Operational closure and stability

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Abstract

In this article we introduce and study the notion of operational closure: a transitive set d is called operationally closed iff it contains all constants of OST and any operation $f \in d$ applied to an element $a \in d$ yields an element $fa \in d$, provided that f applied to a has a value at all. We will show that there is a direct relationship between operational closure and stability in the sense that operationally closed sets behave like Σ_1 substructures of the universe. This leads to our final result that OST plus the axiom (OLim), claiming that any set is element of an operationally closed set, is proof-theoretically equivalent to the system $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ of Kripke-Platek set theory with infinity and Σ_1 separation. We also characterize the system OST plus the existence of one operationally closed set in terms of Kripke-Platek set theory with infinity and a parameter-free version of Σ_1 separation.

Keywords: Operational set theory, operational closure, Kripke-Platek set theory, Σ_1 separation, stability, proof theory.

1 Introduction

Operational set theory is an alternative approach to set theory, introduced in Feferman [4] and motivated by a, as he writes, *wider program whose aim is to provide a common framework for analogues of large cardinal notions that have appeared in admissible set theory, admissible recursion theory, constructive set theory, constructive type theory, explicit mathematics, and systems of recursive ordinal notations that have been used in proof theory.* See Feferman [5] for a detailed description of this program and some principal results concerning the central axiomatic system OST, including a model of OST which interprets the underlying applicative structure of OST in the codes for functions that are Σ_1 definable in parameters.

Jäger [8] provides an inductive model construction for OST, studies several extensions of OST, and presents, among other theories, a system which is equiconsistent with ZFC. Full operational set theory $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ with unbounded existential quantification and power set and several of its subsystems have been analyzed in Jäger [9]. For example, making use of results in Jäger and Krähenbühl [11], we obtain that $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ is proof-theoretically equivalent to the system $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{AC})$, which extends von Neumann-Bernays-Gödel set theory NBG by Σ^1_1 choice for classes. The survey paper Jäger [10] places systems of operational set theory into the general set-theoretic landscape and describes model constructions based on inductive extensions of ZFC. Finally, Jäger and Zumbrunnen [12] is about the notion of regularity in operational set theory.

In this article we introduce and study the notion of operational closure: a transitive set d is called operationally closed iff it contains all constants of OST and any operation $f \in d$ applied to an element $a \in d$ yields an element $fa \in d$, provided that f applied to a has a value at all. We will show that there is a direct relationship between operational closure and stability in the sense that operationally closed sets behave like Σ_1 substructures of the universe. This leads to our final result that OST plus the axiom (OLim), claiming that any set is element of an operationally closed set, is proof-theoretically equivalent to the system $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ of Kripke-Platek set theory with infinity and Σ_1 separation. We also characterize the system OST plus the existence of one operationally closed set in terms of Kripke-Platek set theory with infinity and a parameter-free version of Σ_1 separation.

2 Operational closure

We begin this section with briefly recapitulating the syntax of Feferman's theory OST. Then we introduce the notion of an operationally closed set and prove some properties of such sets. Finally, we turn to the operational limit axiom (OLim) which states that every set is element of an operationally closed set.

In introducing the system OST, we follow Jäger [8, 9, 12] very closely and even use the same formulations whenever it seems adequate. So let $\mathcal{L}$ be a typical language of first order set theory with a symbol for the element relation as its only relation symbol and countably many set variables $a, b, c, f, g, u, v, w, x, y, z, \ldots$ (possibly with subscripts). In addition, we assume that $\mathcal{L}$ has a constant for every finite von Neumann ordinal and the constant ω for the collection of all finite von Neumann ordinals. The formulas of $\mathcal{L}$ are defined as usual.

$\mathcal{L}^\circ$, the language of OST, augments $\mathcal{L}$ by the binary function symbol $\circ$ for partial term application, the unary relation symbol $\downarrow$ (defined) and the following constants: (i) the combinators **k** and **s**; (ii) $\top$, $\bot$, **el**, **non**, **dis**, and **e** for logical operations; (iii) $\mathbb{S}$, $\mathbb{R}$, and $\mathbb{C}$ for set-theoretic operations. The meaning of these constants follows from the axioms below.

The *terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. The variables and constants of $\mathcal{L}^\circ$ are terms of $\mathcal{L}^\circ$.

2. If s and t are terms of $\mathcal{L}^\circ$, then so is $\circ(s, t)$.

In the following we often abbreviate $\circ(s, t)$ as $(s \circ t)$, (st) or simply as st. We also adopt the convention of association to the left so that $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2) \ldots s_n)$. In addition, we often write $s(t_1, \ldots, t_n)$ for $st_1 \ldots t_n$ if this seems more intuitive. Moreover, we frequently make use of the vector notation $\vec{s}$ as shorthand for a finite string $s_1, \ldots, s_n$ of $\mathcal{L}^\circ$ terms whose length is either not important or evident from the context.

Self-application is possible and meaningful, but it is not necessarily total, and there may be terms which do not denote an object. We make use of the definedness predicate $\downarrow$ to single out those which do, and $(t\downarrow)$ is read "t is defined" or "t has a value".

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. All expressions of the form $(s \in t)$ and $(t\downarrow)$ are formulas of $\mathcal{L}^\circ$, the so-called *atomic* formulas.

2. If A and B are formulas of $\mathcal{L}^\circ$, then so are $\neg A$, $(A \vee B)$, and $(A \wedge B)$.

3. If A is a formula of $\mathcal{L}^\circ$ and if t is a term of $\mathcal{L}^\circ$ which does not contain x, then $(\exists x \in t)A$, $(\forall x \in t)A$, $\exists x A$, and $\forall x A$ are formulas of $\mathcal{L}^\circ$.

Since we will be working within classical logic, the remaining logical connectives can be defined as usual. We will often omit parentheses and brackets whenever there is no danger of confusion. The free variables of t and A are defined in the conventional way; the closed $\mathcal{L}^\circ$ terms and closed $\mathcal{L}^\circ$ formulas, also called $\mathcal{L}^\circ$ sentences, are those which do not contain free variables.

Given an $\mathcal{L}^\circ$ formula A and a variable u not occurring in A, we write A^u for the result of replacing each unbounded set quantifier $\exists x(\ldots)$ and $\forall x(\ldots)$ in A by $(\exists x \in u)(\ldots)$ and $(\forall x \in u)(\ldots)$, respectively. Equality of sets is introduced by

$$(s = t) \;:=\; (s\downarrow) \,\wedge\, (t\downarrow) \,\wedge\, (\forall x \in s)(x \in t) \,\wedge\, (\forall x \in t)(x \in s).$$

Suppose now that $\vec{u} = u_1, \ldots, u_n$ and $\vec{s} = s_1, \ldots, s_n$. Then $A[\vec{s}/\vec{u}]$ is the $\mathcal{L}^\circ$ formula which is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the $\mathcal{L}^\circ$ terms $\vec{s}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. If the $\mathcal{L}^\circ$ formula A is written as $B[\vec{u}]$, then we often simply write $B[\vec{s}]$ instead of $B[\vec{s}/\vec{u}]$. Further variants of this notation will be obvious.

The logic of OST is the classical *logic of partial terms* due to Beeson and Feferman with the usual strictness axioms (cf. Beeson [2, 3]), including the common equality axioms. Partial equality of terms is introduced by

$$(s \simeq t) \;:=\; (s{\downarrow} \vee t{\downarrow} \;\rightarrow\; s = t)$$

and says that if either s or t denotes anything, then they both denote the same object.

The non-logical axioms of OST comprise axioms about the applicative structure of the universe, some basic set-theoretic properties, the representation of elementary logical connectives as operations, and operational set existence axioms. They divide into four groups.

I. Applicative axioms.

(A1) $\mathbf{k} \neq \mathbf{s}$,

(A2) $\mathbf{k}xy = x$,

(A3) $\mathbf{s}xy{\downarrow} \wedge \mathbf{s}xyz \simeq (xz)(yz)$.

Thus the universe is a partial combinatory algebra. We have λ-abstraction and thus can introduce for each $\mathcal{L}^\circ$ term t a term $(\lambda x.t)$ whose free variables are those of t other than x such that

$$(\lambda x.t){\downarrow} \;\wedge\; (\lambda x.t)y \simeq t[y/x].$$

As usual, we can generalize λ-abstraction to several arguments by simply iterating abstraction for one argument. Accordingly, we set for all $\mathcal{L}^\circ$ terms t and all variables $x_1, \ldots, x_n$,

$$(\lambda x_1 \ldots x_n.t) \;:=\; (\lambda x_1.(\ldots(\lambda x_n.t)\ldots)).$$

Often the term $(\lambda x_1 \ldots x_n.t)$ is simply written as $\lambda x_1 \ldots x_n.t$. If $\vec{x}$ is the sequence $x_1, \ldots, x_n$, then $(\lambda \vec{x}.t)$ and $\lambda \vec{x}.t$ stand for $\lambda x_1 \ldots x_n.t$.

II. Basic set-theoretic axioms. They comprise: (i) pair and union; (ii) assertions which give the appropriate meaning to the constants for the finite von Neumann ordinals and the constant ω; (iii) $\in$-induction for arbitrary formulas $A[u]$ of $\mathcal{L}^\circ$,

$$(\mathcal{L}^\circ\text{-}\mathsf{I}_\in) \qquad \forall x((\forall y \in x)A[y] \to A[x]) \to \forall x A[x].$$

To increase readability, we will freely use standard set-theoretic terminology. For example, $\{a, b\}$ stands for the unordered pair and $\langle a, b\rangle$ for the ordered pair of the sets a, b; in addition,

$$\mathit{Tran}[a] := (\forall x \in a)(x \subseteq a) \quad \text{and} \quad \mathit{Ord}[a] := \mathit{Tran}[a] \wedge (\forall x \in a)\,\mathit{Tran}[x].$$

Also, if $A[x]$ is an $\mathcal{L}^\circ$ formula, then $\{x : A[x]\}$ denotes the collection of all sets satisfying A; it may be (extensionally equal to) a set, but this is not necessarily the case. In particular, we set

$$\mathbf{V} := \{x : x{\downarrow}\} \quad \text{and} \quad \mathbf{B} := \{x : x = \top \vee x = \bot\}$$

so that $\mathbf{V}$ denotes the collection of all sets, but is not a set itself, and $\mathbf{B}$ stands for the unordered pair consisting of the truth values $\top$ and $\bot$, which is a set by the previous axioms. The following shorthand notation, for n an arbitrary natural number greater 0,

$$(f : a^n \to b) := (\forall x_1, \ldots, x_n \in a)(f(x_1, \ldots, x_n) \in b)$$

expresses that f, in the operational sense, is an n-ary mapping from a to b. It does not say, however, that f is an n-ary function in the set-theoretic sense. In this definition the set variables a and b may be replaced by $\mathbf{V}$ and $\mathbf{B}$. So, for example, $(f : a \to \mathbf{V})$ means that f is total on a, and $(f : \mathbf{V} \to b)$ means that f maps all sets into b.

III. Logical operations axioms.

(L1) $\top \neq \bot$,

(L2) $(\mathbf{el} : \mathbf{V}^2 \to \mathbf{B}) \wedge \forall x \forall y(\mathbf{el}(x, y) = \top \leftrightarrow x \in y)$,

(L3) $(\mathbf{non} : \mathbf{B} \to \mathbf{B}) \wedge (\forall x \in \mathbf{B})(\mathbf{non}(x) = \top \leftrightarrow x = \bot)$,

(L4) $(\mathbf{dis} : \mathbf{B}^2 \to \mathbf{B}) \wedge (\forall x, y \in \mathbf{B})(\mathbf{dis}(x, y) = \top \leftrightarrow (x = \top \vee y = \top))$,

(L5) $(f : a \to \mathbf{B}) \to (\mathbf{e}(f, a) \in \mathbf{B} \wedge (\mathbf{e}(f, a) = \top \leftrightarrow (\exists x \in a)(fx = \top)))$.

The Δ_0 *formulas* of $\mathcal{L}^\circ$ are those $\mathcal{L}^\circ$ formulas which do not contain the function symbol $\circ$, the relation symbol $\downarrow$ or unbounded quantifiers. Hence they are the Δ_0 formulas of traditional set theory, possibly containing additional constants. The logical operations make it possible to represent all Δ_0 formulas by constant $\mathcal{L}^\circ$ terms.

Lemma 1 *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every Δ_0 formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term t_A such that the axioms introduced so far yield*

$$t_A{\downarrow} \wedge (t_A : \mathbf{V}^n \to \mathbf{B}) \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

For a proof of this lemma see Feferman [4, 5]. Now we turn to the operational versions of separation, replacement, and choice.

IV. Set-theoretic operations axioms.

(S1) Separation for definite operations:

$$(f : a \to \mathbf{B}) \to (\mathbb{S}(f,a){\downarrow} \wedge \forall x(x \in \mathbb{S}(f,a) \leftrightarrow (x \in a \wedge fx = \top))).$$

(S2) Replacement:

$$(f : a \to \mathbf{V}) \to (\mathbb{R}(f,a){\downarrow} \wedge \forall x(x \in \mathbb{R}(f,a) \leftrightarrow (\exists y \in a)(x = fy))).$$

(S3) Choice:

$$\exists x(fx = \top) \to (\mathbb{C}f{\downarrow} \wedge f(\mathbb{C}f) = \top).$$

This finishes our description of the system OST. As is known from Feferman [4, 5] and Jäger [8], OST is proof-theoretically equivalent to Kripke-Platek set theory with infinity (see next section for a short introduction of this system and the exact formulation of this equivalence).

In this paper we are primarily interested in operationally closed sets and the effect of adding axioms about their existence to OST. As we will see this leads to an enormous increase of proof-theoretic strength.

Definition 2 (Operational closure)

1. *A set d is called* operationally closed*, in symbols $Opc[d]$, iff d is transitive, contains the constants* $\mathbf{k}$, $\mathbf{s}$, $\top$, $\bot$, $\mathbf{el}$, $\mathbf{non}$, $\mathbf{dis}$, $\mathbf{e}$, $\mathbb{S}$, $\mathbb{R}$, $\mathbb{C}$, *and* ω *as elements and satisfies*

$$(\forall f, x \in d)(fx{\downarrow} \to fx \in d).$$

2. *The* operational limit axiom *states that every set is element of an operational closed set,*

$$(\mathsf{OLim}) \qquad \forall x \exists y(x \in y \wedge Opc[y]).$$

Since every operationally closed set is transitive and contains ω, it also contains the constants for the finite von Neumann ordinals and thus all constants of OST. A further simple observation tells us that any operationally closed set d contains all terms which are defined and closed as well as all λ terms with all its parameters from d.

Lemma 3 *For every closed $\mathcal{L}^\circ$ term t, the theory* OST *proves:*

1. $Opc[d] \wedge t\downarrow \;\rightarrow\; t \in d.$
2. $Opc[d] \wedge \vec{a} \in d \;\rightarrow\; \lambda z.t(\vec{a}, z) \in d.$

Proof. The first assertion is proved by straightforward induction on the buildup of t. For the second, let t be closed and observe that OST proves

$$\lambda z.t(\vec{a}, z) \;=\; (\lambda \vec{x}.(\lambda z.t(\vec{x}, z)))(\vec{a}).$$

$\lambda \vec{x}.(\lambda z.t(\vec{x}, z))$ is closed since t is closed. However, since all λ terms are defined, $\lambda \vec{x}.(\lambda z.t(\vec{x}, z))$ belongs to d according to the first assertion. Hence several applications of the closure conditions of d establish our claim. $\square$

Making essential use of choice, we now obtain a theorem which may be regarded as a stability assertion for operationally closed sets.

Theorem 4 *For any Δ_0 formula $A[u_1, \ldots, u_n, v]$ of the language $\mathcal{L}$ with at most the variables $u_1, \ldots, u_n, v$ free, the theory* OST *proves*

$$Opc[d] \wedge \vec{a} \in d \wedge \exists x A[\vec{a}, x] \;\rightarrow\; (\exists x \in d) A[\vec{a}, x].$$

Proof. We work in OST and assume that $Opc[d]$, $\vec{a} \in d$ and $\exists x A[\vec{a}, x]$. In view of Lemma 1 we have a closed $\mathcal{L}^\circ$ term t_A such that

$$(*) \qquad (t_A : \mathbf{V}^n \rightarrow \mathbf{B}) \;\wedge\; \forall \vec{u} \forall v (A[\vec{u}, v] \leftrightarrow t_A(\vec{u}, v) = \top).$$

Now set $s := \lambda z.t_A(\vec{a}, z)$ and conclude from the previous lemma, our assumptions, and (*) that

$$s \in d \;\wedge\; \exists x (sx = \top).$$

Hence the axiom (S3) about choice yields $\mathbb{C}s\downarrow$ and $s(\mathbb{C}s) = \top$. Since $s \in d$, the operational closure of d implies $\mathbb{C}s \in d$. Together with (*) we thus have

$$\mathbb{C}s \in d \;\wedge\; A[\vec{a}, \mathbb{C}s],$$

and our theorem is proved. $\square$

In Section 4 we will see that this theorem is the crucial step in dealing with Σ_1 separation of ordinary set theory in the context of operational set theory with operationally closed sets.

3 Σ_1 separation and stability

This section begins with briefly recalling the system KP of Kripke-Platek set theory with infinity, the schema of Σ_1 separation, and the notion of a stable ordinal. We then turn to important relationships between Σ_1 separation and stability in order to prepare the ground for establishing proof-theoretic equivalences in the final section of this article. For further reading about KP, its proof-theoretic analysis and some interesting subsystems and extensions consult, for example, Jäger [6, 7] and Rathjen [14].

KP is formulated in our basic language $\mathcal{L}$ with $\in$ as its only relation symbol and equality of sets simply defined by

$$(a = b) \;:=\; (\forall x \in a)(x \in b) \,\wedge\, (\forall x \in b)(x \in a).$$

The collections of Δ_0, Σ_1, Σ, and Π formulas are introduced as usual. If T is a theory in $\mathcal{L}$ containing KP and A a formula of $\mathcal{L}$, then A is Δ over T if there exist a Σ formula B and a Π formula C, both with the same free variables as A, such that T proves the equivalence of A and B plus that of A and C. Also, as in the case of OST, we make use of other standard set-theoretic terminology.

The underlying logic of KP is classical first order logic with equality, its non-logical axioms are: (i) pair, union, (ii) assertions which give the appropriate meaning to the constants for the finite von Neumann ordinals and the set ω, (iii) Δ_0 separation, and Δ_0 collection, i.e.

$$(\Delta_0\text{-}\mathsf{Sep}) \qquad \exists x(x = \{y \in a : B[y]\}),$$

$$(\Delta_0\text{-}\mathsf{Col}) \qquad (\forall x \in a)\exists y C[x,y] \;\rightarrow\; \exists z(\forall x \in a)(\exists y \in z)C[x,y]$$

for arbitrary Δ_0 formulas $B[u]$ and $C[u,v]$ of $\mathcal{L}$, as well as (iv) $\in$-induction for arbitrary formulas $A[u]$ of $\mathcal{L}$,

$$(\mathcal{L}\text{-}\mathsf{I}_\in) \qquad \forall x((\forall y \in x)A[y] \rightarrow A[x]) \;\rightarrow\; \forall x A[x].$$

Clearly, the formula $Ord[a]$, which says that a is an ordinal, is a Δ_0 formula of $\mathcal{L}$, and we use the lower case Greek letters $\alpha, \beta, \gamma, \kappa, \lambda, \zeta, \eta, \xi$ (possibly with subscripts) to range over the ordinals, as we do in OST. In the following we will often be working within the constructible universe, but cannot introduce it here. Most relevant details about constructible sets can be found, for example, in Barwise [1] and Kunen [13].

Very briefly, $(a \in L_\alpha)$ states that the set a is an element of the αth level L_α of the constructible hierarchy, and $a \in \mathbf{L}$ is short for $\exists\alpha(a \in L_\alpha)$. The *axiom of constructibility* is the statement $(\mathbf{V}{=}\mathbf{L})$, i.e. $\forall x \exists \alpha(x \in L_\alpha)$, and we write $\mathsf{KP}\mathbf{L}$ for the theory $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$. It is well-known that the assertions $(a \in L_\alpha)$ and $(a <_\mathbf{L} b)$ are Δ over KP and that the systems KP and $\mathsf{KP}\mathbf{L}$ are of the same consistency strength; both systems prove the same absolute sentences.

Now we can state the exact relationship between the theories OST, KP, and $\mathsf{KP}\mathbf{L}$. The following theorem is proved in Feferman [4, 5], Jäger [8], and Jäger and Zumbrunnen [12].

Theorem 5 *The theories* OST, KP, *and* $\mathsf{KP}\mathbf{L}$ *are of the same consistency strength; in particular, we have:*

1. $\mathsf{KP} \subseteq \mathsf{OST}$.

2. OST *is interpretable in* $\mathsf{KP}\mathbf{L}$.

Adding forms of Σ_1 separation to KP provides an enormous increase of consistency strength. Σ_1 separation is the comprehension principle

$$(\Sigma_1\text{-}\mathsf{Sep}) \qquad \forall x \exists y(y = \{z \in x : A[z]\})$$

for $A[u]$ a Σ_1 formula of $\mathcal{L}$. We will also be interested in parameter-free Σ_1 separation on ω,

$$(\Sigma_1\text{-}\mathsf{Sep})^-_\omega \qquad \exists y(y = \{z \in \omega : B[z]\}),$$

where $B[u]$ is a Σ_1 formula of $\mathcal{L}$ with u as its only free variable. It is well-known, see, for example, Rathjen [15], that $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ proves the same sentences of second order arithmetic as the system $(\Pi^1_2\text{-}\mathsf{CA}) + (\mathsf{BI})$.

Definition 6 (Stability)

1. *Let d be a set with $\omega \cup \{\omega\} \subseteq d \in \mathbf{L}$. Then we say that $\langle d, \in \cap\, d^2\rangle$ is a* Σ_1-elementary substructure of $\mathbf{L}$, *in symbols $\langle d, \in \cap\, d^2\rangle \prec_1 \mathbf{L}$, iff for all Σ_1 formulas $A[\vec{u}]$ of $\mathcal{L}$ and all $\vec{a} \in d$,*

$$\langle d, \in \cap\, d^2\rangle \models A[\vec{a}] \quad \Longleftrightarrow \quad \mathbf{L} \models A[\vec{a}].$$

2. *If d is a transitive set with $\omega \in d$, we often simply write $d \prec_1 \mathbf{L}$ instead of $\langle d, \in \cap\, d^2\rangle \prec_1 \mathbf{L}$.*

3. *An ordinal σ is said to be* stable *iff $L_\sigma \prec_1 \mathbf{L}$.*

In order to formalize the notion of stability within KP, we follow Barwise [1] and let Sat_n be a Σ_1 formula of $\mathcal{L}$ with free variables $u, v_1, \ldots, v_n$ such that for any Σ formula $A[w_1, \ldots, w_n]$ with at most the variables $w_1, \ldots, w_n$ free the following is a theorem of KP:

$$\forall x_1 \ldots \forall x_n (A[x_1, \ldots, x_n] \leftrightarrow Sat_n[e, x_1, \ldots, x_n]),$$

where e is the Gödel number of the formula $A[w_1, \ldots, w_n]$. We can also assume that the Gödel numbering of the $\mathcal{L}$ formulas is so that for any natural number n there exists a Δ_0 definable set $DF_n \subseteq \omega$ whose elements are the Gödel numbers of the Δ_0 formulas of $\mathcal{L}$ with n free variables.

Definition 7 *For any ordinal σ, we set*

$$Stab[\sigma] := (\forall e \in DF_2)(\forall a \in L_\sigma)(\exists x Sat_2[e, x, a] \to (\exists x \in L_\sigma) Sat_2[e, x, a]).$$

Clearly, this is the formalized version of stability; the restriction to one parameter only is not significant. Since our language $\mathcal{L}$ contains the constant ω and constants for all finite von Neumann ordinals we have $\omega < \sigma$ for any stable ordinal σ.

It is easy to show that all instances of $(\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ and $(\Sigma_1\text{-}\mathsf{Sep})$ can be derived in $\mathsf{KP}\mathbf{L} + \exists\sigma Stab[\sigma]$ and $\mathsf{KP}\mathbf{L} + \forall\alpha\exists\sigma(\alpha < \sigma \wedge Stab[\sigma])$, respectively. However, we omit proving these two results here since they are immediate consequences of Theorem 9 and Theorem 10 below. Instead, we turn to the converse directions.

The proof of the following theorem relies very much on the treatment of stability in Barwise [1]. Its second part also follows from results mentioned in Rathjen [15].

Theorem 8

1. $\mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega \vdash \exists\xi Stab[\xi]$.
2. $\mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep}) \vdash \forall\eta\exists\xi(\eta < \xi \wedge Stab[\xi])$.

PROOF. To show the first assertion we work informally within our theory $\mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$. First we use $(\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ to introduce the sets

$$a := \{e \in DF_1 : \exists x Sat_1[e, x]\}$$
$$b := \{e \in DF_1 : \neg\forall x\forall y(Sat_1[e, x] \wedge Sat_1[e, y] \to x = y)\}$$

and then $(\Delta_0\text{-}\mathsf{Sep})$ to form the set $c := a \cap (DF_1 \setminus b)$. Then we have

$$(\forall e \in c)\exists! x Sat_1[e, x].$$

Hence by Σ collection there exists a set d such that for all x,

$$x \in d \leftrightarrow (\exists e \in c)Sat_1[e, x].$$

The next steps are as in Barwise [1], proof of Theorem V.7.8: Following the arguments given there, we first establish $\langle d, \in \cap d^2 \rangle \prec_1 \mathbf{L}$ and then show $d = L_\sigma$, where σ is the least ordinal not in d.

It remains to check that $Stab[\sigma]$. So let $e \in DF_2$ and $a \in L_\sigma$ and assume $\exists x Sat_2[e, x, a]$. Since $\exists x Sat_2[e, x, a]$ is (equivalent to) a Σ_1 formula of $\mathcal{L}$ and $L_\sigma = d \prec_1 \mathbf{L}$, we immediately obtain $(\exists x \in L_\sigma)Sat_2[e, x, a]$, as required.

Turning to the second assertion we work within $\mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep})$ and pick an arbitrary ordinal α. Because of $(\Sigma_1\text{-}\mathsf{Sep})$ we can now build the sets

$$a := \{\langle e, u \rangle \in DF_2 \times L_{\alpha+1} : \exists x Sat_2[e, x, u]\}$$

$$b := \{\langle e, u \rangle \in DF_2 \times L_{\alpha+1} : \neg \forall x \forall y (Sat_2[e, x, u] \wedge Sat_2[e, y, u] \rightarrow x = y)\}$$

and then proceed (more or less) as above. This gives us an L_σ such that $L_\alpha \in L_\sigma$ and $Stab[\sigma]$. $\square$

4 Proof-theoretic equivalences

Now we turn to the proof-theoretic characterizations of $\mathsf{OST} + \exists y\, Opc[y]$ and $\mathsf{OST} + (\mathsf{OLim})$ in terms of extensions of Kripke-Platek set theory by forms of Σ_1 separation and stability assertions.

Theorem 9 (Lower bounds)

1. $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega \subseteq \mathsf{OST} + \exists y\, Opc[y]$.
2. $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep}) \subseteq \mathsf{OST} + (\mathsf{OLim})$.

Proof. According to Theorem 5, OST proves all axioms of KP. To deal with $(\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ in $\mathsf{OST} + \exists y\, Opc[y]$, let d be an operationally closed set and choose a Σ_1 formula $A[u]$ of $\mathcal{L}$ with u as its only free variable. Since $\omega \in d$, Theorem 4 implies

$$(\forall x \in \omega)(A[x] \leftrightarrow A^d[x]). \tag{1}$$

By $(\Delta_0\text{-}\mathsf{Sep})$ there exists the set $a := \{x \in \omega : A^d[x]\}$, and according to (1) we have $a = \{x \in \omega : A[x]\}$, taking care of this instance of $(\Sigma_1\text{-}\mathsf{Sep})^-_\omega$.

Next we have to take care of $(\Sigma_1\text{-}\mathsf{Sep})$ in $\mathsf{OST} + (\mathsf{OLim})$. So let $A[u, \vec{v}]$ be a Σ_1 formula with $u, \vec{v}$ as its only free variables and select arbitrary sets $\vec{a}, b$.

Because of (OLim) there exists an operationally closed set d with $\vec{a}, b \in d$. Now we apply Theorem 4 again and obtain

$$(\forall x \in b)(A[x, \vec{a}] \leftrightarrow A^d[x, \vec{a}]). \tag{2}$$

As before, (Δ_0-Sep) provides for the set $c := \{x \in b : A^d[x, \vec{a}]\}$, and by (2) we thus have $c = \{x \in b : A[x, \vec{a}]\}$, finishing our proof. $\square$

For interpreting $\mathsf{OST} + \exists y \, Opc[y]$ and $\mathsf{OST} + (\mathsf{OLim})$ in extensions of Kripke-Platek set theory we make use of the inductive model construction presented in Jäger and Zumbrunnen [12]. For doing this, we may assume that the constants $\mathbf{k}$, $\mathbf{s}$, $\top$, $\bot$, $\mathbf{el}$, $\mathbf{non}$, $\mathbf{dis}$, $\mathbf{e}$, $\mathbb{S}$, $\mathbb{R}$, and $\mathbb{C}$ of $\mathcal{L}^\circ$ are coded as elements of L_ω.

The decisive point of this interpretation is that there exists a Σ_1 formula $Ap[u, v, w]$ of $\mathcal{L}$ with three free variables u, v, w which takes care of the application of $\mathcal{L}^\circ$ in the sense that the $\mathcal{L}^\circ$ formula $(fx = y)$ translates into $Ap[f, x, y]$ and the $\mathcal{L}^\circ$ formula $(fx\downarrow)$ into $\exists y Ap[f, x, y]$.

Based on this translation of application, a Σ formula $[\![t]\!](u)$ of $\mathcal{L}$ is associated to each term t of $\mathcal{L}^\circ$ expressing that u is the value of t under this interpretation of the operational application. This treatment of terms then leads to a canonical embedding of $\mathcal{L}^\circ$ into $\mathcal{L}$, translating any $\mathcal{L}^\circ$ formula A into an $\mathcal{L}$ formula A^*. See Jäger and Zumbrunnen [12] for all details.

Theorem 10 (Upper bounds) *For all formulas A of $\mathcal{L}^\circ$ we have:*

1. $\mathsf{OST} + \exists y \, Opc[y] \vdash A \quad \Longrightarrow \quad \mathsf{KP}\mathbf{L} + \exists \xi \, Stab[\xi] \vdash A^*$.
2. $\mathsf{OST} + (\mathsf{OLim}) \vdash A \quad \Longrightarrow \quad \mathsf{KP}\mathbf{L} + \forall \eta \exists \xi (\eta < \xi \wedge Stab[\xi]) \vdash A^*$.

Proof. From Jäger and Zumbrunnen [12] we know that $\mathsf{KP}\mathbf{L}$ proves A^* for every axiom A of OST. In order to validate (the translation of) $\exists y \, Opc[y]$ within $\mathsf{KP}\mathbf{L} + \exists \xi Stab[\xi]$, let σ be a stable ordinal. Then L_σ contains the interpretations of all constants of $\mathcal{L}^\circ$, and for all $f, x \in L_\sigma$ we have

$$\exists y Ap[f, x, y] \;\rightarrow\; (\exists y \in L_\sigma) Ap[f, x, y].$$

Hence if f and x are elements of L_σ and if – modulo our interpretation – fx has a value at all, this value belongs to L_σ. This implies that L_σ can act as a witness for an operationally closed set.

To handle (OLim) in the second embedding assertion, pick an arbitrary set a and choose an α such that $a \in L_\alpha$. Then we make use of the assumption $\forall \eta \exists \xi (\eta < \xi \wedge Stab[\xi])$ and know that there exists a stable ordinal σ greater than α. As above, it is easily verified that this L_σ is a possible witness for an operationally closed set containing a. $\square$

The following corollary is an immediate consequence of the previous embedding result and Theorem 8.

Corollary 11 *For all formulas A of $\mathcal{L}^\circ$ we have:*

1. $\mathsf{OST} + \exists y\, Opc[y] \vdash A \quad \Longrightarrow \quad \mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega \vdash A^*$.
2. $\mathsf{OST} + (\mathsf{OLim}) \vdash A \quad \Longrightarrow \quad \mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep}) \vdash A^*$.

To conclude our proof-theoretic considerations we remark that adding the axiom $(\mathbf{V}{=}\mathbf{L})$ to the theories $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ and $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ does not increase their respective proof-theoretic strengths. For completeness, we state the corresponding lemma. Given an $\mathcal{L}$ formula A, we write $A^{\mathbf{L}}$ for the $\mathcal{L}$ formula obtained from A by restricting all unrestricted quantifiers in A to $\mathbf{L}$.

Lemma 12 *Let T be one of the theories $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ or $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$. Then we have for the universal closure A of any axiom of T that $T \vdash A^{\mathbf{L}}$.*

Proof. From Barwise [1], Theorem II.5.5 we know that our assertion is true for the universal closures of all axioms of KP. Now suppose that $A[u]$ is a Σ_1 formula of $\mathcal{L}$ with u as its only free variable. Then, working within $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$, the schema $(\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ yields that

$$a \;:=\; \{x \in \omega : A^{\mathbf{L}}[x]\}$$

is a set. Consequently, we have

$$(\forall x \in a)\exists\xi A^{L_\xi}[x],$$

and $(\Delta_0\text{-}\mathsf{Col})$ implies that there is an ordinal α such that

$$(\forall x \in a)(\exists\xi < \alpha)A^{L_\xi}[x].$$

It follows immediately that $a = \{x \in \omega : A^{L_\alpha}[x]\}$ and therefore an element of $\mathbf{L}$. So we find

$$(\exists y(y = \{x \in \omega : A[x]\}))^{\mathbf{L}}.$$

Therefore, the $\mathbf{L}$-interpretations of the instances of $(\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ are provable in $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$.

The $\mathbf{L}$-interpretations of all instances of $(\Sigma_1\text{-}\mathsf{Sep})$ are treated within the theory $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ analogously. $\Box$

In view of this lemma it is clear that $\mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ is conservative over $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$ and $\mathsf{KP}\mathbf{L} + (\Sigma_1\text{-}\mathsf{Sep})$ conservative over $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$, in both cases for absolute formulas. Combining all our results we thus obtain the following final result.

Theorem 13 *We have the following proof-theoretic equivalences and equiconsistency results:*

1. $\mathsf{OST} + \exists y\, \mathit{Opc}[y] \;\equiv\; \mathsf{KP}\mathbf{L} + \exists \xi\, \mathit{Stab}[\xi] \;\equiv\; \mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})^-_\omega$.

2. $\mathsf{OST} + (\mathsf{OLim}) \;\equiv\; \mathsf{KP}\mathbf{L} + \forall \eta \exists \xi(\eta < \xi \,\wedge\, \mathit{Stab}[\xi]) \;\equiv\; \mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$.

This theorem tells us that our notion of operational closure is a proof-theoretically extremely powerful. For a future publication it is planned to study operationally closed sets and models of operational set theory from a different perspective and to look at several variants.

Let us end this paper with remarking that a uniform version of operational closure leads to inconsistency. Assume that the language $\mathcal{L}^\circ$ comprises a further constant $\mathcal{O}$ with the axiom

(Uniform OLim) $$\forall x(x \in \mathcal{O}x \,\wedge\, \mathit{Opc}[\mathcal{O}x]).$$

Then we have $\mathcal{O} \in \mathcal{O}\mathcal{O}$, and strictness implies that $\mathcal{O}\mathcal{O}{\downarrow}$. Since $\mathcal{O}\mathcal{O}$ is operationally closed, we then have $\mathcal{O}\mathcal{O} \in \mathcal{O}\mathcal{O}$, contradicting the well-foundedness of the $\in$-relation.

References

[1] K. J. Barwise, *Admissible Sets and Structures*, Perspectives in Mathematical Logic, vol. 7, Springer, 1975.

[2] M. J. Beeson, *Foundations of Constructive Mathematics: Metamathematical Studies*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3/6, Springer, 1985.

[3] ———, *Proving programs and programming proofs*, Logic, Methodology, and Philosophy of Science VII (R. Barcan Marcus, G. J. W. Dorn, and P. Weingartner, eds.), Studies in Logic and the Foundations of Mathematics, vol. 114, North-Holland, 1986, pp. 51–82.

[4] S. Feferman, *Notes on Operational Set Theory, I. Generalization of "small" large cardinals in classical and admissible set theory*, Technical Notes, 2001, available online at `http://math.stanford.edu/~feferman/papers/OperationalST-I.pdf`".

[5] ———, *Operational set theory and small large cardinals*, Information and Computation **207** (2009), 971–979.

[6] G. Jäger, *Zur Beweistheorie der Kripke-Platek-Mengenlehre über den natürlichen Zahlen*, Archiv für Mathematische Logik und Grundlagenforschung **22** (1982), 121–139.

[7] ______, *Theories for Admissible Sets: A Unifying Approach to Proof Theory*, Studies in Proof Theory, Lecture Notes, vol. 2, Bibliopolis, 1986.

[8] ______, *On Feferman's operational set theory* OST, Annals of Pure and Applied Logic **150** (2007), no. 1–3, 19–39.

[9] ______, *Full operational set theory with unbounded existential quantification and power set*, Annals of Pure and Applied Logic **160** (2009), no. 1, 33–52.

[10] ______, *Operations, sets and classes*, Logic, Methodology and Philosophy of Science - Proceedings of the Thirteenth International Congress (C. Glymour, W. Wei, and D. Westerståhl, eds.), College Publications, 2009, pp. 74–96.

[11] G. Jäger and J. Krähenbühl, Σ^1_1 *Choice in a Theory of Sets and Classes*, Ways of Proof Theory (R. Schindler, ed.), Ontos Verlag, 2010, pp. 283–314.

[12] G. Jäger and R. Zumbrunnen, *About the strength of operational regularity*, Technical Notes, 2011.

[13] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, 1980.

[14] M. Rathjen, *Fragments of Kripke-Platek set theory*, Proof Theory (P. Aczel, H. Simmons, and S. Wainer, eds.), Cambridge University Press, 1992, pp. 251–273.

[15] ______, *Explicit mathematics with the monotone fixed point principle II: models*, The Journal of Symbolic Logic **64** (1999), no. 2, 517–550.

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