin k(Z)$, and so both $Z$ and $Y$ are $\ll$-minimal; also note that if $u$ is a root of $k(Y)$ not equal to $Y$ (or of $k(Z)$ not equal to $Z$), then $u$ is contained in a root of $k(X)$. Apply the induction hypothesis to one of the cuts, without loss of generality $Y$. The important thing to note is that, since $Z$ is not in $k(Y)$, it is not duplicated by eliminating $Y$, though it may be deleted, since it is in $ce(Y)$. If it is deleted, we are done: otherwise, apply the induction hypothesis a second time to $Z$.

Finally, suppose that the cut has the form $t \bowtie_X s$, where $s = (w_1 + w_2 + \cdots + w_m)$. Since $\ll$ is a partial order on the nodes $F$, at least one of these witnesses will be $\ll$-maximal. Suppose, without loss of generality, that $w_1$ is $\ll$-maximal among the $w_i$'s. Then apply the duplication reduction to the kingdom of $X$, with the decomposition $s = (w_1) + (w_2 + \cdots + w_m)$; we obtain an *AC* forest $G', t^L \bowtie_Y (w_1), t^R \bowtie_Z (w_2 + \cdots + w_m)$. Now apply the induction hypothesis to $Z$, to obtain an *AC* forest $G'', t' \bowtie (w_1')$: crucially, since $w_1$ was not in $k(Z)$, the positive width of this cut does not change after $Z$ is eliminated. We may thus apply the induction hypothesis again to complete the proof.

**Theorem 6 (Cut elimination).** *If $F$ is an expansion net with type $\Gamma$, there is a cut-free expansion net $F'$, reachable by the cut-reduction operations from $F$, such that the type $\Delta$ of $F'$ is a subsequent of $\Gamma$.*

*Proof.* By successive applications of the principal lemma, we can remove all the cuts from $F$, the result being an AC typed forest whose type $\Delta$ is a closure of a subsequent of the type of $F$: since the closure of a classical sequent is just the sequent itself, $\Delta$ is a subsequent of the type of $F$, and so $F'$ is an expansion-net.

## 8 Conclusion

Expansion-nets provide a class of abstract proof objects for classical propositional logic which satisfy our checklist of good properties. There is a sequent calculus ($\mathbf{LK}^*$) with a canonical function from proofs in that calculus to expansion-nets (given in Definition 9) There is a correctness criterion (Definition 10) which can be checked in polynomial time, such that the correct proof structures are precisely the expansion nets. We have sequentialization into $\mathbf{LK}^*$ (Theorem 5), and weakly normalizing cut-elimination directly on expansion-nets (Theorem 6). The last two of these results are new to the paper (although the former was sketched in [23]);

their proofs rely on the characterization of subnets of expansion nets, including the new notion of contiguous subnet defined in this paper. In addition to these properties, expansion-nets also identify a more natural set of sequent derivations than do the previously existing notions of abstract proof.

We mention some further directions:

**Beyond propositional logic** The terminology *expansion* deliberately recalls Miller [24], whose *expansion tree proofs* can be seen as a prototype notion of proof-net for classical logic. The paper [22] makes this connection explicit in the case of first-order prenex formulae; the paper introduces a notion of *Herbrand net* using Girard's notion of a quantifier jump, in which provability at the propositional level is treated as trivial — propositional axioms are replaced by arbitrary propositional tautologies. Expansion-tree proofs themselves do not provide a good notion of proof-net when we move beyond sequents of prenex formulae: they lack the fine-grained propositional structure of expansion-nets and so do not seem to have well-behaved cut-elimination. However, we foresee no major obstacles in combining Herbrand nets with the results of the current paper to capture nets for first- or higher-order classical quantifiers, including cut-elimination.

**Nets for additively formulated classical logic** The correctness/sequentialization results for our nets are heavily tied to the multiplicatively formulated sequent calculus. It is, of course, possible to extract an ed-net from a proof in an additively formulated calculus, but there are natural identities in those calculi which are not validated by our nets. Taking the view that the additive classical connectives are essentially different operations (that happen to coincide at the level of provability), we look for natural notions of proof net for additively formulated classical logic.

## References


1. Gianluigi Bellin. Subnets of proof-nets in multiplicative linear logic with MIX. *Mathematical Structures in Computer Science*, 7(6):663–699, 1997.
2. Gianluigi Bellin, Martin Hyland, Edmund Robinson, and Christian Urban. Categorical proof theory of classical propositional calculus. *Theor. Comput. Sci.*, 364(2):146–165, 2006.
3. Gianluigi. Bellin and Jacques. van de Wiele. Subnets of proof-nets in MLL-. In *Proceedings of the workshop on Advances in linear logic*, pages 249–270, New York, NY, USA, 1995. Cambridge University Press.
4. Kai Brünnler. *Deep Inference and Symmetry in Classical Proofs.* PhD thesis, Technische Universität Dresden, 2003.
5. Kai Brünnler. Locality for classical logic. *Notre Dame Journal of Formal Logic*, 47:557–580, 2006.
6. Stephen A. Cook, Robert, and A. Reckhow. The relative efficiency of propositional proof systems. *Journal of Symbolic Logic*, 44:36–50, 1979.
7. V. Danos and L. Regnier. The structure of multiplicatives. *Archive for Mathematical Logic*, 28:181–203, 1989.
8. Vincent Danos. *La logique lineaire appliquee a letude de divers processus de normalisation et principalement du lambda calcul.* PhD thesis, Univ. de Paris, 1990.
9. Vincent Danos and Roberto di Cosmo. The linear logic primer. Available at `http://www.dicosmo.org/CourseNotes/LinLog/`.
10. Carsten Führmann and David Pym. Order-enriched categorical models of the classical sequent calculus. *Journal of Pure and Applied Algebra*, 204(1):21 – 78, 2006.
11. Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1934.
12. Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987.
13. Jean-Yves Girard. A new constructive logic: Classical logic. *Mathematical Structures in Computer Science*, 1(3):255–296, 1991.
14. Jean-Yves Girard. Proof-nets: The parallel syntax for proof-theory. In *Logic and Algebra*, pages 97–124. Marcel Dekker, 1996.
15. Alessio Guglielmi, Tom Gundersen, and Lutz Straßburger. Breaking paths in atomic flows for classical logic. In *LICS*, pages 284–293. IEEE Computer Society, 2010.
16. Dominic J. D. Hughes. Towards Hilbert's 24th problem: Combinatorial proof invariants. *Electron. Notes Theor. Comput. Sci.*, 165:37–63, 2006.

17. Dominic J.D. Hughes. Proofs Without Syntax. *Annals of Mathematics*, 143(3):1065–1076, November 2006.
18. Dominic J.D. Hughes. A minimal classical sequent calculus free of structural rules. Archived as math.LO/0506463 at arXiv.org, July 2010.
19. F Lamarche and L Strassburger. Naming proofs in classical logic. In *Proceedings of TLCA '05*. Springer-Verlag, 2005.
20. Francois Lamarche and Lutz Strassburger. Constructing free boolean categories. In *LICS '05: Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science*, pages 209—218, Washington, DC, USA, 2005. IEEE Computer Society.
21. Patrick Lincoln and Timothy Winkler. Constant-only multiplicative linear logic is np-complete. *Theoretical Computer Science*, 135:135–155, 1992.
22. Richard McKinley. Proof nets for herbrands theorem. Accepted for publication, ACM Transactions on Computational Logic.
23. Richard McKinley. Expansion nets: proof-nets for propositional classical logic. In *Proceedings of the 17th international conference on Logic for programming, artificial intelligence, and reasoning*, LPAR'10, pages 535–549, Berlin, Heidelberg, 2010. Springer-Verlag.
24. Dale Miller. A compact representation of proofs. *Studia Logica*, 46(4):347–370, 1987.
25. Edmund Robinson. Proof nets for classical logic. *Journal of Logic and Computation*, 13(5):777–797, 2003.
26. Rdiger Thiele. Hilbert's twenty-fourth problem. *American Mathematical Monthly*, 110:2003, 2001.
27. Todd Trimble. *Linear logic, bimodules, and full coherence for autonomous categories.* PhD thesis, Rutgers University, 1994.