Product-free Lambek Calculus is NP-complete

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Abstract

In this paper we prove that the derivability problems for product-free Lambek calculus and product-free Lambek calculus allowing empty premises are NP-complete. Also we introduce a new derivability characterization for these calculi.

Keywords: Lambek calculus, algorithmic complexity, proof nets.
2010 MSC: 03B47, 03F52

Introduction

Lambek calculus L was first introduced in [3]. Lambek calculus uses syntactic types that are built from primitive types using three binary connectives: multiplication, left division, and right division. Natural fragments of Lambek calculus are the product-free Lambek calculus $L(\backslash, /)$, which does not use multiplication, and the unidirectional Lambek calculi, which have only one connective left: a division (left or right).

For the non-associative variant of Lambek calculus the derivability can be checked in polynomial time as shown in [2] (for the product-free fragment of the non-associative Lambek calculus this was proved already in [1]).

In [5] NP-completeness was proved for the derivability problem for full associative Lambek calculus. In [6] there was presented a polynomial algorithm for its unidirectional fragments.

We show that the classical satisfiability problem *SAT* is polynomial time reducible to the $L(\backslash, /)$-derivability problem and thus $L(\backslash, /)$ is NP-complete.

After first presenting this result, the author was pointed to [4], where a very similar (but more complex) technique to explore the derivability for product-free Lambek calculus was presented, though without proving any complexity results.

1. Product-free Lambek Calculus

Product-free Lambek calculus $\mathrm{L}(\backslash, /)$ can be constructed as follows. Let $\mathbf{P} = \{p_0, p_1, \ldots\}$ be a countable set of what we call *primitive types*. Let Tp be the set of *types* constructed from primitive types with two binary connectives $/$, $\backslash$. We will denote primitive types by small letters (p, q, $r, \ldots$) and types by capital letters (A, B, $C, \ldots$). By capital greek letters (Π, Γ, Δ,...) we will denote finite (possibly empty) sequences of types. Expressions like $\Pi \to A$, where Π is not empty, are called *sequents*.

Axioms and rules of $\mathrm{L}(\backslash, /)$:

$$A \to A, \qquad \frac{\Phi \to B \quad \Gamma B \Delta \to A}{\Gamma \Phi \Delta \to A}\ (\mathrm{CUT}),$$

$$\frac{\Pi A \to B}{\Pi \to (B/A)}\ (\to /), \qquad \frac{\Phi \to A \quad \Gamma B \Delta \to C}{\Gamma (B/A) \Phi \Delta \to C}\ (/ \to),$$

$$\frac{A \Pi \to B}{\Pi \to (A \backslash B)}\ (\to \backslash), \qquad \frac{\Phi \to A \quad \Gamma B \Delta \to C}{\Gamma \Phi (A \backslash B) \Delta \to C}\ (\backslash \to),$$

(Here Γ and Δ can be empty.)

In this paper we will consider two calculi — $\mathrm{L}(\backslash, /)$ and $\mathrm{L}^*(\backslash, /)$, called product-free Lambek calculus allowing empty premises. In $\mathrm{L}^*(\backslash, /)$ we allow the antecedent of a sequent to be empty.

It can be shown that in these calculi every derivable sequent has a cut-free derivation where all instances of the axiom are of the form $p \to p$ where $p \in \mathbf{P}$.

2. Reduction from *SAT*

Let $c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1 \ldots c_m$ and variables $x_1 \ldots x_n$. The reduction maps the formula to a sequent, which is derivable in $\mathrm{L}(\backslash, /)$ (and in $\mathrm{L}^*(\backslash, /)$) if and only if the formula $c_1 \wedge \ldots \wedge c_m$ is satisfiable.

For any Boolean variable x_i let $\neg_0 x_i$ stand for the literal $\neg x_i$ and $\neg_1 x_i$ stand for the literal x_i.

Note that $\langle t_1, \ldots, t_n \rangle \in \{0,1\}^n$ is a satisfying assignment for the Boolean formula $c_1 \wedge \ldots \wedge c_m$ if and only if for every $j \leq m$ there exists $i \leq n$ such that the literal $\neg_{t_i} x_i$ appears in the clause c_j (as usual, 1 stands for "true" and 0 stands for "false").

Let $p_i^j, q_i^j, a_i^j, b_i^j; 0 \leq i \leq n, 0 \leq j \leq m$ be distinct primitive types from $\mathbf{P}$.

We define the following families of types:

$$
\begin{aligned}
G^0 &\rightleftharpoons (p_0^0 \backslash p_n^0),\\
G^j &\rightleftharpoons (q_n^j / ((q_0^j \backslash p_0^j) \backslash G^{j-1})) \backslash p_n^j, \quad G \rightleftharpoons G^m\\
A_i^0 &\rightleftharpoons (a_i^0 \backslash p_i^0),\\
A_i^j &\rightleftharpoons (q_i^j / ((b_i^j \backslash a_i^j) \backslash A_i^{j-1})) \backslash p_i^j, \quad A_i \rightleftharpoons A_i^m,\\
E_i^0(t) &\rightleftharpoons p_{i-1}^0,\\
E_i^j(t) &\rightleftharpoons \begin{cases} q_i^j / (((q_{i-1}^j / E_i^{j-1}(t)) \backslash p_{i-1}^j) \backslash p_i^{j-1}), & \text{if } \neg_t x_i \text{ appears in } c_j\\ (q_{i-1}^j / (q_i^j / (E_i^{j-1}(t) \backslash p_i^{j-1}))) \backslash p_{i-1}^j, & \text{if } \neg_t x_i \text{ does not appear in } c_j, \end{cases}\\
F_i(t) &\rightleftharpoons (E_i^m(t) \backslash p_i^m),\\
B_i^0 &\rightleftharpoons a_i^0,\\
B_i^j &\rightleftharpoons q_{i-1}^j / (((b_i^j / B_i^{j-1}) \backslash a_i^j) \backslash p_{i-1}^{j-1}), \quad B_i \rightleftharpoons B_i^m \backslash p_{i-1}^m.
\end{aligned}
$$

Let Π_i denote the following sequences of types:

$$(F_i(0)/(B_i \backslash A_i))\ F_i(0)\ (F_i(0) \backslash F_i(1)).$$

Theorem 2.1. *The following statements are equivalent:*

1. $c_1 \wedge \ldots \wedge c_m$ *is satisfiable.*
2. $\mathrm{L}(\backslash, /) \vdash \Pi_1 \ldots \Pi_n \to G$.
3. $\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \ldots \Pi_n \to G$.

This theorem will be proven in section 6.

3. Derivability Characterization

Let At be the set of *atoms* or *primitive types with superscripts*, $\{p^{\langle i \rangle} | p \in \mathbf{P}, i \in \mathbb{Z}\}$. Let FS be the free monoid (the set of all finite strings) generated by elements of At. We will denote elements of FS by $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and so on, by ε we will denote the empty string.

Consider two mappings:

$$t : \mathrm{FS} \to \mathbf{P}, \quad t(\mathbb{A}p^{\langle i \rangle}) = p; \quad d : \mathrm{FS} \to \mathbb{Z}, \quad d(\mathbb{A}p^{\langle i \rangle}) = i.$$

Let $\mathbb{A} \sqsubset \mathbb{B}$ denote that $\mathbb{A}$ is a strict prefix of $\mathbb{B}$ (i.e. there is $\mathbb{C} \neq \varepsilon \in \mathrm{FS}$ such that $\mathbb{B} = \mathbb{A}\mathbb{C}$). We will denote such $\mathbb{C}$ as $\mathbb{A} \diagdown \mathbb{B}$. By $\mathbb{A} \sqsubseteq \mathbb{B}$ we will denote that either $\mathbb{A} \sqsubset \mathbb{B}$ or $\mathbb{A} = \mathbb{B}$. We can define in the usual way the following notions: $\min_\sqsubset$, $\max_\sqsubset$, $\inf_\sqsubset$, $\sup_\sqsubset$, $[\mathbb{A}, \mathbb{B}]_\sqsubset$, and $(\mathbb{A}, \mathbb{B}]_\sqsubset$.

For $\mathbb{A} \in \mathrm{FS}, \mathbb{A} \neq \varepsilon$ let $\mathcal{P}_\mathbb{A} = \{\mathbb{B} \mid \mathbb{B} \sqsubseteq \mathbb{A}, \mathbb{B} \neq \varepsilon\}$. The relation $\sqsubseteq$ is a total order on $\mathcal{P}_\mathbb{A}$.

Let α be a partial function on $\mathcal{P}_\mathbb{A}$. For each such function we can define the following:

$$\begin{gathered}
\mathbb{B} <_\alpha \mathbb{C} \Leftrightarrow \exists n \geq 1, \alpha^n(\mathbb{B}) = \mathbb{C}, \\
\mathbb{B} \leq_\alpha \mathbb{C} \Leftrightarrow \mathbb{B} <_\alpha \mathbb{C} \vee \mathbb{B} = \mathbb{C}, \\
\mu^-_\alpha(\mathbb{B}) = \min_\sqsubset(\mathbb{B}, \alpha(\mathbb{B})), \\
\mu^+_\alpha(\mathbb{B}) = \max_\sqsubset(\mathbb{B}, \alpha(\mathbb{B})), \\
\mathcal{F}_\alpha(\mathbb{B}) = \{\mathbb{C} \mid \mathbb{C} \leq_\alpha \mathbb{B}\}, \\
\nu^-_\alpha(\mathbb{B}) = \inf_\sqsubset(\mathcal{F}_\alpha(\mathbb{B})), \\
\nu^+_\alpha(\mathbb{B}) = \sup_\sqsubset(\mathcal{F}_\alpha(\mathbb{B})).
\end{gathered}$$

A function $f \colon X \to X$ is an antiendomorphism if $\forall a, b \in X, f(ab) = f(b)f(a)$. In a free monoid it can be defined by its actions on the generators. Consider two antiendomorphisms $(\cdot)^\leftarrow$ and $(\cdot)^\rightarrow$ on FS defined by

$$\begin{gathered}
(p^{\langle 0 \rangle})^\leftarrow = p^{\langle -1 \rangle}, \quad (p^{\langle 0 \rangle})^\rightarrow = p^{\langle 1 \rangle}, \\
(p^{\langle i \rangle})^\leftarrow = (p^{\langle i \rangle})^\rightarrow = p^{\langle -i-sgn(i) \rangle}, \text{for } i \neq 0.
\end{gathered}$$

Consider $[\![\cdot]\!] \colon \mathrm{Tp} \to \mathrm{FS}$, a mapping from Lambek types to elements of the free monoid defined by

$$[\![p]\!] = p^{\langle 0 \rangle}, \quad [\![(A/B)]\!] = [\![B]\!]^\rightarrow [\![A]\!], \quad [\![(A\backslash B)]\!] = [\![B]\!][\![A]\!]^\leftarrow.$$

Let $A \in \mathrm{Tp}$. Let us define φ — the partial function on $\mathcal{P}_{[\![A]\!]}$ that reflects the structure of A:

$$\varphi(\mathbb{A}) = \begin{cases} \inf_\sqsubset\{\mathbb{B} \mid \mathbb{A} \sqsubset \mathbb{B}, |d(\mathbb{B})| = |d(\mathbb{A})| - 1\}, & \text{if } d(\mathbb{A}) > 0; \\ \sup_\sqsubset\{\mathbb{B} \mid \mathbb{B} \sqsubset \mathbb{A}, |d(\mathbb{B})| = |d(\mathbb{A})| - 1\}, & \text{if } d(\mathbb{A}) < 0. \end{cases}$$

It can be easily shown that the following facts hold:

1. There is a unique $\mathbb{A}_0 \in \mathcal{P}_{[\![A]\!]}$ such that $d(\mathbb{A}_0) = 0$.
2. $\varphi(\mathbb{A})$ is defined for every $\mathbb{A} \neq \mathbb{A}_0$.
3. $\leq_\varphi$ is a partial order on $\mathcal{P}_{[\![A]\!]}$.
4. For every $i \in \mathbb{N}$ such that $i < |d(\mathbb{A})|$ there exists $\mathbb{B}$ such that $|d(\mathbb{B})| = i$ and $\mathbb{A} <_\varphi \mathbb{B}$, for instance $\mathbb{A} \leq_\varphi \mathbb{A}_0$.
5. If $\mathbb{A} \in [\mu^-_\varphi(\mathbb{B}), \mu^+_\varphi(\mathbb{B})]_\sqsubset$, then $\mathbb{A} \leq_\varphi \varphi(\mathbb{B})$.

Suppose $\mathbb{A}, \mathbb{B} \in \mathcal{P}_{[\![A]\!]}$. There exists $\mathbb{C} \in \mathcal{P}_{[\![A]\!]}$ such that $\mathbb{A} \leq_\varphi \mathbb{C}$, $\mathbb{B} \leq_\varphi \mathbb{C}$, and for all $\mathbb{C}' \in \mathcal{P}_{[\![A]\!]}$ such that $\mathbb{A} <_\varphi \mathbb{C}'$ and $\mathbb{A} \leq_\varphi \mathbb{C}'$, we have $\mathbb{C} \leq_\varphi \mathbb{C}'$. Such $\mathbb{C}$ is called the φ-join of $\mathbb{A}$ and $\mathbb{B}$.

A set $\mathcal{G} \subset \mathcal{P}_{[\![A]\!]}$ is called φ-closed if there is no $\mathbb{A} \notin \mathcal{G}$ such that $\varphi(\mathbb{A}) \in \mathcal{G}$.

Let $\mathcal{N}_\mathbb{A} = \{\mathbb{B} \in \mathcal{P}_\mathbb{A} \mid d(\mathbb{B}) = 2i+1, i \in \mathbb{Z}\}$.

Suppose we have a Lambek sequent $A_1 \ldots A_n \to B$. Let

$$\mathbb{W} = [\![(\ldots(B/A_n)/\ldots)/A_1]\!] = [\![A_1]\!]^\to \ldots [\![A_n]\!]^\to [\![B]\!].$$

Let π be a function on $\mathcal{P}_\mathbb{W}$, and ψ be a partial function defined by

$$\psi(\mathbb{A}) = \begin{cases} \pi(\mathbb{A}), & \text{if } \mathbb{A} \in \mathcal{N}_\mathbb{W}; \\ \varphi(\mathbb{A}), & \text{if } \mathbb{A} \notin \mathcal{N}_\mathbb{W} \text{ and } d(\mathbb{A}) \neq 0. \end{cases}$$

To characterize derivability of the sequent $A_1 \ldots A_n \to B$ we shall use the following conditions, which we call proof conditions.

1. If $\mathbb{A} \in \mathcal{N}_\mathbb{W}$, then $\pi(\mathbb{A}) \notin \mathcal{N}_\mathbb{W}$ and $\pi^2(\mathbb{A}) = \mathbb{A}$ for all $\mathbb{A} \in \mathcal{P}_\mathbb{W}$.
2. $t(\pi(\mathbb{A})) = t(\mathbb{A})$.
3. $\mu^-_\pi(\mathbb{A}) \sqsubset \mu^-_\pi(\mathbb{B}) \Rightarrow \mu^+_\pi(\mathbb{A}) \sqsubset \mu^-_\pi(\mathbb{B}) \vee \mu^+_\pi(\mathbb{B}) \sqsubset \mu^+_\pi(\mathbb{A})$.
4. $\mathbb{A} \in \mathcal{N}_\mathbb{W} \Longrightarrow \mathbb{A} <_\psi \varphi(\mathbb{A})$ or equivalently $\forall \mathbb{A} \in \mathcal{P}_\mathbb{W}, \mathcal{F}_\varphi(\mathbb{A}) \subset \mathcal{F}_\psi(\mathbb{A})$.
5. $\mathbb{A} \notin \mathcal{N}_\mathbb{W} \wedge \mathbb{A} \neq \mathbb{A}_0 \Longrightarrow \exists \mathbb{B}(\mathbb{B} <_\psi \mathbb{A} \wedge \mathbb{B} \not<_\varphi \mathbb{A})$.

Theorem 3.1 (Derivability Criterion)**.** $\mathrm{L}^*(\backslash, /) \vdash A_1 \ldots A_n \to B$ *if and only if there exists* π *satisfying proof conditions (1)-(4).*

$\mathrm{L}(\backslash, /) \vdash A_1 \ldots A_n \to B$ *if and only if* $n > 0$ *and there exists* π *satisfying proof conditions (1)-(5).*

This theorem will be proven in section 5.

We will call $\mathcal{G} \subset \mathcal{P}_\mathbb{W}$ π-closed if for all $\mathbb{A} \in \mathcal{G}, \pi(\mathbb{A}) \in \mathcal{G}$. It is readily seen that if π satisfies proof conditions (1) and (3), then for every $\mathbb{A} \in \mathcal{N}_\mathbb{W}$, $[\mu^-_\pi(\mathbb{A}), \mu^+_\pi(\mathbb{A})]_\sqsubset$ and $\mathcal{P}_\mathbb{W} \setminus [\mu^-_\pi(\mathbb{A}), \mu^+_\pi(\mathbb{A})]_\sqsubset$ are π-closed. If π satisfies proof conditions (1) and (2), then $\mathcal{G}$ cannot be π-closed if for given $p \in \mathbf{P}$ there are odd number of $\mathbb{A} \in \mathcal{G}$ such that $t(\mathbb{A}) = p$.

Lemma 3.1. *Suppose we have two sequents $A_1 \dots A_n \to B$ and $C_1 \dots C_m \to D$. Let* $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$. *Let* $\mathbb{W} = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![B]\!]$ *and* $\mathbb{W}' = [\![C_1]\!]^{\to} \dots [\![C_m]\!]^{\to}[\![D]\!]$. *Suppose that there is a mapping* $\beta \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}$ *such that the following holds:*

1. β *is injective,*
2. *For all* $\mathbb{A} \in \mathcal{P}_{\mathbb{W}'}$, $t(\beta(\mathbb{A})) = t(\mathbb{A})$, $d(\beta(\mathbb{A})) = d(\mathbb{A})$,
3. *For all* $\mathbb{A}, \mathbb{B} \in \mathbb{P}_{\mathbb{W}'}$, $\mathbb{A} \sqsubset \mathbb{B}$ *if and only if* $\beta(\mathbb{A}) \sqsubset \beta(\mathbb{B})$.

Let $\mathcal{G} = \{\mathbb{A} \in \mathcal{P}_{\mathbb{W}} \mid \neg\exists \mathbb{B} \in \mathcal{P}_{\mathbb{W}'}, \beta(\mathbb{B}) = \mathbb{A}\}$. *If* $\mathcal{G}$ *is* π*-closed and* φ*-closed, then* $\mathrm{L}^*(\backslash, /) \vdash C_1 \dots C_n \to D$.

Proof. Let φ' be φ for $\mathcal{P}_{\mathbb{W}'}$. Since $\mathcal{G}$ is φ-closed, for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}'}$, $\varphi'(\mathbb{A}) = \beta^{-1}(\varphi(\beta(\mathbb{A})))$. Since $\mathcal{G}$ is π-closed, π' defined as $\beta^{-1}\pi\beta$ is defined on all $\mathcal{P}_{\mathbb{W}'}$ and satisfies proof conditions (1)-(4). Therefore by Theorem 3.1

$$\mathrm{L}^*(\backslash, /) \vdash C_1 \dots C_n \to D.$$

□

4. Graphic Representation

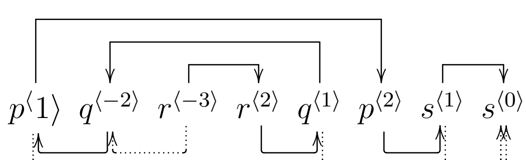
Consider the following Lambek sequent:

$$(p/(r\backslash q))\ (r\backslash q)\ (p\backslash s) \to s.$$

The corresponding element of FS is

$$p^{\langle 1\rangle} q^{\langle -2\rangle} r^{\langle -3\rangle} r^{\langle 2\rangle} q^{\langle 1} p^{\langle 2\rangle} s^{\langle 1\rangle} s^{\langle 0\rangle}.$$

Elements of $\mathcal{P}_{\mathbb{W}}$ correspond to occurences of atoms in the string. So we can draw arrows between such occurences to represent functions φ and ψ. We draw arrows for π for members of $\mathcal{N}_{\mathbb{W}}$ in the upper semiplane of the string and arrows for φ in the lower semiplane. Dotted arrows denote parts of φ that are not part of ψ. Consider the following diagram:



Such diagrams are called proof nets.

Proof nets provide useful intuition about proof conditions. For example proof condition (3) is equivalent to the statement "arrows in the upper semiplane can be drawn without intersections". Proof condition (4) states that for every dotted arrow if we start at its origin and follow solid arrows we will reach its destination.

It is readily seen that this proofnet satisfies proof conditions (1)-(5) and thus $\mathrm{L}(\backslash, /) \vdash (p/(r\backslash q))(r\backslash q)(p\backslash s) \to s$.

5. Proof of the Derivability Criterion

Suppose we have a sequent $A_1 \dots A_n \to B$. Let $\mathbb{W} = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![B]\!]$. Proof conditions:

1. If $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}$, then $\pi(\mathbb{A}) \notin \mathcal{N}_{\mathbb{W}}$ and $\pi^2(\mathbb{A}) = \mathbb{A}$ for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}}$.
2. $t(\pi(\mathbb{A})) = t(\mathbb{A})$.
3. $\mu_\pi^-(\mathbb{A}) \sqsubset \mu_\pi^-(\mathbb{B}) \Rightarrow \mu_\pi^+(\mathbb{A}) \sqsubset \mu_\pi^-(\mathbb{B}) \vee \mu_\pi^+(\mathbb{B}) \sqsubset \mu_\pi^+(\mathbb{A})$.
4. $\mathbb{A} \in \mathcal{N}_{\mathbb{W}} \Longrightarrow \mathbb{A} <_\psi \varphi(\mathbb{A})$ or equivalently $\forall \mathbb{A} \in \mathcal{P}_{\mathbb{W}}, \mathcal{F}_\varphi(\mathbb{A}) \subset \mathcal{F}_\psi(\mathbb{A})$.
5. $\mathbb{A} \notin \mathcal{N}_{\mathbb{W}} \wedge \mathbb{A} \neq \mathbb{A}_0 \Longrightarrow \exists \mathbb{B}(\mathbb{B} <_\psi \mathbb{A} \wedge \mathbb{B} \not<_\varphi \mathbb{A})$.

Lemma 5.1. *If $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$, then there exists π on $\mathcal{P}_{\mathbb{W}}$ satisfying proof conditions (1)-(4).*

If $\mathrm{L}(\backslash, /) \vdash A_1 \dots A_n \to B$, then there exists π on $\mathcal{P}_{\mathbb{W}}$ satisfying proof conditions (1)-(5).

Proof. Suppose that $\mathrm{L}^{(*)}(\backslash, /) \vdash A_1 \dots A_n \to B$. Induction on the length of the derivation.

If the sequent is of the form $p \to p$, then $\mathbb{W} = p^{\langle 1\rangle}p^{\langle 0\rangle}$, $\mathcal{P}_{\mathbb{W}} = \{p^{\langle 1\rangle}, p^{\langle 1\rangle}p^{\langle 0\rangle}\}$, $\mathcal{N}_{\mathbb{W}} = \{p^{\langle 1\rangle}\}$ and π such that $\pi(p^{\langle 1\rangle}) = p^{\langle 1\rangle}p^{\langle 0\rangle}$ and $\pi(p^{\langle 1\rangle}p^{\langle 0\rangle}) = p^{\langle 1\rangle}$ satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(\to /)$. Then $B = (C/D)$, $\mathrm{L}^{(*)}(\backslash, /) \vdash A_1 \dots A_n D \to C$ and for $\mathcal{P}_{\mathbb{W}'}$, where $\mathbb{W}' = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!]^{\to}[\![C]\!]$ there exists π' satisfying all necessary proof conditions. But in this case $\mathbb{W} = \mathbb{W}'$, and therefore this π' works for the sequent $A_1 \dots A_n \to B$ too.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(\to \backslash)$. Then $B = (C\backslash D)$, $\mathbb{W} = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!][\![C]\!]^{\leftarrow}$, $\mathrm{L}^{(*)}(\backslash, /) \vdash CA_1 \dots A_n \to D$, and by induction hypothesis for $\mathcal{P}_{\mathbb{W}'}$, where

$$\mathbb{W}' = [\![C]\!]^{\to}[\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!]$$

there exists π' satisfying all necessary proof conditions. Consider

$$\beta : \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta(\mathbb{A}) = \begin{cases} [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to} [\![D]\!] (\mathbb{A}^{\to^{-1}})^{\leftarrow}, & \text{if } \mathbb{A} \sqsubseteq [\![C]\!]^{\to}; \\ [\![C]\!]^{\to} \diagdown \mathbb{A}, & \text{if } [\![C]\!]^{\to} \sqsubset \mathbb{A}. \end{cases}$$

Let $\pi(\mathbb{A}) = \beta(\pi'(\beta^{-1}(\mathbb{A})))$. Such π satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(/ \to)$. Then $A_1 \dots A_n \to B$ is of the form

$$C_1 \dots (C_i/D) D_1 \dots D_k C_{i+1} \dots C_l \to C$$

so that $\mathrm{L}^{(*)}(\backslash, /) \vdash C_1 \dots C_l \to C$ and $\mathrm{L}^{(*)}(\backslash, /) \vdash D_1 \dots D_k \to D$.

Consider $\mathbb{W}' = [\![C_1]\!]^{\to} \dots [\![C_l]\!]^{\to} [\![C]\!]$ and $\mathbb{W}'' = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to} [\![D]\!]$. By induction hypothesis there are π' and π'' — functions on $\mathcal{P}_{\mathbb{W}'}$ and $\mathcal{P}_{\mathbb{W}''}$ respectively, satisfying all necessary proof conditions.

Let $\mathbb{C} = [\![C_1]\!]^{\to} \dots [\![C_i]\!]^{\to}$ and $\mathbb{D} = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to}$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A}, & \text{if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{D}(\mathbb{C} \diagdown \mathbb{A}), & \text{if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases}$$

$$\text{and } \beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{A}, & \text{if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}((\mathbb{D} \diagdown \mathbb{A})^{\to})^{\to}, & \text{if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}.$$

$$\text{Let } \pi(\mathbb{A}) = \begin{cases} \beta'(\pi'(\beta'^{-1}(\mathbb{A}))), & \text{if } \mathbb{A} \sqsubseteq \mathbb{C} \text{ or } \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{D} \sqsubset \mathbb{A}; \\ \beta''(\pi''(\beta''^{-1}(\mathbb{A}))), & \text{if } \mathbb{C} \sqsubset \mathbb{A} \sqsubseteq \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{D}; \end{cases}.$$

Such π satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(\backslash \to)$. Then $A_1 \dots A_n \to B$ is of the form

$$C_1 \dots C_{i-1} D_1 \dots D_k (D \backslash C_i) \dots C_l \to C$$

so that $\mathrm{L}^{(*)}(\backslash, /) \vdash C_1 \dots C_l \to C$ and $\mathrm{L}^{(*)}(\backslash, /) \vdash D_1 \dots D_k \to D$.

Consider $\mathbb{W}' = [\![C_1]\!]^{\to} \dots [\![C_l]\!]^{\to} [\![C]\!]$ and $\mathbb{W}'' = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to} [\![D]\!]$. By induction hypothesis there are π' and π'' — functions on $\mathcal{P}_{\mathbb{W}'}$ and $\mathcal{P}_{\mathbb{W}''}$ respectively, satisfying all necessary proof conditions.

Let $\mathbb{C} = [\![C_1]\!]^{\to} \dots [\![C_{i-1}]\!]^{\to}$ and $\mathbb{D} = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to}$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A}, & \text{if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\to}(\mathbb{C}\diagdown\mathbb{A}), & \text{if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases}$$

$$\text{and } \beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}\mathbb{A}, & \text{if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}\mathbb{D}((\mathbb{D}\diagdown\mathbb{A})^{\leftarrow})^{\to}, & \text{if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}.$$

$$\text{Let } \pi(\mathbb{A}) = \begin{cases} \beta'(\pi'(\beta'^{-1}(\mathbb{A}))), & \text{if } \mathbb{A} \sqsubseteq \mathbb{C} \text{ or } \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\to} \sqsubset \mathbb{A}; \\ \beta''(\pi''(\beta''^{-1}(\mathbb{A}))), & \text{if } \mathbb{C} \sqsubset \mathbb{A} \sqsubseteq \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\to}; \end{cases}.$$

Such π satisfies all necessary proof conditions.

Thus the lemma is fully proven. □

Now suppose that for the given sequent $A_1 \dots A_n \to B$, $n > 0$, and for $\mathcal{P}_{\mathbb{W}}$ there exists π satisfying proof conditions (1)-(4).

Lemma 5.2. *The relation $\leq_\psi$ is a partial order on $\mathcal{P}_{\mathbb{W}}$.*

Proof. Reflexivity and transitivity directly follow from the definition of $\leq_\psi$.

Now lets prove antisymmetry. Suppose that there are $\mathbb{B}, \mathbb{C} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{B} \leq_\psi \mathbb{C}$ and $\mathbb{C} \leq_\psi \mathbb{B}$. If $\mathbb{B} \neq \mathbb{C}$ then there is $i > 0$ such that $\psi^i(\mathbb{B}) = \mathbb{B}$ and thus for all $j > 0$, $\psi^j(\mathbb{B})$ is defined.

If π satisfies proof condition (4) and $\mathbb{A} \leq_\varphi \mathbb{B}$, then $\mathbb{A} \leq_\psi \mathbb{B}$. There is $\mathbb{A}_0 \in \mathcal{P}_{\mathbb{W}}$ such that $d(\mathbb{A}_0) = 0$, and for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}}$, $\mathbb{A} \leq_\varphi \mathbb{A}_0$. This means that $\mathbb{B} \leq_\varphi \mathbb{A}_0$ and thus $\mathbb{B} \leq_\psi \mathbb{A}_0$. The function ψ is not defined on $\mathbb{A}_0$. Contradiction. □

Lemma 5.3. *If $\mathbb{A} <_\psi \mathbb{B}$ and $\mathbb{C}$ is the φ-join of $\mathbb{A}$ and $\mathbb{B}$, then $\mathbb{C} \notin \mathcal{N}_{\mathbb{W}}$.*

Proof. Suppose that $\mathbb{C} \in \mathcal{N}_{\mathbb{W}}$. There is $\mathbb{C}_1$ such that $\mathbb{A} \leq_\varphi \mathbb{C}_1$ and $\varphi(\mathbb{C}_1) = \mathbb{C}$. There is $\mathbb{C}_2 \neq \mathbb{C}_1$ such that $\mathbb{B} \leq_\varphi \mathbb{C}_2$ and $\varphi(\mathbb{C}_2) = \mathbb{C}$. This means that $\mathbb{A} \leq_\psi \mathbb{C}_1$, $\mathbb{B} \leq_\psi \mathbb{C}_2$, and since $\mathbb{A} \leq_\psi \mathbb{B}$, either $\mathbb{C}_1 <_\psi \mathbb{C}_2$ or $\mathbb{C}_2 <_\psi \mathbb{C}_1$. But since $\psi(\mathbb{C}_1) = \psi(\mathbb{C}_2) = \mathbb{C}$, we get $\mathbb{C} <_\psi \mathbb{C}$. Contradiction. □

Consider the following abbreviations:

- $\mathbb{A}_i = [\![A_1]\!]^{\to} \dots [\![A_i]\!]^{\to}$.
- If $A_i = A'_i/A''_i$, then $\mathbb{A}'_i = [\![A_1]\!]^{\to} \dots [\![A'_i]\!]^{\to}$.
- If $A_i = A''_i \backslash A'_i$, then $\mathbb{A}'_i = [\![A_1]\!]^{\to} \dots ([\![A''_i]\!]^{\leftarrow})^{\to}$.

Lemma 5.4. $\mathrm{L}^*(\backslash,/) \vdash A_1 \ldots A_n \to B$.

Proof. Induction on total number of connectives in the sequent.

If there are no connectives, the sequent is of the form $p_1 \ldots p_n \to q$ and $\mathbb{W} = p_1^{\langle 1 \rangle} \ldots p_n^{\langle 1 \rangle} q^{\langle 0 \rangle}$. The function π satisfies proof condition (1), thus $|\mathcal{N}_{\mathbb{W}}| = |\mathcal{P}_{\mathbb{W}} \backslash \mathcal{N}_{\mathbb{W}}|$. This means that $n = 1$. So $\mathcal{P}_{\mathbb{W}} = \{p_1^{\langle 1 \rangle}, p_1^{\langle 1 \rangle} q^{\langle 0 \rangle}\}$ and $\mathcal{N}_{\mathbb{W}} = \{p_1^{\langle 1 \rangle}\}$. The function π satisfies proof condition (2), therefore $p_1 = q$, and the sequent is an axiom.

If $B = (C/D)$, then the sequent $A_1 \ldots A_n D \to C$ has less connectives then the original sequent, but $[\![A_1]\!]^{\to} \ldots [\![A_n]\!]^{\to} [\![D]\!]^{\to} [\![C]\!] = \mathbb{W}$, and therefore π satisfies all necessary proof conditions for the new sequent. By induction hypothesis this means that $\mathrm{L}^*(\backslash,/) \vdash A_1 \ldots A_n D \to C$ and by applying the rule $(\to /)$ we get $\mathrm{L}^*(\backslash,/) \vdash A_1 \ldots A_n \to B$.

If $B = (C \backslash D)$, then the sequent $C A_1 \ldots A_n \to D$ has less connectives then the original sequent.

Let $\mathbb{W}' = [\![C]\!]^{\to} \mathbb{A}_n [\![D]\!]$. Consider

$$\beta : \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta(\mathbb{B}) = \begin{cases} \mathbb{A}_n [\![D]\!] (\mathbb{B}^{\to^{-1}})^{\leftarrow}, & \text{if } \mathbb{B} \sqsubseteq [\![C]\!]^{\to}; \\ [\![C]\!]^{\to} \diagdown \mathbb{B}, & \text{if } [\![C]\!]^{\to} \sqsubset \mathbb{B}; \end{cases}.$$

Let $\pi'(\mathbb{B}) = \beta^{-1}(\pi(\beta(\mathbb{B})))$. Such π' satisfies all necessary proof conditions. By induction hypothesis this means that $\mathrm{L}^*(\backslash,/) \vdash C A_1 \ldots A_n \to D$, and by applying the rule $(\to \backslash)$ we get $\mathrm{L}^*(\backslash,/) \vdash A_1 \ldots A_n \to B$.

Now we can only consider sequents of the form $A_1 \ldots A_n \to p$. This means that $\mathbb{W} = \mathbb{A}_n p^{\langle 0 \rangle}$. Let $\mathbb{B}_1 = \pi(\mathbb{W})$. Since π satisfies proof condition (4) and ψ is not defined on $\mathbb{W}$, $\varphi(\mathbb{B}_1) = \mathbb{W}$. Therefore $d(\mathbb{B}_1) = 1$ and for every $\mathbb{C} \sqsubset \mathbb{W}$ we have $\mathbb{C} \leq_{\psi} \mathbb{B}_1$. There is no $\mathbb{C} \in \mathcal{P}_{\mathbb{W}}$ such that $\mu^-_{\psi}(\mathbb{C}) \sqsubset \mathbb{B}_1 \sqsubset \mu^+_{\psi}(\mathbb{C})$. There exists $i \leq n$ such that $\mathbb{B}_1 \in (\mathbb{A}_{i-1}, \mathbb{A}_i]$.

Suppose that $A_i = (A_i'/A_i'')$. There exists a unique $\mathbb{D} \in \mathcal{P}_{[\![A_i'']\!]}$ such that $d(\mathbb{D}) = 0$. Consider $\mathbb{B}_2 = \mathbb{A}_i'(\mathbb{D}^{\to})^{\to} \in \mathcal{P}_{\mathbb{W}}$. Obviously $d(\mathbb{B}_2) = -2$, $\varphi(\mathbb{B}_2) = \mathbb{B}_1$, $\psi^2(\mathbb{B}_2) = \mathbb{W}$, and there is no $\mathbb{C} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{B}_2 \sqsubset \mathbb{C}$ and $\varphi(\mathbb{C}) = \mathbb{A}_1$.

Also $\mathcal{F}_{\psi}(\mathbb{B}_2) = [\nu^-_{\psi}(\mathbb{B}_2), \nu^+_{\psi}(\mathbb{B}_2)]_{\sqsubset} = (\mathbb{A}_i', \mathbb{A}_l]_{\sqsubset}$ for some $l \geq i$.

Let us prove this statement. There are no $\mathbb{C} \in \mathcal{F}_{\psi}(\mathbb{B}_2)$ such that $\mathbb{C} \sqsubset \mathbb{B}_1$. There are no $\mathbb{C} \in \mathcal{F}_{\psi}(\mathbb{B}_2)$ such that $\mathbb{C} \in (\mathbb{B}_1, \mathbb{A}_i']$, because in this case φ-join of $\mathbb{C}$ and $\mathbb{B}_2$ is $\mathbb{B}_1 \in \mathcal{N}_{\mathbb{W}}$. Since $(\mathbb{A}_i', \mathbb{A}_i]_{\sqsubset} = \mathcal{F}_{\varphi}(\mathbb{B}_2) \subset \mathcal{F}_{\psi}(\mathbb{B}_2)$, we have $\nu^-_{\psi}(\mathbb{B}_2) = \nu^-_{\varphi}(\mathbb{B}_2)$ and $\mathbb{A}_i \sqsubseteq \nu^+_{\psi}(\mathbb{B}_2)$. If $\mathbb{C} <_{\varphi} \mathbb{D}$, then $\mathbb{C} <_{\psi} \mathbb{D}$. This means that if $\mathbb{C} \in \mathcal{F}_{\psi}(\mathbb{B}_2)$, then either $\varphi(\mathbb{C}) \in \mathcal{F}_{\psi}(\mathbb{B}_2)$, or $\varphi(\mathbb{C}) = \mathbb{B}_1$ and

$\mathbb{C} = \mathbb{B}_2$, or $\varphi(\mathbb{C}) = \mathbb{W}$ and $d(\mathbb{C}) = 1$. Since $\mathcal{F}_\psi(\mathbb{B}_2)$ is φ-closed, this means that $\nu^+_\psi(\mathbb{B}_2) = \mathbb{A}_l$ for some $l \geq i$. Consider $\mathbb{C} \in (\mathbb{A}_i, \mathbb{A}_l]_\sqsubset$. There exists $\mathbb{C}' \in (\mathbb{A}_i, \mathbb{A}_l]_\sqsubset$, such that $\mathbb{C} \leq_\varphi \mathbb{C}'$ and $d(\mathbb{C}') = 1$. If $\mathbb{C}' <_\psi \mathbb{B}_2$, then $\mathbb{C} <_\psi \mathbb{B}_2$. Otherwise there exists $\mathbb{D} \in \mathcal{F}_\psi(\mathbb{B}_2) \cap \mathcal{N}_\mathbb{W}$ such that $\mathbb{C}' \in [\mu^-_\pi(\mathbb{D}), \mu^+_\pi(\mathbb{D})]_\sqsubset$. Since $\mathbb{D} \not<_\varphi \mathbb{C}'$, we have $\mathbb{C} \in [\mu^-_\pi(\mathbb{D}), \mu^+_\pi(\mathbb{D})]_\sqsubset$. Thus for all $\mathbb{C} \in (\mathbb{A}_i, \mathbb{A}_l]_\sqsubset$ we have $\psi(\mathbb{C}) \in (\mathbb{A}'_i, \mathbb{A}_l]_\sqsubset$. Thus the only element $\mathbb{E} \in [\nu^-_\psi(\mathbb{B}_2), \nu^+_\psi(\mathbb{B}_2)]_\sqsubset$ such that $\psi(\mathbb{E}) \notin [\nu^-_\psi(\mathbb{B}_2), \nu^+_\psi(\mathbb{B}_2)]_\sqsubset$ is $\mathbb{B}_2$. Since $\mathbb{C} <_\psi \mathbb{B}_1$, this means that $\mathbb{C} <_\psi \mathbb{B}_2$.

Consider $\mathbb{W}' = \mathbb{A}'_i [\![A_{l+1}]\!]^\rightarrow \dots [\![A_n]\!]^\rightarrow p^{\langle 0 \rangle}$ and $\mathbb{W}'' = [\![A_{i+1}]\!]^\rightarrow \dots [\![A_l]\!]^\rightarrow [\![A''_i]\!]$. Let $\mathbb{C} = [\![A_1]\!]^\rightarrow \dots [\![A_{i-1}]\!]^\rightarrow [\![C]\!]^\rightarrow$ and $\mathbb{D} = [\![A_{i+1}]\!]^\rightarrow \dots [\![A_l]\!]^\rightarrow$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_\mathbb{W}, \qquad \beta'(\mathbb{B}) = \begin{cases} \mathbb{B}, & \text{if } \mathbb{B} \sqsubseteq \mathbb{A}'_i; \\ \mathbb{A}'_i([\![A''_i]\!]^\rightarrow)^\rightarrow \mathbb{D}(\mathbb{A}'_i \diagdown \mathbb{B}), & \text{if } \mathbb{A}'_i \sqsubset \mathbb{B}; \end{cases},$$

$$\beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_\mathbb{W}, \qquad \beta''(\mathbb{B}) = \begin{cases} \mathbb{A}'_i([\![A''_i]\!]^\rightarrow)^\rightarrow \mathbb{B}, & \text{if } \mathbb{B} \sqsubseteq \mathbb{D}; \\ \mathbb{A}_i, ((\mathbb{D} \diagdown \mathbb{B})^\rightarrow)^\rightarrow, & \text{if } \mathbb{D} \sqsubset \mathbb{B}; \end{cases}.$$

The functions $\pi' = \beta'^{-1} \pi \beta'$ and $\pi'' = \beta''^{-1} \pi \beta''$ satisfy all necessary proof conditions. By induction hypothesis this means that

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_{i-1} A'_i A_{l+1} \dots A_n \to p$$

and $\mathrm{L}^*(\backslash, /) \vdash A_{i+1} \dots A_l \to A''_i$. By applying the rule $(/ \to)$ we get

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to p.$$

Suppose that $A_i = (A''_i \backslash A'_i)$. There exists a unique $\mathbb{D} \in \mathcal{P}_{[\![A''_i]\!]}$ such that $d(\mathbb{D}) = 0$. Let $\mathbb{B}_2 = \mathbb{A}_{i-1}(\mathbb{D}^\leftarrow)^\rightarrow \in \mathcal{P}_\mathbb{W}$. Obviously $d(\mathbb{B}_2) = 2$, $\varphi(\mathbb{B}_2) = \mathbb{B}_1$, $\psi^2(\mathbb{B}_2) = \mathbb{W}$, and there is no $\mathbb{C} \in \mathcal{P}_\mathbb{W}$ such that $\mathbb{C} \sqsubset \mathbb{B}_2$ and $\varphi(\mathbb{C}) = \mathbb{B}_1$. Like in the previous case we can say that $\mathcal{F}_\psi(\mathbb{A}_2) = [\nu^-_\psi(\mathbb{A}_2), \nu^+_\psi(\mathbb{A}_2)]_\sqsubset = (\mathbb{A}_l, \mathbb{A}'_i]_\sqsubset$ for some $l \leq i-1$.

Consider $\mathbb{W}' = \mathbb{A}_l [\![A'_i]\!]^\rightarrow [\![A_{i+1}]\!]^\rightarrow \dots [\![A_n]\!]^\rightarrow p^{\langle 0 \rangle}$ and

$$\mathbb{W}'' = [\![A_{l+1}]\!]^\rightarrow \dots [\![A_{i-1}]\!]^\rightarrow [\![A''_i]\!].$$

Let $\mathbb{D} = [\![A_{l+1}]\!]^\rightarrow \dots [\![A_{i-1}]\!]^\rightarrow$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \qquad \beta'(\mathbb{B}) = \begin{cases} \mathbb{B}, & \text{if } \mathbb{B} \sqsubseteq \mathbb{A}_l; \\ \mathbb{A}_l \mathbb{D}(\llbracket A_i'' \rrbracket^{\leftarrow})^{\rightarrow}(\mathbb{A}_l \diagdown \mathbb{B}), & \text{if } \mathbb{A}_l \sqsubset \mathbb{B}; \end{cases},$$

$$\beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \qquad \beta''(\mathbb{B}) = \begin{cases} \mathbb{A}_l \mathbb{B}, & \text{if } \mathbb{B} \sqsubseteq \mathbb{D}; \\ \mathbb{A}_l \mathbb{D}((\mathbb{D} \diagdown \mathbb{B})^{\leftarrow})^{\rightarrow}, & \text{if } \mathbb{D} \sqsubset \mathbb{B}; \end{cases}.$$

The functions $\pi' = \beta'^{-1}\pi\beta'$ and $\pi'' = \beta''^{-1}\pi\beta''$ satisfy all necessary proof conditions. By induction hypothesis this means that

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \ldots A_l A_i' A_{i+1} \ldots A_n \to p$$

and $\mathrm{L}^*(\backslash, /) \vdash A_{l+1} \ldots A_{i-1} \to A_i''$. By applying the rule $(\backslash \to)$ we get

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \ldots A_n \to p.$$

The lemma is fully proven. □

Lemma 5.5. *If π also satisfies proof condition (5), then*

$$\mathrm{L}(\backslash, /) \vdash A_1 \ldots A_n \to B.$$

Proof. By Lemma 5.4 we have $\mathrm{L}^*(\backslash, /) \vdash A_1 \ldots A_n \to B$. The construction given in the proof of Lemma 5.4 provides us with a possible last step of the derivation. Hence we can construct a derivation. If π satisfies proof condition (5), then there will be no $\mathbb{B}_2$ such that $\mathcal{F}_\psi(\mathbb{B}_2) = \mathcal{F}_\varphi(\mathbb{B}_2)$, and thus there will be no steps in derivation that require sequents of the form $\to A$. This means that $\mathrm{L}(\backslash, /) \vdash A_1 \ldots A_n \to B$. □

Lemmas 5.1, 5.4, and 5.5 together gives us Theorem 3.1.

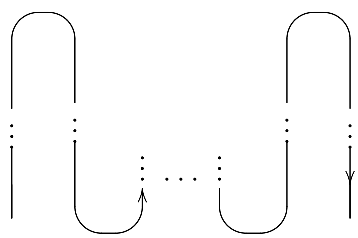
6. Proof of the Main Theorem

By definition of $[\![\cdot]\!]$ we have:

$$
\begin{aligned}
[\![G^0]\!] &= p_n^{0\langle 0\rangle} p_0^{0\langle -1\rangle}\\
[\![G^j]\!] &= p_n^{j\langle 0\rangle} q_n^{j\langle -1\rangle} ([\![G^{j-1}]\!]^{\rightarrow})^{\leftarrow} q_0^{j\langle 4\rangle} p_0^{j\langle -3\rangle}\\
[\![G]\!] &= [\![G^m]\!]\\
[\![E_i^0(t)]\!] &= p_{i-1}^{0\langle 0\rangle}\\
[\![E_i^j(t)]\!] &= \begin{cases} p_{i-1}^{j\langle 2\rangle} q_{i-1}^{j\langle -3\rangle} ((([\![E_i^{j-1}(t)]\!]^{\rightarrow})^{\leftarrow})^{\leftarrow})^{\rightarrow} p_i^{j-1\langle 1\rangle} q_i^{j\langle 0\rangle}, & \text{if } \neg_t x_i \text{ appears in } c_j\\ p_{i-1}^{j\langle 0\rangle} q_{i-1}^{j\langle -1\rangle} ((([\![E_i^{j-1}(t)]\!]^{\leftarrow})^{\rightarrow})^{\rightarrow})^{\leftarrow} p_i^{j-1\langle 3\rangle} q_i^{j\langle -2\rangle}, & \text{if } \neg_t x_i \text{ does not appear in } c_j \end{cases}\\
[\![F_i(t)]\!]^{\rightarrow} &= ([\![E_i^m(t)]\!]^{\leftarrow})^{\rightarrow} p_i^{m\langle 1\rangle}
\end{aligned}
$$

Consider $\mathbb{W} = [\![F_1(t_1)]\!]^{\rightarrow} \dots [\![F_n(t_n)]\!]^{\rightarrow}[\![G]\!]$.

For these sequents it is convienient to use different type of proofnet. Let us write $\mathbb{W}$ like this



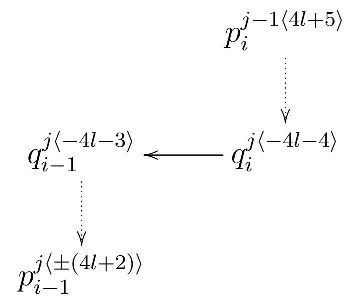
Starting from lower left corner, one atom per cell in a matrix with $2m+1$ rows and $2n+2$ columns.

If a primitive type occurs in the sequent $F_1(t_1)\dots F_n(t_n) \to G$, it occurs exactly twice. Let $\mathbb{P}_i^{j+}$ be the element of $\mathcal{N}_{\mathbb{W}}$ such that $t(\mathbb{P}_i^{j+}) = p_i^j$ (the corresponding atom occurence in the matrix is at row $2j+1$ and column $2i$ for $i > 0$ and $2n+2$ for $i = 0$) and $\mathbb{P}_i^{j-}$ be the element of $\mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}$ such that $t(\mathbb{P}_i^{j-}) = p_i^j$ (row $2j+1$, column $2i+1$). In the same way we define $\mathbb{Q}_i^{j+}$ (row $2j$, column $2j+1$) and $\mathbb{Q}_i^{j-}$ (row $2j$, column $2i$ for $i > 0$ and $2n+2$ for $i = 0$).

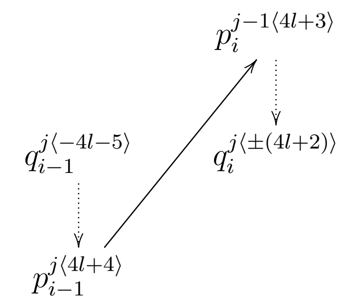
The following facts hold:

1. $d(\mathbb{P}_n^{m-}) = 0$.
2. If $\neg_{t_i} x_i$ does not appear in the clause c_j, then $\varphi^3(\mathbb{P}_i^{j-1+}) = \varphi^2(\mathbb{Q}_i^{j-}) =$

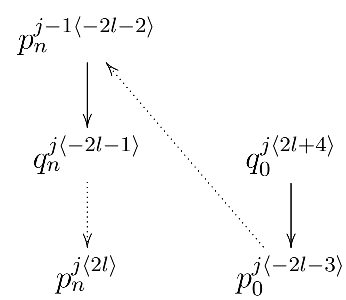
$\varphi(\mathbb{Q}_{i-1}^{j+}) = \mathbb{P}_{i-1}^{j-}$.



3. If $\neg_{t_i} x_i$ appears in clause c_j, then $\varphi^3(\mathbb{Q}_{i-1}^{j+}) = \varphi^2(\mathbb{P}_{i-1}^{j-}) = \varphi(\mathbb{P}_i^{j-1+}) = \mathbb{Q}_i^{j-}$.



4. $\varphi^4(\mathbb{Q}_0^{j-}) = \varphi^3(\mathbb{P}_0^{j+}) = \varphi^2(\mathbb{P}_n^{j-1-}) = \varphi(\mathbb{Q}_n^{j+}) = \mathbb{P}_n^{j-}$.

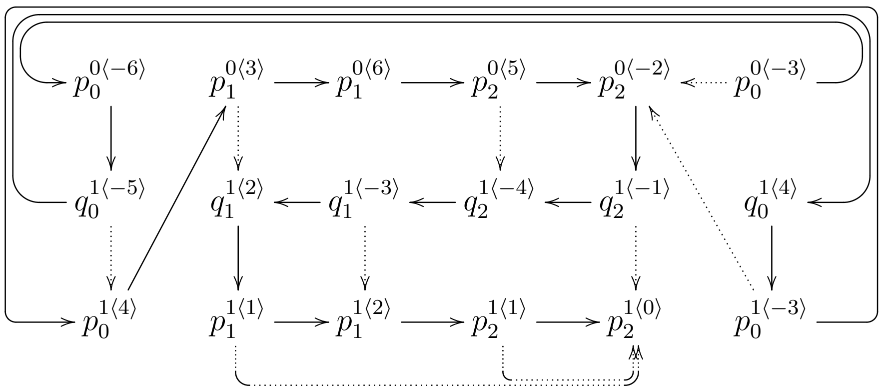


Here $l = m - j$.

The function π can only satisfy proof conditions (1) and (2) if for every i and j, $\pi(\mathbb{P}_i^{j+}) = \mathbb{P}_i^{j-}$ and $\pi(\mathbb{Q}_i^{j+}) = \mathbb{Q}_i^{j-}$. If it is so, then π satisfies proof conditions (3) and (5).

Example 6.1. *Consider the boolean formula $x_1 \vee x_2$.*

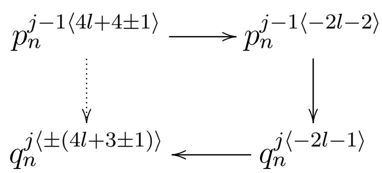
The proof net for $F_1(1)F_2(0) \to G$ will be the following:



Lemma 6.1. *For every $0 < i \leq n$ and $j > 0$, $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.*

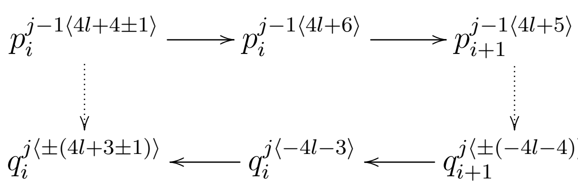
Proof. For $i = n$ this is true, because

$$\psi^3(\mathbb{P}_n^{j-1+}) = \pi\varphi\pi(\mathbb{P}_n^{j-1+}) = \pi\varphi(\mathbb{P}_n^{j-1-}) = \pi(\mathbb{Q}_n^{j+}) = \mathbb{Q}_n^{j-}.$$

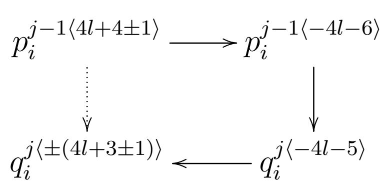


Now suppose that for all $i' > i$ this was already proven. There are four possibilities:

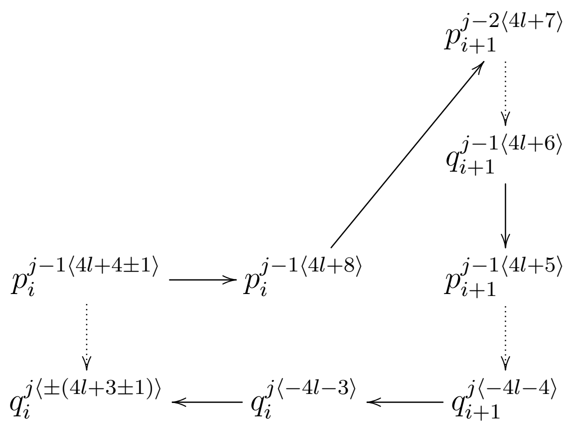
1. If $\neg_{t_{i+1}} x_{i+1}$ does not appear in the clauses c_{j-1} and c_j, then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-1+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-}) = \mathbb{Q}_i^{j-}$, and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{Q}_{i+1}^{j-}$. Thus $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.



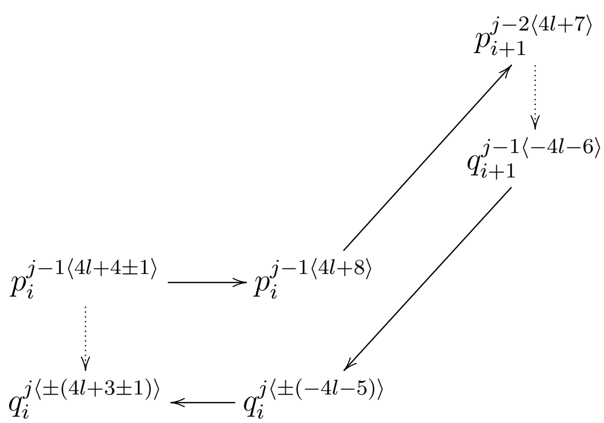
2. If $\neg_{t_{i+1}} x_{i+1}$ does not appear in the clause c_{j-1}, but appears in c_j, then $\psi^3(\mathbb{P}_i^{j-1+}) = \pi\varphi\pi(\mathbb{P}_i^{j-1+}) = \pi\varphi(\mathbb{P}_i^{j-1-}) = \pi(\mathbb{Q}_i^{j+}) = \mathbb{Q}_i^{j-}$.



3. If $\neg_{t_{i+1}} x_{i+1}$ appears in the clause c_{j-1}, but does not appear in c_j, then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-2+}$, $\psi^2(\mathbb{Q}_{i+1}^{j}) = \mathbb{Q}_i^{j-}$, $\varphi(\mathbb{Q}_{i+1}^{j-1+}) = \mathbb{P}_{i+1}^{j-1+}$, $\mathbb{P}_{i+1}^{j-2+} <_\psi \mathbb{Q}_{i+1}^{j-1-}$, and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{Q}_{i+1}^{j-}$. Thus $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.



4. If $\neg_{t_{i+1}} x_{i+1}$ appears in both clauses c_{j-1} and c_j, then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-2+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-1-}) = \mathbb{Q}_i^{j-}$, and $\mathbb{P}_{i+1}^{j-2+} <_\psi \mathbb{Q}_{i+1}^{j-1-}$. Thus $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.

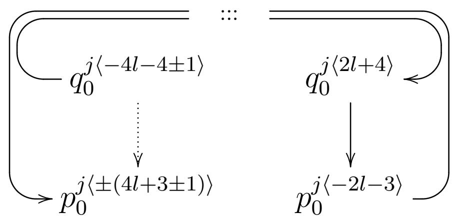


□

Lemma 6.2. *For every $0 \le i < n$ and $j > 0$, $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.*

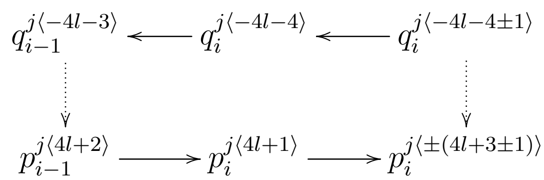
Proof. For $i = 0$ this is true, because

$$\psi^3(\mathbb{Q}_0^{j+}) = \pi\varphi\pi(\mathbb{Q}_0^{j+}) = \pi\varphi(\mathbb{Q}_0^{j-}) = \pi(\mathbb{P}_0^{j+}) = \mathbb{P}_0^{j-}.$$

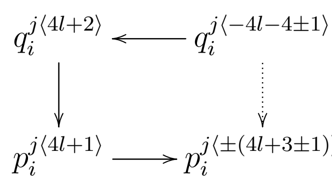


Now suppose that for all $i' < i$ this was already proven. There are four possibilities:

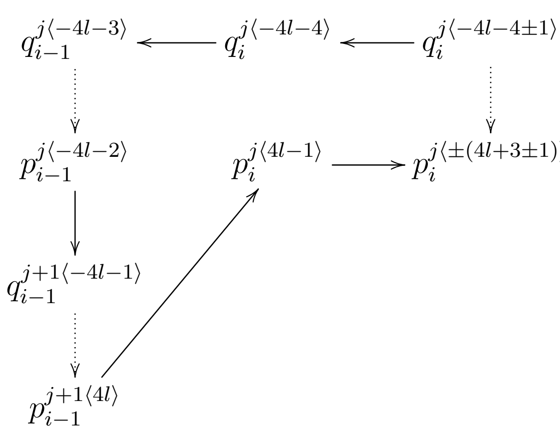
1. If $\neg_{t_i} x_i$ does not appear in the clauses c_{j+1} and c_j, then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+}$, $\psi^2(\mathbb{P}_{i-1}^{j-}) = \mathbb{P}_i^{j-}$, and $\mathbb{Q}_{i-1}^{j+} <_\psi \mathbb{P}_{i-1}^{j-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.



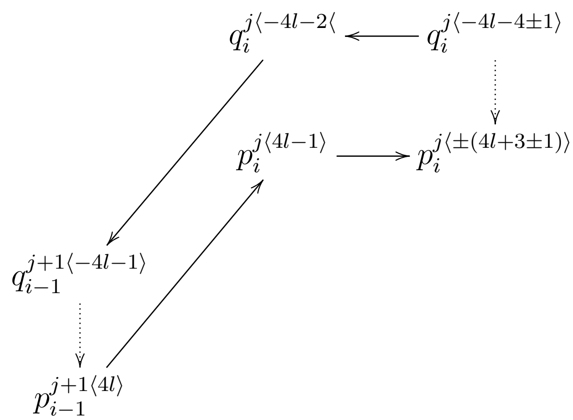
2. If $\neg_{t_i} x_i$ does not appear in the clause c_{j+1}, but appears in c_j, then $\psi^3(\mathbb{Q}_i^{j+}) = \pi\varphi\pi(\mathbb{Q}_i^{j+}) = \pi\varphi(\mathbb{Q}_i^{j-}) = \pi(\mathbb{P}_i^{j+}) = \mathbb{P}_i^{j-}$.



3. If $\neg_{t_i} x_i$ appears in the clause c_{j+1}, but does not appear in c_j, then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+}$, $\psi^2(\mathbb{P}_{i-1}^{j+1-}) = \mathbb{P}_i^{j-}$, $\varphi(\mathbb{P}_{i-1}^{j+}) = \mathbb{Q}_{i-1}^{j+1+}$, $\mathbb{Q}_{i-1}^{j+} <_\psi \mathbb{P}_{i-1}^{j-}$, and $\mathbb{Q}_{i-1}^{j+1+} <_\psi \mathbb{P}_{i-1}^{j+1-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.



4. If $\neg_{t_i} x_i$ appears in both clauses c_{j+1} and c_j, then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+1+}$, $\psi^2(\mathbb{P}_{i-1}^{j+1-}) = \mathbb{P}_i^{j-}$, and $\mathbb{Q}_{i-1}^{j+1+} <_\psi \mathbb{P}_{i-1}^{j+1-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.



□

From lemmas 6.1 and 6.2 we can conclude that if $i > 0$ and $j \leq j'$ then $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_i^{j'+}$.

Lemma 6.3. *If $i < i'$, then $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i'}^{j+}$.*

Proof. If $\neg_{t_{i+1}} x_{i+1}$ appears in clause c_j, then $\psi^2(\mathbb{P}_i^{j+}) = \mathbb{P}_{i+1}^{j-1+}$ and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{P}_{i+1}^{j+}$. If $\neg_{t_{i+1}} x_{i+1}$ appears in clause c_{j+1}, then $\psi(\mathbb{P}_i^{j+1-}) = \mathbb{P}_{i+1}^{j+}$ and $\mathbb{P}_i^{j-} <_\psi \mathbb{P}_i^{j+1+}$. If neither of this is the case, then $\psi^2(\mathbb{P}_i^{j+}) = \mathbb{P}_{i+1}^{j+}$. This means that $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i+1}^{j+}$ and thus $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i'}^{j+}$. □

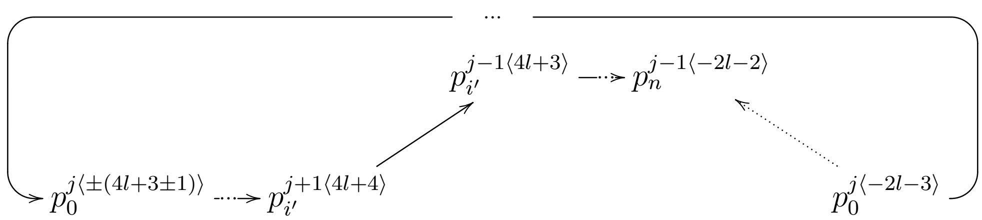
Lemma 6.4. *$\langle t_1, \ldots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$ if and only if $\mathrm{L}^*(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G$ and if and only if*

$$\mathrm{L}(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G.$$

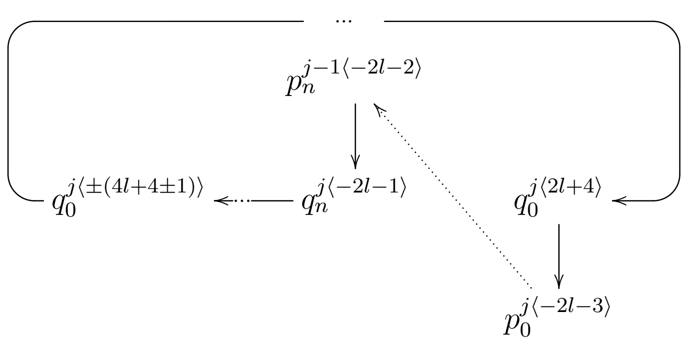
Proof. Suppose that $\langle t_1, \ldots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$. In view of lemmas 6.1 and 6.2 and the fact that for $\mathbb{P}_i^{m+}$ where $i > 0$ proof condition (4) is satisfied automatically, because $\varphi(\mathbb{P}_i^{m+}) = \mathbb{P}_n^{m-}$, the only members of $\mathcal{N}_{\mathbb{W}}$ for which we have not proved that π satisfies proof condition (4) are $\mathbb{P}_0^{j+}$.

We now prove that for every $j > 0$, $\mathbb{P}_0^{j+} <_\psi \varphi(\mathbb{P}_0^{j+}) = \mathbb{P}_n^{j-1-}$. There exists i such that $\neg_{t_i} x_i$ appears in the clause c_j. This means that $\psi(\mathbb{P}_{i-1}^{j-}) = \mathbb{P}_i^{j-1+}$

and by lemma 6.3 $\mathbb{P}_0^{j+} <_\psi \mathbb{P}_i^{j+}$ and $\mathbb{P}_i^{j-1+} <_\psi \mathbb{P}_n^{j-1+}$. Thus $\mathbb{P}_0^{j+} <_\psi \varphi(\mathbb{P}_0^{j+}) = \mathbb{P}_n^{j-1-}$ and by lemma 3.1 we can now say that $\mathrm{L}^*(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G$.



Suppose that $\langle t_1, \ldots, t_n\rangle$ is not a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$. There exists j such that no $\neg_{t_i} x_i$ appear in the clause c_j. This means that for $i \leq n$, $\psi^{2i}(\mathbb{Q}_n^{j+}) = \mathbb{Q}_{n-i}^{j+}$, $\psi(\mathbb{P}_n^{j-1-}) = \mathbb{Q}_n^{j+}$, and $\psi(\mathbb{Q}_0^{j-}) = \mathbb{P}_0^{j+}$. Thus $\mathbb{P}_n^{j-1-} <_\psi \mathbb{P}_0^{j+}$. This means that π cannot satisfy proof condition (4). Thus by lemma 2.1 $\mathrm{L}^*(\backslash, /) \nvdash F_1(t_1) \ldots F_n(t_n) \to G$.



Since π satisfies proof condition (5),

$$\mathrm{L}(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G \Leftrightarrow \mathrm{L}^*(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G$$

and thus the lemma is fully proven. □

Lemma 6.5. *If* $\mathrm{L}(\backslash, /) \vdash \Pi \to A$ *and* $\Pi' \to A'$ *is the result of replacing all instances of primitive type* p *by primitive type* q*, then* $\mathrm{L}(\backslash, /) \vdash \Pi' \to A'$.

Proof. If we replace p by q throughout the derivation of $\Pi \to A$, we will get the derivation of $\Pi' \to A'$. □

Lemma 6.6.

$$\mathrm{L}(\backslash, /) \vdash F_i(1) \to (B_i \backslash A_i),$$
$$\mathrm{L}(\backslash, /) \vdash F_i(0) \to (B_i \backslash A_i).$$

Proof. Consider the boolean formula $c'_1 \wedge \ldots \wedge c'_m$, where

$$c'_i = \begin{cases} (x_1 \vee x_2), & \text{if the literal } \neg_1 x_i \text{ appears in } c_j \\ x_1, & \text{if the literal } \neg_1 x_i \text{ doesn't appear in } c_j. \end{cases}$$

Let $F'_1(1)F'_2(1) \to G'$ be the sequent constructed for this formula. By Lemma 6.4 we can say that $\mathrm{L}(\backslash, /) \vdash F'_1(1)F'_2(1) \to G'$.

By replacing p^j_0 by a^j_i, q^j_0 by b^j_i, p^j_1 by p^j_{i-1}, q^j_1 by q^j_{i-1}, p^j_2 by p^j_i, and q^j_2 by q^j_i, we get $B_iF_i(1) \to A_i$. By Lemma 6.5 we get $\mathrm{L}(\backslash, /) \vdash B_iF_i(1) \to A_i$. Therefore $\mathrm{L}(\backslash, /) \vdash F_i(1) \to (B_i \backslash A_i)$.

Doing the same for the boolean formula $c'_1 \wedge \ldots \wedge c'_m$, where

$$c'_i = \begin{cases} (x_1 \vee x_2), & \text{if the literal } \neg_0 x_i \text{ appears in } c_j \\ x_1, & \text{if the literal } \neg_0 x_i \text{ doesn't appear in } c_j, \end{cases}$$

we get $\mathrm{L}(\backslash, /) \vdash B_iF_i(0) \to A_i$. Therefore $\mathrm{L}(\backslash, /) \vdash F_i(0) \to (B_i \backslash A_i)$. □

Lemma 6.7. $\mathrm{L}(\backslash, /) \vdash \Pi_i \to F_i(t_i)$, *where* $t_i \in \{0, 1\}$.

Proof. Using Lemma 6.6 we get

$$\dfrac{\dfrac{F_i(0) \to F_i(0) \quad F_i(1) \to (B_i \backslash A_i)}{F_i(0)(F_i(0) \backslash F_i(1)) \to (B_i \backslash A_i)}\,(\backslash \to) \quad F_i(0) \to F_i(0)}{(F_i(0)/(B_i \backslash A_i))F_i(0)(F_i(0) \backslash F_i(1)) \to F_i(0)}\,(/ \to)$$

and

$$\dfrac{\dfrac{F_i(0) \to (B_i \backslash A_i) \quad F_i(0) \to F_i(0)}{F_i(0)/(B_i \backslash A_i)F_i(0) \to F_i(0)}\,(/ \to) \quad F_i(1) \to F_i(1)}{(F_i(0)/(B_i \backslash A_i))F_i(0)(F_i(0) \backslash F_i(1)) \to F_i(1)}\,(\backslash \to)$$

Thus $\mathrm{L}(\backslash, /) \vdash \Pi_i \to F_i(0)$ and $\mathrm{L}(\backslash, /) \vdash \Pi_i \to F_i(1)$. □

Lemma 6.8. *If the formula* $c_1 \wedge \ldots \wedge c_m$ *is satifiable, then* $\mathrm{L}(\backslash, /) \vdash \Pi_1 \ldots \Pi_n \to G$.

Proof. Suppose $\langle t_1, \ldots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$. According to Lemma 6.4 $\mathrm{L}(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G$. Now we apply Lemma 6.7 and the cut rule n times. □

Suppose that $\mathrm{L}^*(\backslash,/) \vdash \Pi_1 \dots \Pi_n \to G$. Consider

$$\mathbb{W} = [\![(F_1(0)/(B_1\backslash A_1))]\!]^{\to}[\![F_1(0)]\!]^{\to}[\![(F_1(0)\backslash F_1(1))]\!]^{\to} \dots \\ \dots [\![(F_n(0)/(B_n\backslash A_n))]\!]^{\to}[\![F_n(0)]\!]^{\to}[\![(F_n(0)\backslash F_n(1))]\!]^{\to}[\![G]\!].$$

By Lemma 3.1 for $\mathcal{P}_{\mathbb{W}}$ there exists π satisfying proof conditions (1)-(4). Consider the following abbreviations:

$$\mathbb{F}_i^0 = [\![(F_1(0)/(B_1\backslash A_1))]\!]^{\to}[\![F_1(0)]\!]^{\to}[\![(F_1(0)\backslash F_1(1))]\!]^{\to} \dots [\![F_i(0)]\!]^{\to}$$

$$\begin{aligned} & & \mathbb{F}_i^{0\prime} &= [\![F_i(0)]\!]^{\to} \\ \mathbb{A}_i &= \mathbb{F}_i^0([\![A_i]\!]^{\to})^{\to} & \mathbb{A}_i' &= ([\![A_i]\!]^{\to})^{\to} \\ \mathbb{B}_i &= \mathbb{A}_i(([\![B_i]\!]^{\leftarrow})^{\to})^{\to} & \mathbb{B}_i' &= (([\![B_i]\!]^{\leftarrow})^{\to})^{\to} \\ \mathbb{H}_i &= \mathbb{B}_i[\![F_i(0)]\!]^{\to} & & \\ \mathbb{C}_i &= \mathbb{H}_i([\![F_i(0)]\!]^{\leftarrow})^{\to} & \mathbb{F}_i^{0\prime\prime} &= ([\![F_i(0)]\!]^{\leftarrow})^{\to} \\ \mathbb{F}_i^1 &= \mathbb{C}_i[\![F_i(1)]\!]^{\to} & \mathbb{F}_i^{1\prime} &= [\![F_i(1)]\!]^{\to} \end{aligned}$$

Lemma 6.9. *If* $\mathrm{L}^*(\backslash,/) \vdash \Pi_1 \dots \Pi_i F_{i+1}(t_{i+1}) \dots F_n(t_n) \to G$*, then there is* $t_i \in \{0,1\}$ *such that* $\mathrm{L}^*(\backslash,/) \vdash \Pi_1 \dots \Pi_{i-1} F_i(t_i) \dots F_n(t_n) \to G$

Proof. Consider $\mathbb{W}' = \mathbb{F}_i^1\mathbb{W}''$, where $\mathbb{W}'' = [\![F_{i+1}(t_{i+1})]\!]^{\to} \dots [\![F_n(t_n)]\!]^{\to}[\![G]\!]$. By Lemma 3.1 for $\mathcal{P}_{\mathbb{W}'}$ there exists π satisfying proof conditions (1)-(5).

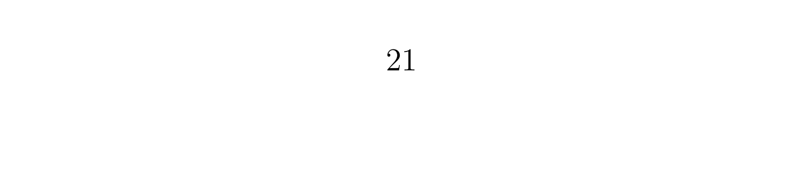
Let $\mathbb{W}_0' = \mathbb{F}_{i-1}^1[\![F_i(0)]\!]^{\to}\mathbb{W}''$ and $\mathbb{W}_1' = \mathbb{F}_{i-1}^1[\![F_i(1)]\!]^{\to}\mathbb{W}''$.

For each j there are only two elements of $\mathcal{P}_{\mathbb{W}'}$ such that $t(\mathbb{A}) = a_i^j$ and two elements such that $t(\mathbb{A}) = b_i^j$. This means that these pairs of elements are π-closed.

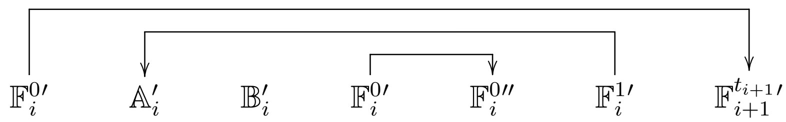
For each j there are six elements of $\mathcal{P}_{\mathbb{W}'}$ such that $t(\mathbb{A}) = p_i^0$. Let us denote them by $\mathbb{P}_1, \dots, \mathbb{P}_6$ so that $\mathbb{P}_1 \sqsubset \dots \sqsubset \mathbb{P}_6$. The following holds:

$$\mathbb{F}_{i-1}^1 \sqsubset \mathbb{P}_1 \sqsubset \mathbb{F}_i^0 \sqsubset \mathbb{P}_2 \sqsubset \mathbb{A}_i \sqsubset \mathbb{B}_i \sqsubset \mathbb{P}_3 \sqsubset \mathbb{H}_i \sqsubset \mathbb{P}_4 \sqsubset \mathbb{C}_i \sqsubset \mathbb{P}_5 \sqsubset \mathbb{F}_i^1 \sqsubset \mathbb{P}_6.$$

$\{\mathbb{P}_1, \dots, \mathbb{P}_6\}$ is π-closed. $\mathbb{P}_1, \mathbb{P}_3, \mathbb{P}_5 \in \mathcal{N}_{\mathbb{W}}$. $[\mathbb{P}_1, \mathbb{P}_2]_{\sqsubset}$,$[\mathbb{P}_3, \mathbb{P}_6]_{\sqsubset}$, and $[\mathbb{P}_4, \mathbb{P}_5]_{\sqsubset}$ cannot be π-closed, therefore there are only two possibilities: either $\pi(\mathbb{P}_1) = \mathbb{P}_4$, $\pi(\mathbb{P}_3) = \mathbb{P}_2$, and $\pi(\mathbb{P}_5) = \mathbb{P}_6$,

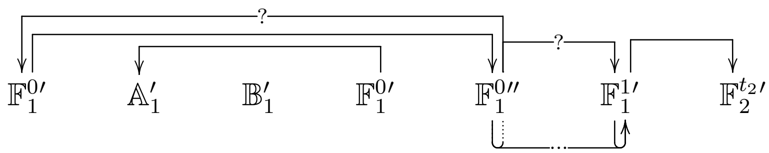


or $\pi(\mathbb{P}_1) = \mathbb{P}_6$, $\pi(\mathbb{P}_3) = \mathbb{P}_4$, and $\pi(\mathbb{P}_5) = \mathbb{P}_2$.

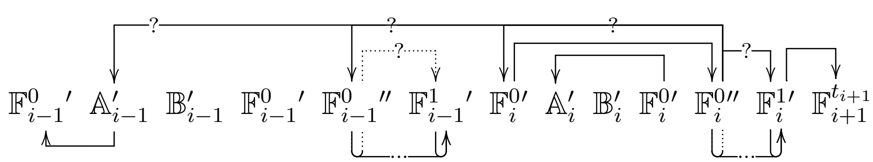


Suppose that $\pi(\mathbb{P}_1) = \mathbb{P}_4$, $\pi(\mathbb{P}_3) = \mathbb{P}_2$, and $\pi(\mathbb{P}_5) = \mathbb{P}_6$. Notice that $t(\mathbb{C}_i) = p^m_{i-1}$ and $\mathbb{C}_i \in \mathcal{N}_{\mathbb{W}'}$.

If $i = 1$, then there are only two variants for $\pi(\mathbb{C}_i)$: one is $p_0^{m\langle l\rangle}$ and the other one is $\mathbb{C}_1 p_0^{m\langle l\rangle}$, where $l = 2$ or $l = 4$. Therefore, since the φ-join of $\mathbb{C}_1$ and $\mathbb{C}_1 p_0^{m\langle l\rangle}$ is $\mathbb{F}^1_1 \in \mathcal{N}_{\mathbb{W}'}$, $\pi(\mathbb{C}_1) = p_0^{m\langle l\rangle}$ and $[p_0^{m\langle l\rangle}, \mathbb{C}_1]_\sqsubset$ is π-closed.



If $i > 1$, then there are four variants for $\pi(\mathbb{C}_i)$: $\mathbb{F}^1_{i-1} p_{i-1}^{m\langle l\rangle}$, $\mathbb{C}_i p_{i-1}^{m\langle l\rangle}$, where $l = 2$ or $l = 4$, $\mathbb{H}_{i-1} p_{i-1}^{m\langle 2\rangle}$, and $\mathbb{F}^0_{i-1} p_{i-1}^{m\langle -2\rangle}$. The second variant is ruled out. If $\pi(\mathbb{C}_i) = \mathbb{H}_{i-1} p_{i-1}^{m\langle 2\rangle}$, then $\pi(\mathbb{C}_{i-1}) = \mathbb{C}_{i-1} p_{i-2}^{m\langle l\rangle}$, where $l = 2$ or $l = 4$, and the φ-join of $\mathbb{C}_{i-1}$ and $\mathbb{C}_{i-1} p_{i-2}^{m\langle l\rangle}$ is $\mathbb{F}^1_{i-1} \in \mathcal{N}_{\mathbb{W}'}$. If $\pi(\mathbb{C}_i) = \mathbb{F}^0_{i-1} p_{i-1}^{m\langle -2\rangle}$, then since the segment $(\mathbb{F}^0_{i-1}, \mathbb{C}_i]_\sqsubset$ is φ-closed and π-closed, $\mathbb{G} \not\leq_\psi \mathbb{F}^0_{i-1} p_{i-1}^{m\langle -2\rangle}$ for all $\mathbb{G} \notin (\mathbb{F}^0_{i-1}, \mathbb{C}_i]_\sqsubset$. But $\psi^2(\mathbb{C}_i) = \varphi(\pi(\mathbb{C}_i)) = \varphi(\mathbb{F}^0_{i-1} p_{i-1}^{m\langle -2\rangle}) = \mathbb{F}^0_{i-1} \notin (\mathbb{F}^0_{i-1}, \mathbb{C}_i]_\sqsubset$. Therefore $\mathbb{C}_i \not<_\psi \mathbb{H}_i p_i^{m\langle 2\rangle}$, but $\mathbb{C}_i <_\varphi \mathbb{H}_i p_i^{m\langle 2\rangle}$ and thus proof condition (4) is not satisfied. Therefore $\pi(\mathbb{C}_i) = \mathbb{F}^1_{i-1} p_{i-1}^{m\langle l\rangle}$ and $(\mathbb{F}^1_{i-1}, \mathbb{C}_i]_\sqsubset$ is π-closed.

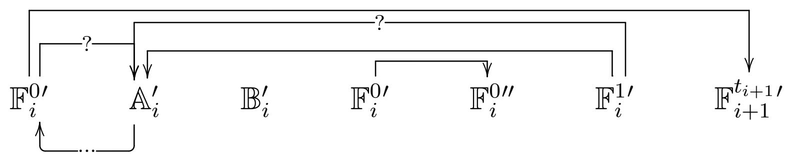


Therefore, since $(\mathbb{F}^1_{i-1}, \mathbb{C}_i]_\sqsubset$ is π-closed and φ-closed, by Lemma 3.1 for $\mathbb{W}'_1$ there is π' satisfying proof conditions (1)-(4) and

$$\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \ldots \Pi_{i-1} F_i(1) \ldots F_n(t_n) \to G.$$

Suppose that $\pi(\mathbb{P}_1) = \mathbb{P}_6$, $\pi(\mathbb{P}_3) = \mathbb{P}_4$, and $\pi(\mathbb{P}_5) = \mathbb{P}_2$. Let $\mathbb{E} = \mathbb{F}^0_i p_{i+1}^{m\langle -2\rangle}$.

There are only two variants for $\pi(\mathbb{E})$: one is $\mathbb{F}_i^0$ and the other one is $\mathbb{F}_i^1$. The φ-join of $\mathbb{E}$ and $\mathbb{F}_i^0$ is $\mathbb{F}_i^0 \in \mathcal{N}_{\mathbb{W}}$. Therefore $\pi(\mathbb{E}) = \mathbb{F}_i^1$ and $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ is π-closed.



Therefore since $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ is π-closed and φ-closed, by Lemma 3.1 for $\mathbb{W}_0'$ there is π' satisfying proof conditions (1)-(4) and

$$\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \ldots \Pi_{i-1} F_i(0) \ldots F_n(t_n) \to G.$$

□

Lemma 6.10. *If* $\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \ldots \Pi_n \to G$*, then the formula* $c_1 \wedge \ldots \wedge c_m$ *is satisfiable.*

Proof. Applying n times Lemma 6.9, we get that there exists $\langle t_1, \ldots, t_n \rangle \in \{0,1\}^n$ such that $\mathrm{L}^*(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G$. By Lemma 6.4 this means that $\langle t_1, \ldots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$. □

Since for all sequents $\mathrm{L}(\backslash, /) \vdash \Pi \to A \Rightarrow \mathrm{L}^*(\backslash, /) \vdash \Pi \to A$, Lemma 6.8 and Lemma 6.10 together give us Theorem 2.1.

Acknowledgements

I am most grateful to Prof. M. Pentus for his constant attention to my work.

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