Contributions to Operational Set Theory

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Contents

Introduction

Solomon Feferman has introduced the system OST of operational set theory and some extensions in [13] and [14]. As the name implies, operational set theories are theories about sets and operations. We use the notion of sets in its traditional set-theoretic meaning. I.e. sets are collections of distinct objects, but not every collection corresponds to a set. Operations, not to be confused with set-theoretic functions, are objects which may map objects to other objects. One important difference between operations and set-theoretic functions is, that operations may have the whole universe as their domain. In a model of an operational set theory, every object is both at the same time, a set and an operation.

Another framework of applicative theories is explicit mathematics. It has been introduced in Feferman [10] as a framework for Bishop style constructive mathematics. The first sort of objects are operations. And operations may in addition be names of types, the objects of the second sort. Types are collections of objects of the first sort (but not all collections form a type). Explicit mathematics and operational set theories have the same so-called applicative axioms. Furthermore they have in common that the existence of some types or sets, respectively, can be proved via some axioms about operations for constructing names of types or sets, respectively.

Different aspects of different variants of operational set theory has been discussed in Cantini [5], Cantini and Crosilla [6, 7, 8], Feferman [13, 14], Jäger [18, 19, 20, 21] as well as Jäger and Zumbrunnen [26]. Furthermore Beeson has presented a system of operational set theory in [4] (operations are called rules there).

The program of operational set theory is motivated and described in Feferman [14] as follows: "A new axiomatic system OST of operational set theory is introduced in which the usual language of set theory is expanded to allow us to talk about (possibly partial) operations applicable both to sets and to operations. OST is equivalent in strength to admissible set theory, and a natural extension of OST is equivalent in strength to ZFC. The language of OST provides a framework in which to express 'small' large cardinal notions — such as those of being an inaccessible cardinal, a Mahlo cardinal, and a weakly compact cardinal — in terms of operational closure conditions that specialize to the analogue notions on admissible sets. This illustrates a wider program whose aim is to provide a common framework for analogues of large cardinal notions that have appeared in admissible set theory, admissible recursion theory, constructive set theory, constructive type theory, explicit mathematics, and systems of recursive ordinal notations that have been used in proof theory."

The theory OST of operational set theory is proof-theoretically equivalent to the theory KP of Kripke-Platek set theory with infinity (c.f. Feferman [13] and [14] and Jäger [18]). There exists also a natural system of operational set theory (a subsystem of OST with in addition axioms for operations representing unbounded existential quantification and creating power sets) which is conservative over Zermelo–Fraenkel set theory with the axiom of choice ZFC for absolute formulas (c.f. Jäger [18]). Full operational set theory, the extension of OST with axioms for operations representing unbounded existential quantification and creating power sets, has the same strength as NBG plus $\in$-induction for arbitrary formulas and a class version of Σ^1_1 choice (c.f. Jäger [19] and Jäger and Krähenbühl [23]). Furthermore a natural extension of OST with the same strength as KP plus Σ_1 separation is presented in Jäger [21].

The main aim of this thesis is to present systems of operational set theory and to determine their proof-theoretic strengths.

In Jäger and Zumbrunnen the notion of operational regularity, a form of regularity with respect to operations, is introduced. Feferman introduced in [14] the axiom (Inac) which states that for any ordinal there is a larger operationally regular ordinal. One main question in this thesis is: how strong is the theory OST augmented with the axiom (Inac)? We will answer this question and show that the system OST + (Inac) is distinctly stronger than expected. The same answer to this question is also presented in Jäger and Zumbrunnen [26].

One axiom of the theory OST ensures the existence of the choice operation $\mathbb{C}$. This axiom claims that whenever there is an object x for some operation f such that $f(x)$ is the boolean value t, then $\mathbb{C}(f)$ is an instance of such an object. A second main question in this thesis is: what happens if we drop that axiom about the choice operation $\mathbb{C}$ from OST? We will proof, the same way as done in Sato and Zumbrunnen [31], that we get a system of the same strength as OST if we do so. For that purpose we will firstly use a bisimulation method and a forcing method as presented in Avigad [1] in order to embed the classical set theory KP into an intuitionistic variant IKP^- of the latter. Secondly we will combine these methods with a well-known realisation method for embedding intuitionistic theories into applicative theories. Furthermore, we will see that these methods can also be used to prove analogous results about some extensions of OST without the axiom about the choice operation.

An additional small task of this thesis is to discuss difficulties if we want to introduce concepts of operational set theory in explicit mathematics and vice versa. One of the discussed concepts is the existence of a choice operation as well as a global choice operation in operational set theory. Another one is the concept of an inverse image operation as we have it in explicit mathematics.

In the first chapter we will introduce all systems of set theory, operational set theory and explicit mathematics used in this thesis and present some basic properties of them which we will use later. In the second chapter we will present different interpretations of pure set theories in other pure set theories. We will use many theories discussed in this chapter only as intermediate theories, i.e. we are not interested in them directly. We will embed some pure set theories in operational set theories in the third chapter. In section 3.3 we will use the results of the sections 2.2, 2.3 and 2.4. In the fourth chapter we will treat the converse direction. I.e., we will see how our systems of operational set theory can be interpreted in the corresponding pure set theories. Finally, in the fifth chapter, we will turn our attention to the described concepts of operational set theory in explicit mathematics and vice versa.

1 Definition and Properties of the used Theories

1.1 Pure Set Theories

In this section we introduce different set theories in the traditional sense, i.e. theories formulated in languages in which we can only speak about sets (and not about operations). In the first subsection we introduce (first order) languages of set theory and define plenty of abbreviations which we will use later. In the second subsection we define all the axioms and theories of pure set theories used in this thesis, and in the third one we present and proof some basic properties of them. We will use a lot of notations, phrases and proofs of the papers Jäger and Zumbrunnen [26] and Sato and Zumbrunnen [31].

1.1.1 Languages of Set Theory

We denote by $\mathcal{L}$ the relational extension of the language of set theory. $\mathcal{L}$ has one sort of variables, we use the letters a, b, c, d, f, g, h, m, n, p, q, u, v, w, x, y, $z, ...$ (possibly with subscripts) to denote them. $\mathcal{L}$ is a language without constants and function symbols. It contains the binary relation symbols $=$ and $\in$, the 0-ary relation symbol $\perp$ and for each natural number i countably infinitely many relation symbols R^i_j of arity i.

The *formulas* of $\mathcal{L}$ are built up as usual using the connectives $\wedge$, $\vee$ and $\rightarrow$ and the quantifiers $\forall$ and $\exists$. We will also work with intuitionistic set theories. Therefore we define negation as

$$\neg A \;:=\; A \rightarrow \perp,$$

for any formula A. Furthermore we define

$$A \leftrightarrow B \;:=\; (A \rightarrow B) \wedge (B \rightarrow A).$$

We appoint that negation takes precedence over $\wedge$, $\vee$, and $\rightarrow$ and that $\wedge$ and $\vee$ take precedence over $\rightarrow$ and $\leftrightarrow$.

Also the bounded quantifiers $(\forall x \in a)$ and $(\exists x \in a)$ are abbreviations and are defined as usual. For any formula A we write A^a for the formula which we get if we replace all unbounded quantifiers Qv by unbounded quantifiers of the form $(Qv \in a)$, where Q is any of the two quantifiers.

Formulas without free variables are called *sentences*.

We will often write $Qx_0, ..., x_n(...)$ for $Qx_0Qx_1...Qx_n(...)$ and

$$(Qx_0, ..., x_n \in a)(...) \text{ for } (Qx_0 \in a)...(Qx_n \in a)(...),$$

where Q is any of the quantifiers. And if $\vec{x}$ is the finite sequence of variables $x_0, ..., x_n$, we will often write $Q\vec{x}(...)$ and $(Q\vec{x} \in a)(...)$ for $Qx_0, ..., x_n(...)$ and $(Qx_0, ..., x_n \in a)(...)$, respectively.

If $\vec{u} = u_1, ..., u_n$ and $\vec{t} = t_1, ..., t_n$ are finite sequences of variables and terms (we will use the notation introduced here also in other languages in which not all terms are variables), respectively, we write $A[\vec{t}/\vec{u}]$ for the formula which is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the terms $\vec{t}$ (in order to avoid collision of variables, a renaming of bound variables may be necessary). If A is written as $A[\vec{u}]$ we just write $A[\vec{t}]$ for $A[\vec{t}/\vec{u}]$.

We will use the abbreviation $\vec{x} \in \vec{y}$ for the formula

$$x_0 \in y_0 \wedge ... \wedge x_n \in y_n,$$

if $\vec{x}$ and $\vec{y}$ are finite sequences of variables of the form $x_0, ..., x_n$ and $y_0, ..., y_n$, respectively. We will write $\vec{x} \in y$ for $\vec{x} \in \vec{y}$ if $\vec{y}$ is the sequence $y, ..., y$.

As usual we use the symbol ! after an existential quantifier for expressing uniqueness. I.e., if A is a formula, $\exists! x A[x]$ abbreviates the formula

$$\exists x A[x] \wedge \forall x, y(A[x] \wedge A[y] \rightarrow x = y).$$

In the following we introduce as usual some classes of formulas.

Definition 1.1 (Δ_0 formulas)**.** We call formulas of $\mathcal{L}$ without any occurrences of unbounded quantifiers Δ_0 *formulas* of $\mathcal{L}$.

Definition 1.2 (Σ and Π formulas)**.** The class of Σ *formulas* of $\mathcal{L}$ is the smallest class containing the Δ_0 formulas of $\mathcal{L}$ and that is closed under conjunction, disjunction, bounded quantification and unbounded existential quantification. The class of Π *formulas* of $\mathcal{L}$ is the smallest class containing the Δ_0 formulas of $\mathcal{L}$ and that is closed under conjunction, disjunction, bounded quantification and unbounded universal quantification.

Definition 1.3 (Σ_n and Π_n formulas)**.** The Σ_n *formulas* and Π_n *formulas* of $\mathcal{L}$ are inductively defined for all natural numbers n as follows:

(i) Every Δ_0 formula of $\mathcal{L}$ is a Σ_n formula as well as a Π_n formula of $\mathcal{L}$ for every natural number n.

(ii) If A and B are Σ_n (Π_n) formulas of $\mathcal{L}$, then so are $A \vee B$ and $A \wedge B$.

(iii) If A is a Σ_n (Π_n) formula and B is a Π_n (Σ_n) formula of $\mathcal{L}$, then $A \to B$ is a Π_n (Σ_n) formula of $\mathcal{L}$.

(iv) If A is a Σ_n formula and B is a Π_n formula of $\mathcal{L}$, then $\forall x A$ is a Π_{n+1} formula and $\exists x B$ is a Σ_{n+1} formula of $\mathcal{L}$.

(v) If A is a Σ_n formula, B is a Π_n formula of $\mathcal{L}$ and $n > 0$, then $\exists x A$ is a Σ_n formula and $\forall x B$ is a Π_n formula of $\mathcal{L}$.

Definition 1.4 (Δ and Δ_n formulas)**.** A formula A of $\mathcal{L}$ is called Δ *formula (Δ_n formula)* of $\mathcal{L}$ with respect to a theory $\mathcal{T}$, if there is a Π (Π_n) formula B as well as a Σ (Σ_n) formula C of $\mathcal{L}$, so that $\mathcal{T}$ proves $A \leftrightarrow B$ as well as $A \leftrightarrow C$.

Formulas without free variables which are Δ w.r.t. KP (c.f. Definition 1.8) are also called *absolute sentences*.

When we will work with intuitionistic logic we will use the notions of negative and strongly negative formulas as well as the notion of double-negation interpretation of a formula.

Definition 1.5 (Negative and strongly negative formulas)**.** We call $\mathcal{L}$ formulas *negative* if they are built up from atomic formulas by means of the connectives $\land$ and $\to$ and the quantifier $\forall$. The *strongly negative* formulas of $\mathcal{L}$ are inductively defined as follows:

(i) $\bot$ is a strongly negative formula.

(ii) If A is an atomic formula and B a strongly negative formula of $\mathcal{L}$, then also $A \to B$ is strongly negative.

(iii) If the $\mathcal{L}$ formulas A and B are strongly negative, then so are $A \to B$, $A \land B$ as well as $\forall x A$.

For instance, $x \in y$ is due to this definition negative but not strongly negative.

Definition 1.6 (Double-negation interpretation)**.** The *double-negation interpretation* A^N of each $\mathcal{L}$ formula A is inductively defined as follows:

(i) If A is an atomic formula, then A^N is the formula $\neg\neg A$.

(ii) If A is the formula $B \land C$, then A^N is the formula $B^N \land C^N$.

(iii) If A is the formula $B \lor C$, then A^N is the formula $\neg(\neg B^N \land \neg C^N)$.

(iv) If A is the formula $B \to C$, then A^N is the formula $B^N \to C^N$.

(v) If A is the formula $\exists x B$, then A^N is the formula $\neg\forall x \neg B^N$.

(vi) If A is the formula $\forall x B$, then A^N is the formula $\forall x B^N$.

Notice that we have for all formulas A: A^N is a strongly negative formula which is classically equivalent to A. It is well known, that if A is classically valid, then A^N is intuitionistically valid (c.f. Theorem 5.2.8 in van Dalen [33]).

We will often use standard set-theoretic notations. That is, different assertions as $x = \{x_0, ..., x_n\}$, $x = \langle y, z\rangle x \subseteq y$, $x = \cup y$, $x = y \cup z$, $x = y \cap z$, $x = y \times z$ and so on are abbreviations of Δ_0 formulas of $\mathcal{L}$ and are defined as usual (see for instance Chapter I in Part A of Barwise [2]); and we will often write $\emptyset$ for the empty set. Notice that we consider *ordered pairs* as Kuratowski pairs, i.e. $\langle y, z\rangle$ corresponds to the set $\{\{y\}, \{y, z\}\}$ and *ordered n-tuples* for $n > 2$ are inductively defined as

$$\langle x_0, ..., x_{n-1}\rangle \ := \ \langle x_0, \langle x_1, ..., x_{n-1}\rangle\rangle.$$

By $x = \{x_0, ..., x_n\}$ we denote for instance the Δ_0 formula

$$x_0 \in x \wedge ... \wedge x_n \in x \wedge (\forall v \in x)(v = x_0 \vee ... \vee v = x_n).$$

We will write $x = \{y, z\}$ and $x = \cup y$ and so on also if we work with theories without extensionality. We do this although the x in this abbreviations might not be unique. I.e. this abbreviations mean "x is some set containing exactly y and z" and "x is some set corresponding to the union of the set y", respectively.

In $\mathcal{L}$ we write $x \neq y$ for the formula $\neg(x = y)$, as well as $x \notin y$ for the formula $\neg(x \in y)$.

We will sometimes use abbreviations of the form $\forall x{\subseteq}a(...)$ and $\exists x{\subseteq}a(...)$ for formulas of the form $\forall x(x \subseteq a \to ...)$ and $\exists x(x \subseteq a \wedge ...)$, respectively.

If $A[x]$ is a formula we denote by $\{x : A[x]\}$ the *collection* of all sets x satisfying $A[x]$, i.e. $y \in \{x : A[x]\}$ means nothing else than $A[y]$. We will write $\{x \in a : A[x]\}$ for the collection $\{x : x \in a \wedge A[x]\}$. Collections may be (extensionally equal to) a set. If it is the case, we will often write $a = \{x : A[x]\}$ for the formula $\forall x(x \in a \leftrightarrow A[x])$.

We also introduce some other standard formulas which we will often use. The formulas $\mathrm{Rel}[f]$ and $\mathrm{Fun}[f]$ are Δ_0 formulas expressing that f is a set-theoretic *relation* (i.e. a set of ordered pairs) and a set-theoretic *function*, respectively. Furthermore $\mathrm{Dom}[f, a]$ and $\mathrm{Ran}[f, a]$ are Δ_0 formulas expressing that the *domain* and *range*, respectively, of f is a. We will also use the notation $\mathrm{Ran}_{\subseteq}[f, a]$ for a Δ_0 formula expressing that the range of f is a subset of a. We will write $f'x = y$ for a Δ_0 formula expressing that the set theoretic function f applied to x is y. Typically the notation $f(x) = y$ is used for this formula. But since we will use this typical notation later in the context of operations, we have to chose another one. The precise definition of these formulas about relations and functions is given for instance in Chapter I in Part A of Barwise [2].

We will write $\mathrm{Prog}[b, a, r]$ for the Δ_0 formula

$$(\forall x \in a)((\forall y \in a)(\langle y, x\rangle \in r \to y \in b) \to x \in b),$$

expressing that b is *progressive* with respect to a and the relation r. By $\mathrm{WF}[a, r]$ we denote the Π_1 formula

$$\forall b \subseteq a(\mathrm{Prog}[b, a, r] \to a \subseteq b),$$

expressing that r is a *well founded* relation on a.

The Δ_0 formula $\mathrm{Tran}[a]$, expressing that a is *transitive*, is the formula

$$(\forall x \in a)(x \subseteq a),$$

and the Δ_0 formula $\mathrm{Ord}[a]$, expressing that a is an *ordinal number*, is the formula

$$\mathrm{Tran}[a] \land (\forall x \in a)\mathrm{Tran}[x].$$

We will use lower case Greek letters α, β, γ, κ, λ, ξ, η, ... (possibly with subscripts) to range over the ordinals. I.e., for instance, formulas of the form $\forall\xi(...)$ are abbreviations of formulas of the form $\forall x(\mathrm{Ord}[x] \to ...)$. We will write ω for the first infinite ordinal. Then we write $\mathrm{Lim}[a]$ for the Δ_0 formula

$$\mathrm{Ord}[a] \land a \neq 0 \land (\forall x \in a)(\exists y \in a)(y = x \cup \{x\}),$$

expressing that a is a *limit ordinal*. In the context of ordinals we often use the symbol $<$ for the symbol $\in$ and we write 0 instead of $\emptyset$.

For each natural number $n > 0$ we write $\mathrm{Tup}_n[a]$ for a Δ_0 formula expressing that a is an ordered n-tuple and $(a)_i = b$, $(a)_i \in b$ ... for Δ_0 formulas expressing that its i-th component is b, is an element of b ... (for the definition of these formulas see Chapter I in Part A of Barwise [2]). Notice that we start numbering with 0. I.e., if for instance $x = \langle y_0, ..., y_{n-1}\rangle$ we have $\mathrm{Tup}_n[x]$ and $(x)_i = y_i$ for all $0 \leq i < n$.

Definition 1.7 (Functional regularity)**.** We call an ordinal κ *functionally regular*, in symbols $\mathrm{Frg}[\kappa]$, if $\omega < \kappa$ and

$$\forall f(\forall\xi < \kappa)(\mathrm{Fun}[f] \land \mathrm{Dom}[f, \xi] \land \mathrm{Ran}_{\subseteq}[f, \kappa] \to (\exists\eta < \kappa)\mathrm{Ran}_{\subseteq}[f, \eta]).$$

We also introduce some sub-languages of $\mathcal{L}$. The language of set theory, $\mathcal{L}$ without the relation symbols R^i_j, is denoted by $\mathcal{L}_\in$. I.e. the only relation symbols of $\mathcal{L}_\in$ are $\perp$ and $\in$. Let us fix two natural numbers j and k and call the binary and unary relation symbols R^2_j and R^2_k also $\mathcal{P}$ and Ad, respectively. Then $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathsf{Ad}}$ are the language $\mathcal{L}$ but restricted to the relation symbols $\in$, $\perp$ as well as $\mathcal{P}$ and Ad, respectively.

1.1.2 Definitions of Set Theories

First of all we introduce some basic set theoretic axioms and axiom schemas. It may seem as some axioms are formulated here a little strange. However, later it will be important that they (or some of them) have exactly the form given here.

Basic axioms of set theories

(extensionality) $a = b \leftrightarrow (\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$,

(pairing) $\exists x(a \in x \wedge b \in x)$,

(union) $\exists x(\forall y \in a)(\forall z \in y)(z \in x)$,

(Δ_0 separation) $\exists x((\forall y \in x)(y \in a \wedge A[y]) \wedge (\forall y \in a)(A[y] \rightarrow y \in x))$ for all Δ_0 formulas $A[y]$ in which x does not occur,

(Δ_0^- separation) $\exists x((\forall y \in x)(y \in a \wedge A[y]) \wedge (\forall y \in a)(A[y] \rightarrow y \in x))$ for all negative Δ_0 formulas $A[y]$ in which x does not occur,

(separation) $\exists x((\forall y \in x)(y \in a \wedge A[y]) \wedge (\forall y \in a)(A[y] \rightarrow y \in x))$ for all formulas $A[y]$ in which x does not occur,

(Δ_0 collection) $(\forall x \in a)\exists y A[x, y] \rightarrow \exists z(\forall x \in a)(\exists y \in z)A[x, y]$ for all Δ_0 formulas $A[x, y]$ in which z does not occur freely,

(Δ_0 collection$^\sharp$) $(\forall x \in a)\exists y A[x, y] \rightarrow \exists z(\forall x \in a)\neg(\forall y \in z)\neg A[x, y]$ for all negative Δ_0 formulas $A[x, y]$ in which z does not occur freely,

(replacement) $(\forall x \in a)\exists! y A[x, y] \rightarrow \exists z(\forall x \in a)(\exists y \in z)A[x, y]$ for all formulas $A[x, y]$ in which z does not occur freely,

($\in$-induction) $\forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$ for all formulas $A[x]$,

($\in$-induction$^-$) $\forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$ for all negative formulas $A[x]$,

(infinity) $\exists x((\exists y \in x)\mathrm{zero}[y] \wedge (\forall y \in x)(\exists z \in x)\mathrm{succ}[y, z])$,

(N-infinity) $\exists x((\exists y \in x)\mathrm{zero}[y] \wedge (\forall y \in x)(\exists z \in x)\mathrm{succ}[y, z])^N$,

(axiom of choice)
$(\forall x \in a)(x \neq \emptyset) \rightarrow \exists f(\mathrm{Fun}[f] \wedge \mathrm{Dom}[f, a] \wedge (\forall x \in a)(f'x \in x))$,

where $\mathrm{zero}[x]$ is the formula $(\forall y \in x)\bot$ and $\mathrm{succ}[y, z]$ is the conjunction of the formulas $y \in z$, $(\forall u \in y)(u \in z)$ and $(\forall u \in z)(u \in y \vee u = y)$.

Definition 1.8 (The theories KP_0, KP_ω, KP and KP^{int})**.** The theory KP_0 is based on classical first-order logic including the common equality axioms for $=$ and consists of the non-logical axioms extensionality, pairing, union, Δ_0 separation, Δ_0 collection and infinity. The theory KP_ω is the theory KP_0 with in addition $\in$-induction restricted to ω. The theory KP is the theory KP_0 with in addition unrestricted $\in$-induction. And the theory KP^{int} is the theory KP but without the axiom extensionality.

Notice that in literature $\mathsf{KP}\omega$ often denotes the theory which we call KP; KP_ω should not be mixed up with $\mathsf{KP}\omega$. Notice furthermore that in all theories introduced in the previous definition, in contrast to similar theories introduced in Jäger [16], we do not have urelements. The theory introduced in the next definition is the subsystem of the famous theory ZFC without the power set axiom.

Definition 1.9 (The theory ZFC^-)**.** The theory ZFC^- is based on classical first-order logic including the common equality axioms for $=$ and consists of the non-logical axioms extensionality, pairing, union, separation, replacement, $\in$-induction, infinity and the axiom of choice.

In the following two definitions we introduce some intuitionistic set theories which will deal as intermediate theories.

Definition 1.10 (The intensional theories $\mathsf{IKP}_0^\sharp$, $\mathsf{IKP}_\omega^\sharp$ and $\mathsf{IKP}^\sharp$)**.** The theory $\mathsf{IKP}_0^\sharp$ is based on intuitionistic first-order logic including the common equality axioms for $=$ and consists of the non-logical axioms pairing, union, Δ_0^- separation, Δ_0 collection$^\sharp$ and N-infinity. The theory $\mathsf{IKP}_\omega^\sharp$ is the theory IKP_0 with in addition $\in$-induction$^-$ restricted to ω. And the theory $\mathsf{IKP}^\sharp$ is the theory $\mathsf{IKP}_0^\sharp$ with in addition unrestricted $\in$-induction$^-$.

Avigad uses the theory $\mathsf{IKP}^\sharp$ in [1] under the name $\mathsf{IKP}^{int\sharp}$. We will often add the following axiom schema (a set theoretic version of Markov's principle) to the theories introduced in the previous definition.
For all negative Δ_0 formulas $A[x]$ of $\mathcal{L}$:

$$(MP_{res}) \qquad \neg\forall x A[x] \to \exists y \neg(\forall x \in y) A[x].$$

Roughly speaking, (MP_{res}) is a reflection principle for double-negation-interpreted Σ_1 formulas.

Definition 1.11 (The intensional theories IKP_0^-, IKP_ω^- and IKP^-)**.** The theory IKP_0^- is based on intuitionistic first-order logic including the common equality axioms for $=$ and consists of the non-logical axioms pairing, union, Δ_0^- separation, Δ_0 collection and N-infinity. The theory IKP_ω^- is the theory IKP_0 with in addition $\in$-induction restricted to ω. And the theory IKP^- is the theory IKP_0^- with in addition unrestricted $\in$-induction.

We will use abbreviations for formulas expressing assertions about the *constructible hierarchy* $\mathbf{L}$. They can be defined as for instance in the fifth and sixth section of Chapter II in Part A of Barwise [2]. Definitions and properties of the constructible hierarchy can also be found for instance in Devlin [9], Jech [27], Krivine [28] or Kunen [29]. Furthermore we will introduce the constructible hierarchy in operational set theory in section 3.2 (p. 57 et seqq.). We write $a \in L_\alpha$ for a formula expressing that the set a is an element of the αth level L_α of the constructible hierarchy and $a \in \mathbf{L}$ is short for $\exists\alpha(a \in L_\alpha)$. Furthermore, given a set $a \in \mathbf{L}$ we write $\mathrm{od}(a)$ for the least ordinal α such that $a \in L_{\alpha+1}$ and $a <_{\mathbf{L}} b$ for a formula expressing that a is smaller than b according to the well known well-ordering $<_{\mathbf{L}}$ on the constructible universe. The *axiom of constructibility* is given by

$$(\mathbf{V}{=}\mathbf{L}) \qquad \forall x \exists \xi (x \in L_\xi).$$

It is well known that $a \in L_\alpha$, $\mathrm{od}(a) = \alpha$ and $a <_{\mathbf{L}} b$ are Δ formulas w.r.t. KP and that the systems KP and $\mathsf{KP}+(\mathbf{V}{=}\mathbf{L})$ prove the same absolute sentences.

The well known *axiom Beta* is given by

$$(\mathrm{Beta}) \quad \mathrm{WF}[a,r] \to \exists f(\ \mathrm{Fun}[f] \land \mathrm{Dom}[f,a] \land \\ (\forall x \in a)(f'x = \{f'y : y \in a \land \langle y,x\rangle \in r\})\ ).$$

Axiom Beta has not the adequate form that we can apply a theorem later directly (c.f. Theorems 3.49 and 3.53). Therefore we introduce an alternative version:

$$(\mathrm{Beta}') \quad \forall a, r \exists f, b(\ \mathrm{Fun}[f] \land \mathrm{Dom}[f,b] \land b \subseteq a \land \mathrm{Prog}[b,a,r] \\ \land (\forall x \in b)(f'x = \{f'y : y \in b \land \langle y,x\rangle \in r\})\ ).$$

Notice that all axioms of the theories introduced up to here can be formulated in the language $\mathcal{L}_\in$.

The next axiom is the *power set axiom* formulated with a binary relation symbol $\mathcal{P}$ of $\mathcal{L}$. It is given by

$$(\mathcal{P}) \qquad \forall x \exists y \mathcal{P}(x,y) \land \forall x \forall y (\mathcal{P}(x,y) \leftrightarrow \forall z(z \in y \leftrightarrow z \subseteq x)).$$

The theory $\mathsf{KP} + (\mathcal{P})$ is called *theory of power admissible sets*.

We also introduce the set $\mathcal{AD}$ of axioms for the relation symbol Ad expressing that specific sets are admissible. The axioms of $\mathcal{AD}$ are given by

(i) $\mathsf{Ad}(a) \to (\omega \in a \land \mathrm{Tran}[a])$,

(ii) $\mathsf{Ad}(a) \to (\forall \vec{x} \in a) A^a[\vec{x}]$,

(iii) $(\mathsf{Ad}(a) \land \mathsf{Ad}(b)) \to (a \in b \lor a = b \lor b \in a)$, where $A[\vec{u}]$ is an instance of the axioms pairing, Δ_0 separation, Δ_0 collection or Tran and $\vec{u}$ denotes in each case its free variables, and where Tran is the axiom,

$$(\mathsf{Tran}) \qquad \forall x \exists y (x \subseteq y \land \mathrm{Tran}[y]).$$

The axiom (Lim), which states that every set is an element of some admissible set, is given by

$$(\mathsf{Lim}) \qquad \forall x \exists y (x \in y \land \mathsf{Ad}(y)).$$

Definition 1.12 (The theories KPI_0, KPI_ω and KPI)**.** The theories KPI_0, KPI_ω and KPI are the theories KP_0, KP_ω and KP, respectively, with the additional axioms $\mathcal{AD}$ as well as (Lim).

We will additionally use a notion of admissible sets with stronger closure properties than usual admissible sets.

Definition 1.13 (Strong admissible sets)**.** We call a set *a strong admissible*, in symbols $\mathrm{Sd}[a]$, if

$$\begin{aligned} \mathsf{Ad}(a) \ \land \ (\forall x \in a)\forall f(\, &\mathrm{Fun}[f] \land \mathrm{Dom}[f, x] \land \mathrm{Ran}_{\subseteq}[f, a] \\ &\to (\exists y \in a)\mathrm{Ran}_{\subseteq}[f, y]\ ). \end{aligned}$$

The analogous axiom to (Lim), but formulated for strong admissible sets instead of only admissible sets, is then given by the $\mathcal{L}_{\mathsf{Ad}}$ formula

$$(\mathsf{SdLim}) \qquad \forall x \exists y (x \in y \land \mathrm{Sd}(y)).$$

Definition 1.14 (The theory KPSd)**.** The theory KPSd is the theory KP with the additional axioms $\mathcal{AD}$ as well as (SdLim).

Finally, the *strong limit axiom* is given by

$$(\mathsf{SLim}) \qquad \forall \xi \exists \eta (\xi < \eta \land \mathrm{Frg}[\eta]).$$

Definition 1.15 (The theory KPS)**.** The theory KPS is the theory KP with the additional axiom (SLim).

1.1.3 Some Properties of Set Theories

The first lemma can be proved as Theorem 4.5 in Chapter I in Part A of Barwise [2]. Notice that in the proof neither extensionality nor $\in$-induction is used.

Lemma 1.16 (Δ separation)**.** *Let $\mathcal{T}$ be a theory containing KP_0 or KP^{int} and A a Δ formula w.r.t. $\mathcal{T}$. Then $\mathcal{T}$ proves*

$$\exists y(y = \{x \in a : A[x]\}).$$

In the following we present two slightly different versions of so-called Σ *recursion*. To formulate them we will use further abbreviations: If R and S are n+1-ary and n-ary, respectively, relation symbols of $\mathcal{L}$, $A[R]$ and $B[S]$ are formulas of $\mathcal{L}$ and $C[\alpha, \vec{x}]$ a formula of $\mathcal{L}$ with distinguished free variables α and $\vec{x} = x_0, ..., x_{n-1}$, we write $A[C[.]]$ and $B[C[\alpha, .]]$ for the result of substituting $C[\xi, \vec{v}]$ for each occurrence of the form $R(\xi, \vec{v})$ in $A[R]$, and substituting $C[\alpha, \vec{v}]$ for each occurrence of the form $S(\vec{v})$ in $B[S]$, respectively, as well as of renaming bounded variables if necessary to avoid collision in A and in B. We prove the first version of Σ recursion exactly as in Jäger and Zumbrunnen [26] and similar as in Section 6 of Chapter I in Part A of Barwise [2] (definition by Σ recursion).

Proposition 1.17 (Σ recursion 1)**.** *Let R be an n-ary relation symbol and $A[\alpha, \vec{a}, R]$ a Δ formula w.r.t. KP of $\mathcal{L}$ with distinguished free variables α and $a = a_0, ..., a_{n-1}$. Then there exists a Σ formula $B[\alpha, \vec{a}]$ of $\mathcal{L}$, in which R does not occur, such that KP proves*

$$B[\alpha, \vec{a}] \leftrightarrow (\vec{a} \in L_\alpha \wedge A[\alpha, \vec{a}, (\exists \xi < \alpha) B[\xi, .]]).$$

Proof. To simplify the notation we assume w.l.o.g. that $n = 1$. Let $C[f, \alpha, b]$ be the conjunction of the following formulas:

(i) $\mathrm{Fun}[f] \wedge \mathrm{Dom}[f, \alpha]$,

(ii) $(\forall \xi < \alpha)(f'\xi = \{x \in L_\xi : A[\xi, x, \bigcup_{\eta<\xi} f'\eta]\})$,

(iii) $b = \{x \in L_\alpha : A[\alpha, x, \bigcup_{\eta<\alpha} f'\eta]\}$.

It is easy to see that C is a Δ formula w.r.t. KP, so let $D[f, \alpha, b]$ be a formula which is provably equivalent to $C[f, \alpha, b]$ in KP. By transfinite induction on α (which follows from $\in$-induction) it follows

$$D[f, \alpha, b] \wedge D[f, \alpha, c] \to f = g \wedge b = c.$$

Furthermore, by Σ replacement (see Theorem 4.6 in Chapter I in Part A of Barwise [2]), Δ separation and also by transfinite induction, we have

$$\exists f \exists b D[f, \alpha, b].$$

Therefore it is easy to check that if we let $B[\alpha, a]$ be the Σ formula

$$\exists f \exists b(D[f, \alpha, b] \wedge a \in b),$$

then it has the properties stated in the proposition. □

The second version follows directly form the *Second Recursion Theorem* in Barwise [2, p. 157].

Proposition 1.18 (Σ recursion 2)**.** *Let R be an $(n+1)$-ary relation symbol and $A[\alpha, \vec{a}, R]$ a Σ formula of $\mathcal{L}$ with distinguished free variables α as well as $a = a_0, ..., a_{n-1}$ in which R occurs only positively. Then there exists a Σ formula $B[\alpha, \vec{a}]$ of $\mathcal{L}$ in which R does not occur, such that* KP *proves*

$$B[\alpha, \vec{a}] \leftrightarrow A[\alpha, \vec{a}, B[.]].$$

The next proposition tells us, that the theory KPS is contained in the theory KPSd.

Proposition 1.19. *The axiom* (SLim) *is provable in the theory* KPSd*.*

Proof. Let α be any ordinal. We have to prove in KPSd that there is an ordinal $\beta > \alpha$ which is functionally regular. By (SdLim) there is a strong admissible set b with $\alpha \in b$. Let β be the set $\{x \in b : \mathrm{Ord}[x]\}$, which exists by Δ_0 separation. Clearly we have $\alpha \in \beta$. First let us check that β is an ordinal. Clearly every element of β is transitive. Let x be an element of β. Then x is an ordinal and contains therefore only ordinals. Since b is admissible, it is transitive, and contains therefore all elements of x. It follows that also β contains all elements of x. Hence β is transitive and so an ordinal. In order to show that β is functionally regular, assume that ξ is an ordinal with $\xi < \beta$ and f is a function with domain ξ and $\mathrm{Ran}_{\subseteq}[f, \beta]$. Since $\beta \subseteq b$ and b is a strong admissible, this implies that there is a y with $\mathrm{Ran}_{\subseteq}[f, y]$. If we let y' be the set $\{x \in y : \mathrm{Ord}[x]\}$, then it is an element of b since b is closed under Δ_0 separation. And if η is the transitive closure of y', it is clearly also an element of b and obviously an ordinal; so $\eta < \beta$. It is easy to check that $\beta \cap y \subseteq \eta$. Hence we have $\mathrm{Ran}_{\subseteq}[f, \eta]$. We can conclude that β is a functionally regular ordinal. □

The next lemma and its proof are taken from Sato and Zumbrunnen [31].

Lemma 1.20. *There are Δ formulas (w.r.t.* KP*) $A[a, b, n]$ and $B[x, w, m]$ such that* KP *proves*

(i) $A[a, b, n]$ is equivalent to $n \in \omega \rightarrow$ "b is the set a^n".

(ii) $B[x, w, m]$ is equivalent to

$$(\exists n \in \omega)(m < n \wedge \text{“}w \text{ is in } a^n\text{”}) \\ \rightarrow \text{“}x \text{ is the } (m+1)\text{-st component of } w\text{”}.$$

Proof. In Chapter I in Part A of Barwise [2], the Δ_0 formula $x = a \times b$ of $\mathcal{L}_\in$, which expresses that x is the Cartesian product of a and b, is introduced (also $x = a_0 \times ... \times a_n$ is introduced for arbitrary n, but the n is a natural number on the meta-level). Let $C[f, a, n]$ be the Δ_0 formula

$$\begin{aligned}&\mathrm{Fun}[f] \wedge \mathrm{Dom}[f, n{+}1] \wedge f'0 = \{0\} \wedge f'1 = a \\ &\qquad\qquad\qquad\qquad \wedge (\forall k < n)(k > 0 \rightarrow f'(k{+}1) = f'k \times a),\end{aligned}$$

and $A[a, b, n]$ the Σ formula

$$\exists f(C[f, a, n] \wedge f'n = b).$$

Furthermore let $A'[a, b, n]$ be the Π formula

$$\forall f(C[f, a, n] \rightarrow f'n = b).$$

It is easy to prove in KP that there is exactly one f for which $C[f, a, n]$ holds. Therefore KP proves $A[a, b, n] \leftrightarrow A'[a, b, n]$.

For the second assertion let $B[x, w, m]$ be the Σ formula

$$\begin{aligned}&\exists f \exists g(\mathrm{Fun}[f] \wedge \mathrm{Fun}[g] \wedge \mathrm{Dom}[f, m{+}1] \wedge \mathrm{Dom}[g, m{+}1] \\ &\qquad\qquad \wedge g'0 = w \wedge (\forall k < m)(g'(k{+}1) = (g'k)_1) \\ &\qquad\qquad \wedge f'0 = (w)_0 \wedge (\forall k < m)(f'(k{+}1) = (g'k)_0) \wedge f'm = x).\end{aligned}$$

Since the functions f and g are (provable in KP) unique, there is also a Π formula equivalent to $B[x, w, m]$. And since ordered n-tuples have the form

$$\langle x_0, \langle x_1, \langle ... \langle x_{n-2}, x_{n-1} \rangle ... \rangle \rangle \rangle,$$

the formula is as stated in the lemma. □

In the following we write $b = a^n$ for the formula $A[a, b, n]$ and $x = w[m]$ for $B[x, w, m]$ of the previous lemma. Notice that $x = (w)_n$ and $x = w[n]$ are not the same: in the first case n is a natural number on the meta-level and in the second one we have $n \in \omega$.

For the next lemma we introduce the Δ_0 formula $\prec[y, x, a, r]$ given by

$$(\exists n \in \omega)(\exists w \in a^n)(\forall m < n)(\langle w[m], w[m{+}1] \rangle \in r \wedge w[0] = y \wedge w[n] = x),$$

expressing that y is smaller than x w.r.t. the transitive closure of $r \cap (a \times a)$. The Σ formula $\mathrm{WP}[x, a, r]$, expressing by the first and second expression of the next lemma that x is in the *well founded part* of the relation r, is given by

$$\begin{aligned}&\exists v(\ v = \{y \in a : \prec[y, x, a, r]\} \\ &\quad \wedge \exists f(\mathrm{Fun}[f] \wedge \mathrm{Dom}[f, v] \wedge (\forall x \in v)(f'x = \{f'y : y \in v \wedge \langle y, x \rangle \in r\}))\ ).\end{aligned}$$

In the proof of the next lemma we also use the Π formula $\mathrm{WP}'[x,a,r]$ given by

$$\forall v(\ v = \{y \in a : \prec[y,x,a,r]\} \to \mathrm{WF}[v,r]\ ).$$

Furthermore we will informally write $x_{\prec}$ for the set $\{y \in a : \prec[y,x,a,r]\}$. Also the next lemma, the next proposition and their proofs are taken from Sato and Zumbrunnen [31].

Lemma 1.21.

(i) KP *proves that*

$$\mathrm{Fun}[f] \wedge \mathrm{Dom}[f,b] \wedge (\forall x \in b)(f'x = \{f'y : y \in b \wedge \langle y,x\rangle \in r\})$$

implies $\mathrm{WF}[b,r]$.

(ii) $\mathsf{KP} + (\mathrm{Beta})$ *proves that* $\mathrm{WP}[x,a,r]$ *and* $\mathrm{WP}'[x,a,r]$ *are equivalent.*

(iii) $\mathsf{KP} + (\mathrm{Beta})$ *proves that* $\{x \in a : \mathrm{WP}[x,a,r]\}$ *is a set.*

(iv) KP *proves that* $b \subseteq a$ *and* $\mathrm{WF}[a,r]$ *implies* $\mathrm{WF}[b,r]$.

(v) $\mathsf{KP} + (\mathrm{Beta})$ *proves that* $b = \{x \in a : \mathrm{WP}[x,a,r]\}$ *implies that there is a unique function* f *with domain* b *and*

$$(\forall x \in b)(f'x = \{f'y : y \in b \wedge \langle y,x\rangle \in r\})$$

and that b *is progressive w.r.t.* a *and* r.

Proof. For (i) we assume $\mathrm{Fun}[f]$, $\mathrm{Dom}[f,b]$ and

$$(\forall x \in b)(f'x = \{f'y : y \in b \wedge \langle y,x\rangle \in r\}).$$

Furthermore we assume $\mathrm{Prog}[c,b,r]$ for some $c \subseteq b$. It follows for every $x \in b$ that $x \in c$ if $y \in c$ for all $y \in b$ with $f'y \in f'x$. This means, if d is the range of f and a is the set $\{z \in d : (\exists y \in c)(z = f'y)\}$, then

$$(\forall v \in d)((\forall z \in d)(z \in v \to z \in a) \to v \in a).$$

By $\in$-induction it follows

$$(\forall v \in d)(v \in a),$$

which corresponds to $(\forall x \in c)(x \in b)$. All in all this implies $\mathrm{WF}[b,r]$.

Assertion (ii) follows because by (Beta) and (i) we have

$$\mathrm{WF}[v,r] \leftrightarrow \exists f(\mathrm{Fun}[f] \wedge \mathrm{Dom}[f,v] \wedge (\forall x \in v)(f'x = \{f'y : y \in v \wedge \langle y,x\rangle \in r\})),$$

and because by Δ separation (Lemma 1.16) and extensionality there is exactly one set v with $v = \{y \in a : \prec[y,x,a,r]\}$.

Assertion (iii) follows by (ii) and again by Δ separation .

The proof of (iv) is straightforward.

For (v) assume that $b = \{x \in a : \mathrm{WP}[x, a, r]\}$, that is by (ii), b is the set of all $x \in a$ for those r is well-founded on $x_\prec$. First we show that r is well-founded on b: we assume that $c \subseteq b$, $\mathrm{Prog}[c, b, r]$ and $x \in b$ and we have to prove that $x \in c$. From the progressivity of c it follows (directly from the definition of Prog and $x_\prec$) $\mathrm{Prog}[c \cap x_\prec, b \cap x_\prec, r]$. Because $x \in b$ we have $\mathrm{WF}[x_\prec, r]$ and by (iv) also $\mathrm{WF}[b \cap x_\prec, r]$ and so $b \cap x_\prec \subseteq c \cap x_\prec$. By the progressivity of c we get therefore $x \in c$ and r is therefore well-founded on b. So by (Beta) there is a function f with domain b such that $f'x = \{f'y : y \in b \wedge \langle y, x\rangle \in r\}$ for all $x \in b$. That this f is unique can be proved by $\in$-induction. In the following we prove that b is progressive w.r.t. a and r: by (ii) this is the case iff.

$$(\forall z \in a)((\forall z' \in a)(\langle z', z\rangle \in r \to \mathrm{WF}[z'_\prec, r]) \to \mathrm{WF}[z_\prec, r]).$$

Assume $z \in a$, $(\forall z' \in a)(\langle z', z\rangle \in r \to \mathrm{WF}[z'_\prec, r])$, $c \subseteq z_\prec$ and $\mathrm{Prog}[c, z_\prec, r]$. We have to prove $z_\prec \subseteq c$, so assume $v \in z_\prec$ (and we are done if $v \in c$). It is easy to see that we get $\mathrm{Prog}[c, v_\prec, r]$ from $\mathrm{Prog}[c, z_\prec, r]$. From (iv) we get furthermore $\mathrm{WF}[v_\prec, r]$ because $v_\prec \subseteq z'_\prec$ for some z with $\langle z', z\rangle \in r$. So we have $v_\prec \subseteq c$ and therefore $v \in c$ by the progressivity of c w.r.t. $z_\prec$. So we have $\mathrm{Prog}[b, a, r]$ and this finishes the prove of (v). □

Proposition 1.22. KP *proves that* (Beta) *and* (Beta$'$) *are equivalent.*

Proof. The direction from left to right follows by the assertions (iii) and (v) of Lemma 1.21 and the other direction by the definition of WF. □

We end this section with two lemmas about intuitionistic validity of some formulas, which we will use later and which are also used in Sato and Zumbrunnen [31]. The proof of the first one is taken from Sato and Zumbrunnen [31].

Lemma 1.23. *The following formulas are intuitionistically valid for arbitrary $\mathcal{L}$ formulas A, A_0, A_1, and A_2 and any strongly negative formula B:*

(i) $(A_0 \to (A_1 \to A_2)) \leftrightarrow ((A_0 \wedge A_1) \to A_2)$,

(ii) $(A_0 \to (A_1 \to A_2)) \leftrightarrow (A_1 \to (A_0 \to A_2))$,

(iii) $(A_0 \to (A_1 \wedge A_2)) \leftrightarrow ((A_0 \to A_1) \wedge (A_0 \to A_2))$,

(iv) $(A_0 \to (A_1 \to A_2)) \leftrightarrow ((A_0 \to A_1) \to (A_0 \to A_2))$,

(v) $\exists x(A_0 \to A_1) \to (A_0 \to \exists x A_1)$ *if x is not free in A_0,*

(vi) $(\exists x \in y)(A_0 \to A_1) \to (A_0 \to (\exists x \in y)A_1)$ *if x is not free in A_0,*

(vii) $\forall x(A_0 \to A_1) \leftrightarrow (A_0 \to \forall x A_1)$ *if x is not free in A_0,*

(viii) $(\forall x \in y)(A_0 \to A_1) \leftrightarrow (A_0 \to (\forall x \in y)A_1)$ *if x is not free in A_0,*

(ix) $\exists x A \to \neg\forall x \neg A$,

(x) $(\exists x \in y)A \to \neg(\forall x \in y)\neg A$,

(xi) $\exists y \neg(\forall x \in y)A[x] \to \neg\forall x A[x]$,

(xii) $B \leftrightarrow B^N$,

(xiii) $\forall x(A_0[x] \leftrightarrow A_1[x]) \to (\forall x A_0[x] \leftrightarrow \forall x A_1[x])$,

(xiv) $\exists x(A_0[x] \to (\forall y \in z)A_1[y])$ *implies* $\forall y \exists x(A_0[x] \to (y \in z \to A_1[y]))$ *if y is not free in $A_0[x]$,*

(xv) $\forall y \exists x(A_0[x] \to (y \in z \to A_1[y]))$ *implies* $(\forall y \in z)\exists x(A_0[x] \to A_1[y])$.

Proof. The assertions (i), (v), (vii) follows from Lemma 5.2.1 in van Dalen [33]). Assertion (viii) follows by the assertions (ii) and (vii).

We prove assertion (xii) by induction on the length of B. If B is the formula $\perp$, then the assertion follows by Theorem 5.2.6 in van Dalen [33]. If B is of the form $C \to D$, where C is atomic, we have that $C^N \to D^N$ implies $C \to D$ by assertion (7) of Lemma 5.2.1 in van Dalen [33] and the induction hypothesis. And if we apply twice assertion (14) of Lemma 5.2.1 in van Dalen [33] and the induction hypothesis, we get that $C \to D$ implies $C^N \to \neg\neg D^N$. And by Theorem 5.2.6 also in van Dalen [33] we get that $C \to D$ implies $C^N \to D^N$. All the other cases follow directly by induction hypothesis.

All other assertions can be formally proved within suitable Gentzen systems. We prove here for instance assertion (xiv) within the system **G1i** which is presented in Troelstra and Schwichtenberg [32] (notice that this systems is for multi-sets, not for sequences):

$$
\begin{prooftree}
\AxiomC{$A_0[x'] \Rightarrow A_0[x']$}
\AxiomC{$D[x',y'] \Rightarrow D[x',y']$}
\RightLabel{(LW)}
\UnaryInfC{$A_0[x'], D[x',y'] \Rightarrow D[x',y']$}
\RightLabel{(L$\forall$)}
\UnaryInfC{$A_0[x'], \forall y D[x',y] \Rightarrow D[x',y']$}
\RightLabel{(L$\to$)}
\BinaryInfC{$A_0[x'], A_0[x'] \to \forall y D[x',y] \Rightarrow D[x',y']$}
\RightLabel{(R$\to$)}
\UnaryInfC{$A_0[x'] \to \forall y D[x',y] \Rightarrow A_0[x'] \to D[x',y']$}
\RightLabel{(R$\exists$)}
\UnaryInfC{$A_0[x'] \to \forall y D[x',y] \Rightarrow \exists x(A_0[x] \to D[x,y'])$}
\RightLabel{(L$\exists$)}
\UnaryInfC{$\exists x(A_0[x] \to \forall y D[x,y]) \Rightarrow \exists x(A_0[x] \to D[x,y'])$}
\RightLabel{(R$\forall$)}
\UnaryInfC{$\exists x(A_0[x] \to \forall y D[x,y]) \Rightarrow \forall y \exists x(A_0[x] \to D[x,y])$}
\RightLabel{(R$\to$)}
\UnaryInfC{$\Rightarrow\ \exists x(A_0[x] \to \forall y D[x,y]) \to \forall y \exists x(A_0[x] \to D[x,y])$}
\end{prooftree}
$$

where $D[u, v]$ is an abbreviation of the formula $v \in z \to A_1[u, v]$. The remaining assertions can be treated similarly. □

We prove the next Lemma as Lemma 5.1 is proved in Avigad [1].

Lemma 1.24.

(i) The following formulas are intuitionistically valid for an arbitrary $\mathcal{L}$ formula A:

a) $(y \in x \to A^N) \ \leftrightarrow\ (\neg\neg(y \in x) \to A^N)$,

b) $((\forall y \in x)A)^N \ \leftrightarrow\ (\forall y \in x)A^N$,

c) $((\exists y \in x)A)^N \ \leftrightarrow\ \neg(\forall y \in x)\neg A^N$.

(ii) If A is a Δ_0 formula of $\mathcal{L}$ then A^N is intuitionistically equivalent to some strongly negative Δ_0 formula.

Proof. The direction of (i) a) from right to left follows, because $y \in x$ implies $\neg\neg(y \in x)$ intuitionistically. The other direction follows because $y \in x \to A^N$ implies $\neg A^N \to \neg y \in x$ intuitionistically and the latter implies $(\neg\neg(y \in x) \to \neg\neg A^N)$ intuitionistically. Since $\neg\neg A \to A$ is classically valid, $\neg\neg A^N \to A^N$ is intuitionistically valid.

Assertion (i) b) follows from a) since $((\forall y \in x)A)^N$ is by the definition of double-negation interpretation identical with $\forall y(\neg\neg(y \in x) \to A^N)$.

Assertion (i) c) follows from b) and the fact that $(\exists y \in x)A$ is classically equivalent to $\neg(\forall y \in x)\neg A$.

Assertion (ii) is proved by induction on the length of A, using (i). □

1.2 Operational Set Theories

In this section we introduce different versions of operational set theory and some extensions. The structure is analogous to the previous section. In the first subsection we introduce the language(s) of operational set theory and we define some notations and abbreviations. Axioms and theories are presented in the second subsection. And in the third subsection we present and proof some basic properties of the introduced theories. As in the last section, we follow very closely the papers Jäger and Zumbrunnen [26] and Sato and Zumbrunnen [31].

1.2.1 Languages of Operational Set Theory

The language of operational set theory, $\mathcal{L}_\in^\circ$, is the language $\mathcal{L}_\in$ extended by

- the constants ω, k, s, t, f, el, non, dis, e, $\mathbb{D}$, $\mathbb{U}$, $\mathbb{S}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{P}$, $\mathbb{B}$ and $\mathbb{A}$,

- the unary relation symbol $\downarrow$ and
- the binary function symbol $\circ$.

The meaning of all these symbols will become clear in the next subsection. The language $\mathcal{L}^\circ$ is the analogous extension of $\mathcal{L}$.

The *terms* $(r, s, t, ...)$ of $\mathcal{L}^\circ$ are inductively defined as follows: 1) all variables and constants of $\mathcal{L}^\circ$ are $\mathcal{L}^\circ$ terms; 2) if s and t are $\mathcal{L}^\circ$ terms, then so is $\circ(s,t)$. Terms without any occurrence of variables are called *closed terms*. The *formulas* of $\mathcal{L}^\circ$ are built up as the formulas of $\mathcal{L}$ but with the newly introduced terms and the new atomic formulas of the form $t\downarrow$.

Because terms may be undefined, we redefine $\neq$ and $\notin$ for the language $\mathcal{L}^\circ$, namely we write in $\mathcal{L}^\circ$ $s \neq t$ and $s \notin t$ for the formulas

$$\neg(s = t) \wedge s\downarrow \wedge t\downarrow \quad \text{and} \quad \neg(s \in t) \wedge s\downarrow \wedge t\downarrow,$$

respectively.

All the other abbreviations and notations introduced for $\mathcal{L}$ in the previous section are also used for $\mathcal{L}^\circ$. In the following we introduce some new abbreviations and notations which are specific for operational set theory.

Partial equality, $\simeq$, is introduced as follows: for all $\mathcal{L}^\circ$ terms s and t we write

$$(s \simeq t) \quad \text{for} \quad ((s\downarrow \vee t\downarrow) \to s = t).$$

If $\vec{u} = u_1, ..., u_n$ and $\vec{t} = t_1, ..., t_n$ are finite sequences of variables and terms, respectively, we write $s[\vec{t}/\vec{u}]$ for the term which is obtained from s by simultaneously replacing all occurrences of the variables $\vec{u}$ by the terms $\vec{t}$.

For terms s, t, t_0, ... , t_n we will just write st for $\circ(s,t)$, and $st_0...t_n$ as well as $s(t_0, ..., t_n)$ for $((...((st_0)t_2)...)t_n)$.

We will write $\mathbf{B}$ for the collection $\{x : x = \mathsf{t} \vee x = \mathsf{f}\}$ and $\mathbf{V}$ for the collection $\{x : x\downarrow\}$.

For an arbitrary natural number n, arbitrary $\mathcal{L}^\circ$ terms r, s, t, and variables x, $x_0, ..., x_n$ we write

- $(t : r \to s)$ for $(\forall x \in r)(tx \in s)$ and
- $(t : r^{n+1} \to s)$ for $(\forall x_0 \in r)...(\forall x_n \in r)(t(x_0, ..., x_n) \in s)$.

The terms r and/or s may be replaced by $\mathbf{V}$ and/or $\mathbf{B}$. Notice that $f : a \to b$ does not mean that f is a set-theoretic function.

Definition 1.25 (Δ, Π, Σ, Δ_n, Π_n and Σ_n formulas of $\mathcal{L}^\circ$)**.** The Δ*,* Π*,* Σ*,* Δ_n*,* Π_n and Σ_n *formulas* of $\mathcal{L}^\circ$ are the Δ, Π, Σ, Δ_n, Π_n and Σ_n formulas of $\mathcal{L}$ (for any natural number n) but they can contain constants of $\mathcal{L}^\circ$.

Due to this definition, a formula of $\mathcal{L}^\circ$ of one of these classes does not contain the function symbol $\circ$ nor the relation symbol $\downarrow$ (but all the other relation symbols are allowed).

The notion introduced in the next definition is an operational-set-theoretic version of the notion of functional regularity (introduced in Definition 1.7).

Definition 1.26 (Operational regularity)**.** We call an ordinal κ *operationally regular*, in symbols $\mathrm{Org}[\kappa]$, if $\omega < \kappa$ and

$$\forall f (\forall \xi < \kappa)((f : \xi \to \kappa) \to (\exists \eta < \kappa)(f : \xi \to \eta)).$$

1.2.2 Definitions of Operational Set Theories

The underlying logic of all operational set theories is the classical *logic of partial terms* due to Beeson [3] including the common equality axioms for $=$.

In this logic both of the formulas $(st)\downarrow$ and $(s \in t)$ imply $s\downarrow$ as well as $t\downarrow$; and if R is an arbitrary n-ary relation symbol of $\mathcal{L}^\circ$, then $R(t_0, ..., t_{n-1})$ implies $t_k\downarrow$ for each $0 \leq k < n$. The formula $t\downarrow$ means that t is defined in the sense of denoting an object in the universe. Therefore we need the assumption $t\downarrow$ if we want to conclude $\exists x A[x]$ from $A[t]$.

We are now ready to formulate some systems of operational set theory as in Sato and Zumbrunnen [31]. This formulation is similar to the original one in Feferman [13, 14]. There are four groups of non-logical axioms of operational set theory. The so called *applicative axioms* are standard axioms about the combinators k and s.

Applicative axioms

(A1) $\mathsf{k} \neq \mathsf{s}$,

(A2) $\mathsf{k}ab = a$,

(A3) $\mathsf{s}ab\downarrow \wedge\, \mathsf{s}abc \simeq (ac)(bc)$.

In the second group we have the so called *basic set-theoretic axioms*. They are standard set-theoretic axioms.

Basic set-theoretic axioms

The axioms extensionality, $\in$-induction for all formulas of $\mathcal{L}^\circ$ (all formulated as in subsection 1.1.2) as well as the following version of infinity, expressing that ω is the first infinite ordinal:

$$\mathrm{Ord}[\omega] \wedge \mathrm{Lim}[\omega] \wedge (\forall x \in \omega)(\neg \mathrm{Lim}[x]).$$

The axioms of the third group, the *logical operations axioms*, describe the representation of the element relation, the connectives negation and disjunction as well as bounded existential quantification as operations.

Logical operations axioms

(L1) $\mathsf{t} \neq \mathsf{f}$,

(L2) $(\mathsf{el} : \mathbf{V}^2 \to \mathbf{B}) \wedge \forall x \forall y(\mathsf{el}(x, y) = \mathsf{t} \leftrightarrow x \in y)$,

(L3) $(\mathsf{non} : \mathbf{B} \to \mathbf{B}) \wedge (\forall x \in \mathbf{B})(\mathsf{non}(x) = \mathsf{t} \leftrightarrow x = \mathsf{f})$,

(L4) $(\mathsf{dis} : \mathbf{B}^2 \to \mathbf{B}) \wedge (\forall x, y \in \mathbf{B})(\mathsf{dis}(x, y) = \mathsf{t} \leftrightarrow (x = \mathsf{t} \vee y = \mathsf{t}))$,

(L5) $(f : a \to \mathbf{B}) \to (\mathsf{e}(f, a) \in \mathbf{B} \wedge (\mathsf{e}(f, a) = \mathsf{t} \leftrightarrow (\exists x \in a)(fx = \mathsf{t})))$.

The last group of axioms contains axioms about some set-theoretic operations.

Operational set-theoretic axioms

($\mathbb{D}$) Unordered pair (or not necessarily proper doubleton):

$$\forall x \forall y(\mathbb{D}(x, y)\downarrow \wedge\, \mathbb{D}(x, y) = \{x, y\}).$$

($\mathbb{U}$) Union:

$$\forall x(\mathbb{U}(x)\downarrow \wedge\, \mathbb{U}(x) = \cup x).$$

($\mathbb{S}$) Separation for definite operations:

$$(f : a \to \mathbf{B}) \;\to\; \mathbb{S}(f, a)\downarrow \;\wedge\; \mathbb{S}(f, a) = \{x \in a : fx = \mathsf{t}\}.$$

($\mathbb{R}$) Replacement:

$$(f : a \to \mathbf{V}) \;\to\; \mathbb{R}(f, a)\downarrow \;\wedge\; \mathbb{R}(f, a) = \{x : (\exists y \in a)(x = fy)\}.$$

($\mathbb{C}$) Choice:

$$\exists x(fx = \mathsf{t}) \to f(\mathbb{C}f) = \mathsf{t}.$$

Definition 1.27 (The theories OST_0^-, OST_ω^- and OST^-)**.** The theory OST_0^- is based on the classical logic of partial terms and consists of all the applicative axioms, all the basic set-theoretic axioms except $\in$-induction, all the logical operations axioms as well as the operational set-theoretic axioms ($\mathbb{D}$), ($\mathbb{U}$), ($\mathbb{S}$) and ($\mathbb{R}$). The theory OST_ω^- is the theory OST_0^- with in addition $\in$-induction restricted to ω. The theory OST^- is the theory OST_0^- with in addition unrestricted $\in$-induction.

The theory OST was originally formulated in Feferman [13] and [14] without the axioms ($\mathbb{D}$) and ($\mathbb{U}$) but with the set-theoretical axioms pairing and union. Since Feferman proved that there are closed terms of his system for forming unordered pairs and unions, respectively (c.f. Corollary 2 in Feferman [14]), our formulation is equivalent to the original one.

Definition 1.28 (The theory OST)**.** The theory OST is based on the classical logic of partial terms and consists of all the applicative axioms, all the basic set-theoretic axioms, all the logical operations axioms as well as all operational set-theoretic axioms.

In other words, OST is the system OST^- with in addition the axiom $(\mathbb{C})$ for the choice operation.

The introduced operational set theories can be extended by the following axioms. The first axiom, an operational version of axiom Beta, is about the operation $\mathbb{B}$ for creating collapsing functions.

$$(\mathbb{B}) \quad \forall a, r \exists b(\ \mathrm{Fun}[\mathbb{B}(a,r)] \wedge \mathrm{Dom}[\mathbb{B}(a,r), b] \wedge b \subseteq a \wedge \mathrm{Prog}[b,a,r] \\ \wedge (\forall x \in b)(\mathbb{B}(a,r)'x = \{\mathbb{B}(a,r)'y : y \in b \wedge \langle y, x\rangle \in r\})\ ).$$

We also introduce an operational version of $(\mathcal{P})$, an axiom which provides us with the operation $\mathbb{P}$ for creating power sets.

$$(\mathbb{P}) \qquad (\mathbb{P} : \mathbf{V} \to \mathbf{V}) \wedge \forall x \forall y (y \in \mathbb{P}x \leftrightarrow y \subseteq x).$$

The operational version of the axiom (Lim), an axiom about the operation $\mathbb{A}$ for creating admissible sets, is given by

$$(\mathbb{A}) \qquad \forall x (x \in \mathbb{A}(x) \wedge \mathsf{Ad}(\mathbb{A}(x))).$$

The last axiom in this section was introduced in Feferman [13, 14]. It is an operational-set-theoretic version of the axiom (SLim) and given by

$$(\mathsf{Inac}) \qquad \forall \xi \exists \eta (\xi < \eta \wedge \mathrm{Org}[\eta]).$$

1.2.3 Some Properties of Operational Set Theories

The applicative axioms imply that we can introduce λ-abstraction and that the following recursion theorem is available.

Lemma 1.29 (λ-abstraction)**.** *Given an arbitrary variable x and terms s and t of $\mathcal{L}^\circ$, we can introduce a so-called λ-term $(\lambda x.t)$ such that OST_0^- proves the formulas*

$$(\lambda x.t)\downarrow \quad \textit{and} \quad s\downarrow \to (\lambda x.t)s \simeq t[s/x].$$

The variables of $(\lambda x.t)$ are those of t other than x.

We will often write $\lambda x_0...x_n.t$ for $(\lambda x_0.(\lambda x_1.(...(\lambda x_n.t)...)))$, and if $\vec{x}$ is the finite sequence $x_0, \dots, x_n$ of variables we will often write $\lambda\vec{x}.t$ for $(\lambda x_0...x_n.t)$.

The previous lemma can be proved as as for instance in Beeson [3] on p. 101 and the next one as ibidem on p. 103.

Lemma 1.30 (Recursion theorem)**.** *There is a closed term* fix*, a so-called* fixed point operator*, such that* OST_0^- *proves for all variables* x, f *and* g *the formula*

$$\text{fix}(f)\downarrow \wedge (\text{fix}(f) = g \to gx \simeq f(g, x)).$$

The next lemma is available in OST as well as in OST_0^- and is proved as in Feferman [13] and [14]. The proof is also elaborated in Zumbrunnen [34].

Lemma 1.31. *If* $A[\vec{u}]$ *is a* Δ_0 *formula of* $\mathcal{L}_\in^\circ$ *with at most the variables* $\vec{u} = u_0, ..., u_{n-1}$ *free, then there exists a closed* $\mathcal{L}_\in^\circ$ *term* t_A *such that* OST_0^- *proves the formula*

$$t_A\downarrow \wedge (t_A : \mathbf{V}^n \to \mathbf{B}) \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \mathsf{t}).$$

Now we present some facts about OST (and extensions). Later we will show some properties of OST^- which we will use later.

It is easy to verify that $\lambda x.\mathbb{C}(\lambda y.\mathsf{el}(y, x))$ is a global choice operator in OST. Therefore it is also easy to prove the next proposition (as done in Feferman [14]).

Proposition 1.32. *The theory* OST *proves the axiom of choice.*

The consistency strength of OST is the consistency strength of KP. Different proofs of this fact can be found in Feferman [13] and [14] as well as in Jäger [18].

Theorem 1.33. *The theories* OST *and* KP *prove the same absolute sentences of* $\mathcal{L}_\in$*.*

In Jäger [18] also the strength of $\mathsf{OST} + (\mathbb{P})$ is analysed and the assertions formulated in the next theorem are proved there.

Theorem 1.34. *The theory* $\mathsf{KP} + (\mathcal{P})$ *can be embedded into* $\mathsf{OST} + (\mathbb{P})$ *and the latter can be embedded into* $\mathsf{KP} + (\mathcal{P}) + (\mathbf{V}{=}\mathbf{L})$*.*

The result in the previous theorem gives us interesting lower and upper bounds of $\mathsf{KP} + (\mathcal{P})$. Since $\mathsf{KP} + (\mathcal{P}) + (\mathbf{V}{=}\mathbf{L})$ is strictly stronger than $\mathsf{KP} + (\mathcal{P})$ (c.f. Theorem 6.47 in Mathias [30]), this result does not resolve the exact strength of $\mathsf{OST} + (\mathbb{P})$.

Now we turn our attention to OST^- (and OST_0^- and OST_ω^-). In the next two propositions we will introduce closed terms which act as specific operations which we will use later. Analogous assertions, but only for the theory OST, are proved in Feferman [13, 14]. In some of the proofs of these analogous assertions the choice operation $\mathbb{C}$ is used. We will see that it also works without ($\mathbb{C}$). The next two propositions and their proofs are taken from Sato and Zumbrunnen [31].

Proposition 1.35. *There are closed $\mathcal{L}_{\in}^{\circ}$ terms* p, p_0 *and* p_1 *such that* OST_0^- *proves*

(i) $\forall x, y(\mathrm{p}(x, y) = \langle x, y\rangle)$,

(ii) $\forall x, y, z(z = \langle x, y\rangle \to (\mathrm{p}_0(z) = x \wedge \mathrm{p}_1(z) = y))$.

PROOF. Since we regard ordered pairs as Kuratowski pairs, p can be defined as the term $\lambda xy.\mathbb{D}(\mathbb{D}(x, x), \mathbb{D}(x, y))$. For defining p_0 let $A[v, z]$ be the formula $v \in (z)_0$ and let t_A be the corresponding term due to Lemma 1.31. Then we define p_0 as the term $\lambda z.\mathbb{S}(\lambda v.t_A, \mathbb{U}(\mathbb{U}(z)))$ and we get the stated property by extensionality. The term p_1 can be defined analogously. □

The next proposition tells us that we can define closed terms for OST_0^- for creating domains and ranges of relations as well as the Cartesian product of any two sets. Furthermore there are closed terms for OST_0^- for translating set-theoretic functions to operations and vice versa.

Proposition 1.36. *There exist closed $\mathcal{L}_{\in}^{\circ}$ terms* dom, ran, op, prod *and* fun *such that* OST_0^- *proves the following assertions:*

(i) $\mathrm{dom}(f)\downarrow \wedge \mathrm{ran}(f)\downarrow \wedge \mathrm{op}(f)\downarrow$,

(ii) $\mathrm{Rel}[a] \to (\mathrm{Dom}[a, \mathrm{dom}(a)] \wedge \mathrm{Ran}[a, \mathrm{ran}(a)])$,

(iii) $(\mathrm{Fun}[f] \wedge y \in \mathrm{dom}(f)) \to f'y = \mathrm{op}(f, y)$,

(iv) $\forall x \forall y(\mathrm{prod}(x, y) = \{\langle v, w\rangle : v \in x \wedge w \in y\})$ *(i.e.* $\mathrm{prod}(x, y)$ *is* $x \times y$*) and*

(v) $\forall f((f : a \to \mathbf{V}) \to$
$$\mathrm{Fun}[\mathrm{fun}(f, a)] \wedge \mathrm{Dom}[\mathrm{fun}(f, a), a] \wedge (\forall x \in a)(\mathrm{fun}(f, a)'x = fx)).$$

PROOF. That there are closed terms dom and ran with the stated properties can be proved as the corresponding assertion of Lemma 4 in Feferman [14] (the choice operation is not necessary if we can use $\mathbb{U}$). For constructing op without $\mathbb{C}$, let $A[x, y, a, f]$ be the Δ_0 formula

$$(\exists z \in a)(f'y = z \wedge x \in z)$$

and t_A the corresponding term due to Lemma 1.31. Then OST_0^- proves for any set theoretic function f with y in its domain that $x \in f'y$ is equivalent to $t_A(x, y, \mathrm{ran}(f), f) = \mathsf{t}$ and by extensionality

$$f'y = \{x \in \mathbb{U}(\mathrm{ran}(f)) : t_A(x, y, \mathrm{ran}(f), f) = \mathsf{t}\}.$$

So it proves the stated properties for

$$\mathrm{op} := \lambda fy.\mathbb{S}(\lambda x.t_A(x, y, \mathrm{ran}(f), f), \mathbb{U}(\mathrm{ran}(f))).$$

The closed terms prod and fun can be defined as in the proofs of Lemma 3 ($\mathbb{C}$ is not used if our p has not any occurrence of $\mathbb{C}$) and Lemma 5 ($\mathbb{C}$ is not used if our prod has not any occurrence of $\mathbb{C}$), respectively, in Feferman [14]. □

We introduce the well known λ-terms $\overline{0} := \lambda fx.x$, $\overline{1} := \lambda fx.fx$ and iszero $:= \lambda xyz.x(\lambda u.z)y$. Easy computations show that the next lemma holds.

Lemma 1.37. *The applicative axioms of* OST_0^- *prove for all* y, z *that*

$$\mathrm{iszero}(\overline{0}, y, z) = y \text{ and } \mathrm{iszero}(\overline{1}, y, z) = z.$$

We will use the next lemma, in which we introduce a term ite_A corresponding to a if–then–else statement, later. It and its proofs are as in Sato and Zumbrunnen [31].

Lemma 1.38. *Let* $\vec{x}$ *be a sequence of variables* $x_0, ..., x_{n-1}$. *For every* Δ_0 *formula* $A[\vec{x}]$ *of* $\mathcal{L}^\circ_\in$ *with at most the variables* $\vec{x}$ *free there exists a closed* $\mathcal{L}^\circ_\in$ *term* ite_A *such that* OST_0^- *proves for any* y *and* z

$$(\mathrm{ite}_A(y, z) : \mathbf{V}^n \to \{y, z\}) \wedge \\ ((A[\vec{x}] \to \mathrm{ite}_A(y, z, \vec{x}) = y) \ \wedge \ (\neg A[\vec{x}] \to \mathrm{ite}_A(y, z, \vec{x}) = z)).$$

Proof. Let $A[\vec{x}]$ be a Δ_0 formula of $\mathcal{L}^\circ_\in$ with at most the variables $\vec{x}$ free, $B[u, v, w, \vec{x}]$ the Δ_0 formula

$$(u = v \wedge A[\vec{x}]) \vee (u = w \wedge \neg A[\vec{x}])$$

and t_B the term due to Lemma 1.31. Then it is easy to check that if ite_A is the term $\lambda yz\vec{x}.\mathrm{iszero}(\mathbb{U}(\mathbb{S}(\lambda u.t_B(u, \overline{0}, \overline{1}, \vec{x}), \mathbb{D}(\overline{0}, \overline{1}))), y, z)$, it has the stated properties. □

1.3 Explicit Mathematics

Explicit mathematics has been introduced in Feferman [10] and also studied in Feferman [11, 12]. We do not work here with Feferman's original formalisation of explicit mathematics. We use the formalisation in the context of theories of types and names as developed in Jäger [17] and used in many other papers as for instance in Feferman and Jäger [15], Jäger, Kahle and Studer [22], Jäger and Strahm [24] or Jäger and Studer [25]. In such theories individual objects might be names of types. For our purpose it is enough to introduce very briefly only the base theory EET.

The language of EET is the second order language $\mathbb{L}$ about individuals and types. In $\mathbb{L}$ we have the individual variables $a, b, c, f, x, y, z, ...$ as well as the type variables $S, U, V, X, Z, ...$ (both possibly with subscripts). Furthermore $\mathbb{L}$ includes the individual constants k, s (combinators), p, p_0, p_1 (pairing and projections), 0 (zero), $\mathsf{s_N}$, $\mathsf{p_N}$ (successor and predecessor), $\mathsf{d_N}$ (definition by cases), nat (a name of the natural numbers), id, co, int, dom (for creating names of the identity type, of complements, intersections and domains) as well as inv (for creating names of inverse images). As the language of operational set theory also $\mathbb{L}$ contains the binary function symbol $\circ$ for term application and $\downarrow$ for expressing definedness. In addition $\mathbb{L}$ has the unary relation symbol N (natural numbers) and the binary relation symbols $=$, $\in$ and $\Re$ (equality, membership and naming).

The *individual terms* (r,s,t,...) of $\mathbb{L}$ are built up as in the language $\mathcal{L}^\circ$ of operational set theory. The abbreviations w.r.t. term application are defined as for $\mathcal{L}^\circ$ terms (see subsection 1.2.1).

The atomic formulas of $\mathbb{L}$ are the formulas of the type $s\downarrow$, $\mathsf{N}(s)$, $s = t$, $s \in U$ and $\Re(s, U)$ for any individual terms t and s and any type variable U. The *formulas* of $\mathbb{L}$ are then built up as usual.

We will use the abbreviations $s \simeq t$ (as in the language $\mathcal{L}^\circ$ of operational set theory, see p. 21),

$$\begin{array}{rll} s \mathrel{\dot\in} t & \text{for the formula} & \exists X(\Re(t, X) \wedge s \in X), \\ U = V & \text{for the formula} & \forall x(x \in U \leftrightarrow x \in V), \\ \Re(s) & \text{for the formula} & \exists X \Re(s, X) \\ \text{and } \Re(\vec{s}, \vec{U}) & \text{for the formula} & \Re(s_0, U_0) \wedge ... \wedge \Re(s_n, U_n), \end{array}$$

for individual terms s, t, $\vec{s} = s_0, ..., s_n$ and type variables X as well as $\vec{S} = S_0, ..., S_n$. We will often write 1 for $\mathsf{s_N}0$ and (s, t) for the term $\mathsf{p}(s, t)$ for arbitrary terms s and t. Furthermore, if $A[x]$ is an $\mathbb{L}$ formula, we will often informally write $\{x : A[x]\}$ for the type containing exactly those individuals x, for which $A[x]$ holds, if this type exists.

The logic of EET is (as the logic of operational set theory) the (classical) *logic of partial terms* due to Beeson [3] including the common equality axioms for $=$.

The non-logical axioms of EET consists of the following three groups.

Applicative axioms

(1) $\mathsf{k}ab = a$,

(2) $\mathsf{s}ab\downarrow \wedge\, \mathsf{s}abc \simeq (ac)(bc)$,

(3) $\mathsf{p}_0(a, b) = a \wedge \mathsf{p}_1(a, b) = b$,

(4) $\mathsf{N}(0) \wedge \forall x(\mathsf{N}(x) \to \mathsf{N}(\mathsf{s_N}x))$,

(5) $\forall x(\mathsf{N}(x) \rightarrow \mathsf{s_N}x \neq 0 \wedge \mathsf{p_N}(\mathsf{s_N}x) = x)$,

(6) $\forall x(\mathsf{N}(x) \wedge x \neq 0 \rightarrow \mathsf{N}(\mathsf{p_N}x) \wedge \mathsf{s_N}(\mathsf{p_N}x) = x)$,

(7) $\mathsf{N}(a) \wedge \mathsf{N}(b) \wedge a = b \rightarrow \mathsf{d_N}xyab = x$,

(8) $\mathsf{N}(a) \wedge \mathsf{N}(b) \wedge a \neq b \rightarrow \mathsf{d_N}xyab = y$.

Explicit representation and equality

(1) $\exists x \Re(x, U)$,

(2) $\Re(a, U) \wedge \Re(a, V) \rightarrow U = V$,

(3) $U = V \wedge \Re(a, U) \rightarrow \Re(a, V)$.

Basic type existence axioms

(1) $\Re(\mathsf{nat}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{nat} \leftrightarrow \mathsf{N}(x))$,

(2) $\Re(\mathsf{id}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{id} \leftrightarrow \exists y(x = \mathsf{p}(y, y)))$,

(3) $\Re(a) \rightarrow \Re(\mathsf{co}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{co}(a) \leftrightarrow \neg x \mathrel{\dot\in} a)$,

(4) $\Re(a) \wedge \Re(b) \rightarrow \Re(\mathsf{int}(a, b)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{int}(a, b) \leftrightarrow x \mathrel{\dot\in} a \wedge x \mathrel{\dot\in} b)$,

(5) $\Re(a) \rightarrow \Re(\mathsf{dom}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{dom}(a) \leftrightarrow \exists y(\mathsf{p}(x, y) \mathrel{\dot\in} a))$,

(6) $\Re(a) \rightarrow \Re(\mathsf{inv}(a, f)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{inv}(a, f) \leftrightarrow fx \mathrel{\dot\in} a)$.

The first two applicative axioms imply that λ-abstraction and a form of the recursion theorem are available in EET (i. e. the Lemmas 1.29 and 1.30 are also valid for EET).

We call an $\mathbb{L}$ formula *elementary* if it does not contain the relation symbol $\Re$ nor bounded type variables. As stated in Feferman and Jäger [15], the following uniform comprehension principle for elementary formulas is available in EET:

Proposition 1.39 (Elementary comprehension)**.** *Let $A[\vec{z}, \vec{Z}]$ be an elementary $\mathbb{L}$ formula with no individual variables other than $\vec{z} = z_0, ..., z_{m+1}$ and no type variables other than $\vec{Z} = Z_0, ..., Z_n$. Then there exists a closed individual term t_A of $\mathbb{L}$, so that* EET *proves for all $\vec{a} = a_0, ..., a_m$, $\vec{b} = b_0, ..., b_n$ and $\vec{S} = S_0, ..., S_n$:*

(i) $\Re(\vec{b}, \vec{S}) \rightarrow \Re(t_A(\vec{a}, \vec{b}))$ *and*

(ii) $\Re(\vec{b}, \vec{S}) \rightarrow \forall x(x \mathrel{\dot\in} t_A(\vec{a}, \vec{b}) \leftrightarrow A[x, \vec{a}, \vec{S}])$.

2 Interpreting Pure Set Theories in Pure Set Theories

2.1 Interpreting ZFC^- in $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ and the latter in KPS

In this section we will see that L_κ is for any functionally regular ordinal κ, provably in $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$, a model of ZFC^-. Therefore $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ is stronger than ZFC^-. Furthermore we will see that $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ is a conservative extension of KPS for Σ sentences. We will follow in this section Jäger and Zumbrunnen [26], all lemmas and theorems in this section are in a similar form also presented there.

We start the section with a lemma, whose proof is straightforward.

Lemma 2.1. KP *proves that any functionally regular ordinal is a limit ordinal.*

The next lemma can be proved as for example in Krivine [28] or Kunen [29] (all the arguments used there are also available in $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$). We will present essentially the same proof, but elaborated in operational set theory, later in section 3.2 (c.f. the proof of Lemma 3.15).

Lemma 2.2. $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves that there is for every ordinal $\alpha \geq \omega$ a bijection between α and L_α.*

The same assertion which is provable in $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ according to the previous lemma can also be proved in KP alone. This follows from Lemma 6.8 in chapter II of Devlin [9]. However, for our purpose the formulation in our lemma is strong enough.

We prove the next two lemmas exactly as in Jäger and Zumbrunnen [26].

Lemma 2.3. $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves that* $\mathrm{Frg}[\kappa]$ *implies for all a that*

$$a \in L_\kappa \wedge \mathrm{Fun}[f] \wedge \mathrm{Dom}[f, a] \wedge \mathrm{Ran}_\subseteq[f, L_\kappa] \to (\exists b \in L_\kappa)\mathrm{Ran}_\subseteq[f, b].$$

Proof. Assume $a \in L_\kappa$, $\mathrm{Fun}[f]$, $\mathrm{Dom}[f, a]$ $\mathrm{Ran}_\subseteq[f, L_\kappa]$ where κ is functionally regular ordinal. Then there is an α with $\omega < \alpha < \kappa$ and $a \subseteq L_\alpha$. Let

g be some bijection between α and L_α (which exists by the previous lemma). Furthermore let h be the function from α to κ given by

$$h'\xi = \begin{cases} \mathrm{od}(f'(g'\xi)) & \text{if } g'\xi \in a, \\ 0 & \text{else} \end{cases}$$

for all $\xi < \alpha$. It is clear that h exists because $\mathrm{od}(x) = \eta$ is Δ w.r.t. KP and we can therefore apply Δ separation to $\alpha \times \kappa$. Since κ is functionally regular and $\alpha < \kappa$, there must be a $\beta < \kappa$ with $\mathrm{Ran}_\subseteq[h, \beta]$. By the definition of od it follows that $\mathrm{Ran}_\subseteq[f, L_\beta]$. $\square$

Lemma 2.4. *Let $A[\vec{u}, v, w]$ be an $\mathcal{L}_\in$ formula with at most the variables $\vec{u}, v, w$ free. Then* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves*

$$\begin{aligned} &\mathrm{Frg}[\kappa] \wedge a, \vec{u} \in L_\kappa \to \\ &\qquad (\exists b \in L_\kappa)(\forall x \in a)(\ (\exists y \in L_\kappa)A^{L_\kappa}[\vec{u}, x, y] \leftrightarrow (\exists y \in b)A^{L_\kappa}[\vec{u}, x, y]\ ). \end{aligned}$$

Proof. For a non empty set u we write $\min_{\mathbf{L}}(u) = z$ for a Δ formula expressing that z is the least element of u w.r.t. the well ordering $<_{\mathbf{L}}$. Assume $\mathrm{Frg}[\kappa]$ and $a, \vec{u} \in L_\kappa$. Let f be the function from a to L_κ given by

$$f'x = \begin{cases} \min_{\mathbf{L}}(\{z \in \mathbf{L}_\kappa : A^{L_\kappa}[\vec{u}, x, z]\}) & \text{if } (\exists y \in L_\kappa)A^{L_\kappa}[\vec{u}, x, y], \\ \emptyset & \text{else} \end{cases}$$

for all $x \in a$. The function f exists because we can apply Δ separation to $\alpha \times \kappa$. Since κ is functionally regular, there is by the previous lemma a set $b \in L_\kappa$ with $\mathrm{Ran}_\subseteq[f, b]$. Therefore we get

$$(\forall x \in a)((\exists y \in L_\kappa)A^{L_\kappa}[\vec{u}, x, y] \leftrightarrow (\exists y \in b)A^{L_\kappa}[\vec{u}, x, y]).$$

The other direction follows directly because $b \subseteq L_\kappa$. $\square$

The next lemma implies that separation for arbitrary formulas is, provably in $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$, available in L_κ for each functionally regular ordinal κ. In its formulation and its proof we write $\langle\vec{x}\rangle \in \times(\vec{a})$ as an abbreviation for $\langle x_0, ..., x_n\rangle \in a_0 \times ... \times a_n$, if $\vec{x}$ and $\vec{a}$ are the sequences of variables $x_0, ..., x_n$ and $a_0, ..., a_n$, respectively. We prove the lemma exactly as in Jäger and Zumbrunnen [26].

Lemma 2.5. *Let $A[\vec{u}, \vec{v}]$ be an $\mathcal{L}_\in$ formula with at most the variables $\vec{u}, \vec{v}$ free. Then* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves*

$$\mathrm{Frg}[\kappa] \to (\forall\vec{a} \in L_\kappa)(\forall\vec{x} \in L_\kappa)(\exists b \in L_\kappa)(b = \{\langle\vec{y}\rangle \in \times(\vec{a}) : A^{L_\kappa}[\vec{x}, \vec{y}]\}).$$

Proof. We proof the assertion by induction on the length of $A[\vec{u}, \vec{v}]$. Assume $\mathrm{Frg}[\kappa]$. Then $\kappa > \omega$ is by Lemma 2.1 a limit ordinal. Therefore the assertion is obviously true if A is an atomic formula and if A is the result of connecting shorter formulas with a connective, the assertion follows directly from the induction hypothesis.

Assume $A[\vec{u}, v]$ is a formula of the form $\exists z B[\vec{u}, v, z]$. Furthermore assume $\vec{a}, \vec{x} \in L_\kappa$. By the previous lemma there is a $c \in L_\kappa$ such that

$$(\forall \vec{y} \in \vec{a})((\exists z \in L_\kappa) B^{L_\kappa}[\vec{x}, \vec{y}, z] \leftrightarrow (\exists z \in c) B^{L_\kappa}[\vec{x}, \vec{y}, z]). \tag{2.1}$$

By induction hypothesis there is a $b_0 \in L_\kappa$ such that

$$b_0 \;=\; \{\langle \vec{y}, z\rangle \in \times(\vec{a}, c) : B^{L_\kappa}[\vec{x}, \vec{y}, z]\}.$$

Now we can define by Δ_0 separation the set

$$b \;:=\; \{\langle \vec{y}\rangle \in \times(\vec{a}) : (\exists z \in c)(\langle \vec{y}, z\rangle \in b_0)\}.$$

Since κ is a limit ordinal and $\vec{a}, \vec{x}, b_0 \in L_\kappa$, it follows $b \in L_\kappa$. By (2.1) we have that

$$\begin{array}{rcl} \langle \vec{y}\rangle \in b & \leftrightarrow & \vec{y} \in \vec{a} \wedge (\exists z \in c) B^{L_\kappa}[\vec{x}, \vec{y}, z] \\ & \leftrightarrow & \vec{y} \in \vec{a} \wedge (\exists z \in L_\kappa) B^{L_\kappa}[\vec{x}, \vec{y}, z] \\ & \leftrightarrow & \vec{y} \in \vec{a} \wedge A^{L_\kappa}[\vec{x}, \vec{y}], \end{array}$$

and so b has all required properties.

If A is a formula of the form $\forall z B$ it is equivalent to $\neg \exists z \neg B$. The assertion follows therefore from the other cases. □

Theorem 2.6. *If A is the universal closure of an axiom of* ZFC^-*, then* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves that* $\mathrm{Frg}[\kappa]$ *implies* A^{L_κ} *I.e.* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves that L_κ is a standard model of* ZFC^- *if κ is functionally regular.*

Proof. Assume $\mathrm{Frg}[\kappa]$. By Lemma 2.1 κ is a limit ordinal with $\kappa > \omega$. So if A is the universal closure of the axiom extensionality, pairing, union, infinity or an instance of $\in$-induction, A^{L_κ} is clearly provable in KP. If A is the universal closure of an instance of separation, the assertion follows directly from the previous lemma.

Assume that A is the universal closure of an instance of replacement. Then A^{L_κ} has the form

$$\begin{array}{l} (\forall w, \vec{z} \in L_\kappa)(\; (\forall x \in w)(\exists! y \in L_\kappa) B^{L_\kappa}[x, y, w, \vec{z}] \rightarrow \\ \qquad\qquad (\exists v \in L_\kappa)(\forall x \in w)(\exists y \in v) B^{L_\kappa}[x, y, w, \vec{z}]\;), \end{array}$$

where in $B[x, y, w, \vec{z}]$ at most the variables $x, y, w, \vec{z}$ occur freely. By Δ separation we can define the set

$$f \;:=\; \{\langle x, y\rangle \in w \times L_\kappa : B^{L_\kappa}[x, y, w, \vec{z}]\}.$$

If we assume $w \in L_\kappa$ and

$$(\forall x \in w)(\exists! y \in L_\kappa) B^{L_\kappa}[x, y, w, \vec{z}],$$

then we have $\mathrm{Fun}[f]$, $\mathrm{Dom}[f, w]$, $\mathrm{Ran}_\subseteq[f, L_\kappa]$ and therefore there exists by Lemma 2.3 a $v \in L_\kappa$ with $\mathrm{Ran}_\subseteq[f, v]$. It follows

$$(\forall x \in w)(\exists y \in v) B^{L_\kappa}[x, y, w, \vec{z}]$$

by the definition of f.

If $a \in L_\kappa$ is a non-empty set, we can define a choice function with domain a as the set

$$f \;:=\; \{\langle x, y\rangle \in a \times \cup a : y \in x \wedge (\forall z \in x)(z \neq y \to y <_{\mathbf{L}} z)\}.$$

By the previous lemma we have $f \in L_\kappa$. It follows that if A is the universal closure of the axiom of choice, then A^{L_κ} is provable in $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$. □

Remark 1. Assume that $\mathcal{T}$ is any extension of $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ which proves the existence of some functionally regular ordinal. The previous theorem tells us that we can interpret ZFC^- in $\mathcal{T}$ in an obvious way. By Gödel's results it even tells us that $\mathcal{T}$ is stronger than ZFC^-. The theory $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ is an example of such an extension $\mathcal{T}$.

We know now that $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ is strictly stronger than ZFC^-. We end up this section by proving that $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ is proof-theoretically not stronger than KPS. The latter is therefore strictly stronger than ZFC^- too. We prove the next lemma exactly as the corresponding theorem in Jäger and Zumbrunnen [26].

Lemma 2.7. *If A is the universal closure of an axiom of* $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$*, then* KPS *proves A^L.*

Proof. It is well known that KP proves A^L, if A is the universal closure of any axiom of KP or the universal closure of the axiom $(\mathbf{V}{=}\mathbf{L})$ (c.f. Theorem 5.5 of Chapter II in Part A of Barwise [2] and Lemma 2.9 in Chapter II of Devlin [9]). So let A be the axiom (SLim). A^L is therefore

$$(\forall \xi \in \mathbf{L})(\exists \eta \in \mathbf{L})(\xi < \eta \wedge \mathrm{Frg}^{\mathbf{L}}[\eta]).$$

Let $\xi \in \mathbf{L}$ be an arbitrary ordinal. By (SLim) there is a functionally regular $\eta > \xi$. Clearly we have $\eta \in \mathbf{L}$. And trivially $\mathrm{Frg}^{\mathbf{L}}[\eta]$ holds. Hence KPS proves $A^{\mathbf{L}}$. □

Now we can use a well known argument to prove the next theorem.

Theorem 2.8. *The theories* KPS *and* KPS + ($\mathbf{V}$=$\mathbf{L}$) *prove the same* Σ *sentences.*

Proof. Clearly KPS + ($\mathbf{V}$=$\mathbf{L}$) proves every formula which is provable in KPS. If a Σ sentence A is provable in KPS + ($\mathbf{V}$=$\mathbf{L}$), then A^L is provable in KPS by the previous lemma. Therefore A is also provable in KPS by Σ persistency (c.f. for instance Lemma 4.2 of Chapter I in Part A of Barwise [2]). □

2.2 Interpreting KP in KP^{int}

In the end, we will not be interested directly in the theory KP^{int}, but we will use it as an intermediate theory. The method for interpreting KP (and some extensions) in KP^{int} (and some extensions) we use in this section is presented in Avigad [1]. The formulations of all definitions, lemmas, theorems and proofs as well as some comments in this section are taken from Sato and Zumbrunnen [31].

In this section as well as in the next two sections we will work with theories without extensionality. We will nevertheless use the abbreviations $x = \{y_0, ..., y_n\}$ and $x = \cup y$, although the x in this abbreviations might not be unique. In the context of non-extensional theories this abbreviations mean "x is some set containing exactly the sets y_0,...,y_n" and "x is some set corresponding to the union of y", respectively.

Definition 2.9 ($\sim_a$, field(a), Bis, $\sim$ and $\in^*$)**.** By $y \sim_a z$ we denote the Δ_0 formula $\{y, z\} \in a$, by $y \in \mathrm{field}(a)$ the Δ_0 formula

$$(\exists x \in a)(\exists z \in x)(x = \{y, z\}),$$

and by Bis[a] the formula

$$\begin{aligned}&\forall y, z(y, z \in \mathrm{field}(a) \to\\ &\qquad (y \sim_a z \ \leftrightarrow\ (\forall u \in y)(\exists v \in z)(u \sim_a v) \land (\forall v \in z)(\exists u \in y)(v \sim_a u))).\end{aligned}$$

Finally we write $y \sim z$ for the Σ_1 formula $\exists a(\mathrm{Bis}[a] \land y \sim_a z)$ and $y \in^* a$ for the formula $(\exists z \in a)(y \sim z)$.

Sets a with Bis[a] are called *bisimulations*. The next lemma can be proved exactly as Lemma 4.5 in Avigad [1].

Lemma 2.10. KP^{int} *proves that*

$$y \sim z \text{ is equivalent to } \forall a(\mathrm{Bis}[a] \land y \in \mathit{field}(a) \land z \in \mathit{field}(a) \to y \sim_a z),$$

and, more general,

$$(\forall y, z \in b)(\ y \sim z \ \leftrightarrow\ \forall a(\mathrm{Bis}[a] \wedge b \subseteq \mathit{field}(a) \rightarrow y \sim_a z)\ ).$$

In the next definition we want to designate relations for which $\sim$ behaves as an equality relation. The simplest way to do so would be to designate only the relations R for which certain theory proves:

$$\bigwedge_{i=0}^{n-1}(x_i \sim x_i') \ \rightarrow\ (R(x_0, ..., x_{n-1}) \leftrightarrow R(x_0', ..., x_{n-1}')). \tag{2.2}$$

But if we did so, we would not designate the power set relation $\mathcal{P}$: assume $z \neq z'$ but $z \sim z'$. Then $\mathsf{KP}^{int} + (\mathcal{P})$ proves for some y that $\mathcal{P}(\{z\}, y)$ and $\neg\mathcal{P}(\{z'\}, y)$ (because $\{z'\}$ is not a subset of $\{z\}$), whereas it proves $\{z\} \sim \{z'\}$ as well as $y \sim y$. In order to also designate the power set relation, we chose the rather complicated way in the next definition. Notice that in the case $n > 0$ in this definition we demand special properties from the last position of the relation. Just as well we could demand these properties from any other position of the relation (but then we would also have to adjust the case (iii) of Definition 2.12).

Definition 2.11 (Suitable relation symbols and formulas for $*$-translation)**.** Let R be an $(n{+}1)$-ary relation symbol of $\mathcal{L}$ and $\vec{x} = x_0, ..., x_{n-1}$; $\vec{x}' = x_0', ..., x_{n-1}'$; y and y' variables. We call R *suitable for $*$-translation* with respect to a the theory $\mathcal{T}$,

- if either $n = 0$ (and therefore R is a unary relation symbol) and $\mathcal{T}$ proves
$$\forall y, y'(y \sim y' \ \rightarrow\ (R(y) \leftrightarrow R(y'))),$$
- or if $n > 0$ and $\mathcal{T}$ proves
  (i) $\exists y'(y \sim y' \wedge R(\vec{x}, y')) \ \leftrightarrow \forall y'(R(\vec{x}, y') \rightarrow y \sim y')$ as well as
  (ii) $\bigwedge_{i=0}^{n-1}(x_i \sim x_i') \wedge R(\vec{x}, y) \ \rightarrow\ \exists y'(y \sim y' \wedge R(\vec{x}', y')).$

We call a formula *suitable for $*$-translation* with respect to a the theory $\mathcal{T}$, if it contains at most the relation symbols $=$, $\in$ and relation symbols which are suitable for $*$-translation with respect to $\mathcal{T}$.

Notice that an $(n{+}1)$-ary relation which is suitable for $*$-translation can be seen as graph of an n-ary function, if $n > 0$. We can consider each relation, which fulfills condition (2.2) above, as suitable for $*$-translation.

Remark 2. It is straightforward to check that for instance the symbol $\mathcal{P}$ is suitable for $*$-translation with respect to $\mathsf{KP}^{int} + \mathcal{P}$, respectively.

Definition 2.12 (Formula A^*)**.** For any $\mathcal{L}$ formula A we write A^* for the $\mathcal{L}$ formula which we get if we replace in A

(i) every occurrence of the form $x = y$ by $x \sim y$,

(ii) every occurrence of the form $x \in y$ by $x \in^* y$ and

(iii) every occurrence of the form $R(\vec{x}, y)$ by $\exists y'(y \sim y' \land R(\vec{x}, y'))$ for any relation symbol R of arity two or more other than $=$ and $\in$, where $\vec{x}$ is a string of variables of the correct length.

Furthermore, if $\mathcal{A}$ is a set of $\mathcal{L}$ formulas, we write $\mathcal{A}^*$ for the set $\{A^* : A \in \mathcal{A}\}$.

The formula $R(x)^*$ stays $R(x)$ for all unary relation symbols R. If R is suitable for $*$-translation w.r.t. $\mathcal{T}$, then Condition (i) of Definition 2.11 guarantees that $R(\vec{x}, y)^*$ is a Δ_1 formula w.r.t. the a theory $\mathcal{T}$. Therefore, the next lemma can be proved as the corresponding assertions are proved in Avigad [1].

Lemma 2.13. *Let $\mathcal{T}$ be a theory containing KP^{int} and let A be a formula of $\mathcal{L}$ which is suitable for $*$-translation with respect to $\mathcal{T}$.*

(i) KP^{int} *proves that*

a) $\forall x \exists y(\mathrm{Bis}[y] \land (\forall z \in x)(z \in \mathit{field}(y)))$,

b) $\sim$ *is an equivalence relation,*

c) $x \sim y \to (A^*[x] \leftrightarrow A^*[y])$,

d) $(\forall x \in^* y)A^* \leftrightarrow (\forall x \in y)A^*$ *and*

e) $(\exists x \in^* y)A^* \leftrightarrow (\exists x \in y)A^*$.

(ii) If A is Δ_0, then A^* is Δ_1 with respect to $\mathcal{T}$.

Proof. We present here the proofs of (i) b) to d).

In view of (i) a), Lemma 2.10 and the definition of $\mathrm{Bis}[a]$, the proof of (i) b) is straightforward by $\in$-induction.

The proof of (i) c) is by induction on the length of $A[x]$. If $A[x]$ is of the form $v = w$, the assertion follows from (i) b). If $A[x]$ is of the form $v \in w$, the assertion follows from (i) b), the definition of $\in^*$ and the definition of $\mathrm{Bis}[w]$. If $A[x]$ is of the form $R(\vec{v}, w)$ for R suitable for $*$-translation, the assertion follows from (i) b), and the definition of $(R(\vec{v}, w))^*$. If $A[x]$ is not atomic, the assertion follows directly from the induction hypothesis.

By (i) b) and the definition of $\in^*$ we know that $x \in y$ implies $x \in^* y$ and hence the direction from left to right of (i) d) holds. For the other direction assume $(\forall x \in y)A^*[x]$ and $x \in^* y$. Therefore there is an $x' \in y$ with $x' \sim x$. So we have $A^*[x']$ and therefore $A^*[x]$ by (i) c). $\square$

The next theorem can be proved as Theorem 4.9 in Avigad [1].

Theorem 2.14. *Let $\mathcal{A}$ be a set of $\mathcal{L}$ formulas and A an $\mathcal{L}$ formula. Assume that all formulas in $\mathcal{A}$ as well as A are suitable for $*$-translation with respect to $\mathsf{KP}^{int} + \mathcal{A}$. Then we have: If A is provable in $\mathsf{KP} + \mathcal{A}$, then A^* is provable in $\mathsf{KP}^{int} + \mathcal{A}^*$.*

Notice that the $*$-translations of the equality axioms are provable if we work only with relation symbols suitable for $*$-translation.

The relation symbol Ad is not suitable for $*$-translation w.r.t. the theory $\mathsf{KP}^{int} + \mathcal{AD}$. If we wanted to interpret KPl in the corresponding theory without extensionality, we could introduce a truth predicate such that we can express $x \vDash \mathsf{KP}$ (x is a modal of KP) in $\mathcal{L}_\in$, and then define a new translation of $\mathcal{L}_{\mathsf{Ad}}$ formulas to $\mathcal{L}_\in$ formulas by replacing $\mathsf{Ad}(x)$ by the formula $(x \vDash \mathsf{KP})^* \wedge \mathrm{Tran}[x]$. If an $\mathcal{L}_{\mathsf{Ad}}$ formula should be provable in $\mathsf{KP} + \mathcal{AD}$, then the translated formula is provable in KP^{int} plus the translated formulas of $\mathcal{AD}$ regarding this new translation.

2.3 Interpreting KP^{int} in $\mathsf{IKP}^\sharp + (MP_{res})$

In this section we make use of the interpretation of KP^{int} in the intermediate theory $\mathsf{IKP}^\sharp + (MP_{res})$, which is presented in Avigad [1]. The formulations of all definitions, remarks, lemmas, theorems and proofs as well as some comments in this section are taken from Sato and Zumbrunnen [31].

Lemma 2.15. *The double-negation interpretation of each axiom of KP^{int} is provable in $\mathsf{IKP}^\sharp + (MP)_{res}$.*

Remark 3. Notice that $\in$-induction$^-$ is only necessary for proving the double-negation interpretation of instances of $\in$-induction.

Proof of Lemma 2.15. Pairing, union, Δ_0 separation and Δ_0 collection can be handled exactly as in the proof of Lemma 5.2 in Avigad [1]. The double-negation interpretation of an instance of $\in$-induction has the form

$$\forall x(\forall y(\neg\neg(y \in x) \to A^N[y]) \to A^N[x]) \to \forall x A^N[x],$$

where A is an arbitrary formula. This instance is by Lemma 1.24 (i) a) equivalent to an instance of $\in$-induction$^-$ since $A^N[x]$ is clearly negative. The double-negation interpretation of infinity follows by N-infinity and the ninth assertion of Lemma 1.23. □

Notice that in the proof above the axiom schema $(MP)_{res}$ is only used for proving the double-negated instances of Δ_0 collection. It is used for concluding $(\forall x \in a)\exists z \neg(\forall y \in z)\neg A^N[x, y]$ from antecedents of double-negation interpreted instances of Δ_0 collection, which have the form $(\forall x \in a)\neg\forall y \neg A^N[x, y]$ (where $A[x, y]$ is Δ_0).

Definition 2.16 (Weak Σ_1 and very weak Σ_1 formulas)**.** The *(very) weak* Σ_1 *formulas* of $\mathcal{L}$ are the formulas of the form $\exists y \neg(\forall x \in y)A[x]$ where $A[x]$ is a (strongly) negative Δ_0 formula without any occurrence of the variable y.

For reasons of simplification we will call a formula itself (very) weak Σ_1, if it is intuitionistically equivalent to some (very) weak Σ_1 formula, . In the next definition we define a class of formulas, for which KP^{int} is conservative over $\mathsf{IKP}^\sharp + (MP_{res})$.

Definition 2.17 ($\mathcal{C}_{res}$)**.** The set $\mathcal{C}_{res}$ of $\mathcal{L}$ formulas is inductively defined as follows:

(i) Every very weak Σ_1 formula is in $\mathcal{C}_{res}$.

(ii) If A and B are in $\mathcal{C}_{res}$, then also $A \wedge B$ is in $\mathcal{C}_{res}$.

(iii) If A and B are in $\mathcal{C}_{res}$, then also $A \to B$ is in $\mathcal{C}_{res}$.

(iv) If A is in $\mathcal{C}_{res}$, then also $\forall x A$ is in $\mathcal{C}_{res}$.

Later, it will be important that A^N is in $\mathcal{C}_{res}$ for an arbitrary formula A of $\mathcal{L}$.

Lemma 2.18. *For each formula A in $\mathcal{C}_{res}$, there is a strongly negative $\mathcal{L}$ formula A' such that*

(i) KP^{int} *proves that A and A' are equivalent, and*

(ii) (MP_{res}) *implies intuitionistically that A and A' are equivalent.*

Proof. The proof is by induction on the length of A. If A is a very weak Σ_1 formula it is of the form $\exists y \neg(\forall x \in y)C[x]$ where $C[x]$ is a strongly negative Δ_0 formula. Let A' be $\neg\forall x C[x]$ which is strongly negative. Then A is by Lemma 1.23 (xi) and $(MP)_{res}$ equivalent to A'. Furthermore $A \to A'$ is classically valid and KP^{int} proves $A' \to A$ (if there is a z for which $A[z]$ does not hold, then KP^{int} proves $\neg(\forall x \in \{z\})A[x]$ for this z). If A is $C \wedge D$, $C \to D$ or $\forall x C$ for C and D in $\mathcal{C}_{res}$, there are by induction hypothesis strongly negative C' and D' for which KP^{int} proves and (MP_{res}) implies that they are equivalent to C and D, respectively. Therefore KP^{int} also proves and (MP_{res}) also implies that A is equivalent to A' if it is $C' \wedge D'$, $C' \to D'$ or $\forall x C'$, respectively (in the last case by Lemma 1.23), which are all strongly negative too. $\square$

Theorem 2.19. *Let $\mathcal{A}$ be a set of $\mathcal{L}$ formulas such that $C \to C^N$ is provable in $\mathsf{IKP}^\sharp + (MP_{res})$ for all $C \in \mathcal{A}$. If the $\mathcal{L}$ formula A is in $\mathcal{C}_{res}$ and provable in $\mathsf{KP}^{int} + \mathcal{A}$, then it is also provable in $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{A}$.*

The proof of the previous theorem is based on the fact, that if a formula A is classically valid, then A^N is intuitionistically valid. Therefore it is necessary to have the restriction on the set of axioms $\mathcal{A}$ in the formulation of the previous theorem. The restriction ensures that every double-negation interpretation of an axiom of $\mathcal{A}$ is provable in $\mathsf{IKP}^\sharp + MP_{res} + \mathcal{A}$. For instance the restriction seems not to be fulfilled if $\mathcal{A}$ is the set of all instances of Δ_0 collection.

Proof of Theorem 2.19. Assume A is in $\mathcal{C}_{res}$ and provable in $\mathsf{KP}^{int} + \mathcal{A}$. Let A' be as in the previous lemma and let B be the conjunction of all non logical axioms occurring in some proof in $\mathsf{KP}^{int} + \mathcal{A}$ of A'. By the deduction theorem it follows that $B \to A'$ and therefore also $B^N \to A'^N$ are classically valid. The latter must be also intuitionistically valid. By Lemma 2.15 and the property of $\mathcal{A}$ we know that B^N is provable in $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{A}$. The stated assertion follows by modus ponens, Lemma 1.23 (xii) and the properties of A'. □

2.4 Interpreting $\mathsf{IKP}^\sharp + (MP_{res})$ in IKP^-

In this section we will use Avigad's forcing method, which is presented in Avigad [1], to embed the intermediate theory $\mathsf{IKP}^\sharp + (MP_{res})$ (and some extensions) in the intermediate theory IKP^- (and some extensions). The formulations of all definitions, lemmas, theorems and proofs as well as some comments are also in this section taken from Sato and Zumbrunnen [31].

In IKP^- (as well as in $\mathsf{IKP}^\sharp$), we have separation only for negative Δ_0 formulas. Since the abbreviations $a = \{a_0, a_1\}$, $a = \cup b$ and $a = b \cup c$ stand all for non-negative formulas, we have no separation principle which allows us to prove that the corresponding sets (unordered pairs and unions) exist (our formulation of the axioms pairing and union only state that some supersets of them exist). Therefore we change our definition of the abbreviations for $a \cup b$, $\cup a$ and $\{a_0, ..., a_n\}$ for this section as follows:

- $a = \{a_0, ..., a_n\}$ is the formula

$$a_0 \in a \wedge ... \wedge a_n \in a \wedge (\forall x \in a)\neg(x \neq a_0 \wedge ... \wedge x \neq a_n),$$

- $a = \cup b$ is the formula

$$(\forall x \in b)(\forall y \in x)(y \in a) \wedge (\forall y \in a)\neg(\forall x \in b)(\neg y \in x) \text{ and}$$

- $a = b \cup c$ is the formula $a = \cup\{b, c\}$.

Due to this redefinition, the axioms pairing, union and Δ_0^- separation assure that IKP^- proves the existence of sets corresponding to $a \cup b$, $\cup a$ and

$\{a_0, ..., a_n\}$. We will also use abbreviations of the form $A[\{a_0, ..., a_n\}]$, $A[\cup a]$ and $A[a \cup b]$ in the sense of the redefined abbreviations in this section .

We will introduce a separate forcing relation in IKP^- for each finite sequence of negative Δ_0 formulas. Later we will assign one of these forcing relations to each proof in $\mathsf{IKP}^\sharp + (MP_{res})$ of some formula. For defining the forcing relations, we need a kind of truth predicates for finite sets of negative Δ_0 formulas of $\mathcal{L}$, which is introduced in the next definition.

Definition 2.20 ($\mathrm{Tr}_{\mathfrak{S}}$)**.** Let $\mathfrak{S}$ be a finite sequence $D_0[z_0, \vec{y}], ..., D_{n-1}[z_0, \vec{y}]$ of negative Δ_0 formulas with at most the variables z_0, $\vec{y} = y_0, ...y_m$ free. Then $\mathrm{Tr}_{\mathfrak{S}}[p, u]$ is a negative Δ_0 formula equivalent to the conjunction of

$$\bigwedge_{i=0}^{n-1} \forall \vec{y}(\langle i, y_0, ..., y_m\rangle \in p \to (\forall z_0 \in u) D_i[z_0, \vec{y}_0])$$

and

$$\bigwedge_{i=n}^{2n-1} \forall z_0 \forall \vec{y}(\langle i, z_0, y_0, ..., y_m\rangle \in p \to D_i[z_0, \vec{y}_0]).$$

For defining the forcing relations, we will use the notation $p \Vdash_{\mathfrak{S}} A$ introduced in the next definition for each $\mathcal{L}$ formula A. In Avigad's words it means that there is a u which is "sufficiently large to witness the fact that A follows from the formulas in p" [1, p. 20].

Definition 2.21 ($\Vdash_{\mathfrak{S}}$)**.** Let $\mathfrak{S}$ be a finite sequence $D_0[z_0, \vec{y}], ..., D_{n-1}[z_0, \vec{y}]$ of negative Δ_0 formulas with at most the variables z_0, $\vec{y}$ free. For any $\mathcal{L}$ formula A we write $p \Vdash_{\mathfrak{S}} A$ for the $\mathcal{L}$ formula $\exists u(\mathrm{Tr}_{\mathfrak{S}}[p, u] \to A)$.

If $B[z_0, \vec{y}]$ is the $(i{+}1)$-th negative Δ_0 formula in the sequence $\mathfrak{S}$ we write for arbitrary variables $\vec{x} = x_0, ..., x_m$ and z

- $p, \forall z_0 B[z_0, \vec{x}] \Vdash_{\mathfrak{S}} A$ for a negative Δ_0 formula expressing that
$$p \cup \{\langle i, x_0, ..., x_m\rangle\} \Vdash_{\mathfrak{S}} A,$$
and
- $p, B[z, \vec{x}] \Vdash_{\mathfrak{S}} A$ for a negative Δ_0 formula expressing that
$$p \cup \{\langle i{+}n, z, x_0, ..., x_m\rangle\} \Vdash_{\mathfrak{S}} A.$$

Furthermore $p, q \Vdash_{\mathfrak{S}} A$ abbreviates a formula which expresses $p \cup q \Vdash_{\mathfrak{S}} A$ and $\Vdash_{\mathfrak{S}} A$ one which expresses $\emptyset \Vdash_{\mathfrak{S}} A$.

It is obvious that the next lemma holds.

Lemma 2.22. *For an arbitrary $\mathcal{L}$ formula A IKP_0^- proves that $a \subseteq b$ implies:*

(i) $\mathrm{Tr}_{\mathfrak{S}}[p,b]$ *implies* $\mathrm{Tr}_{\mathfrak{S}}[p,a]$ *and*

(ii) $\mathrm{Tr}_{\mathfrak{S}}[p,a] \to A$ *implies* $\mathrm{Tr}_{\mathfrak{S}}[p,b] \to A$.

The next lemma is a list of properties of $\Vdash_{\mathfrak{S}}$ which we will use later.

Lemma 2.23. *Let A and B be arbitrary $\mathcal{L}$ formulas, $C[z_0, \vec{y}]$ a negative Δ_0 formula of $\mathcal{L}$ occurring in $\mathfrak{S}$ and D an arbitrary Δ_0 formula of $\mathcal{L}$. Then* IKP_0^- *proves the following:*

(i) If $p \Vdash_{\mathfrak{S}} A$ and $p \subseteq q$ then $q \Vdash_{\mathfrak{S}} A$.

(ii) $C[z, \vec{x}] \Vdash_{\mathfrak{S}} C[z, \vec{x}]$ for all variables $\vec{x} = x_0, ..., x_m$ and z.

(iii) $p \Vdash_{\mathfrak{S}} (A \wedge B)$ iff. $p \Vdash_{\mathfrak{S}} A$ and $p \Vdash_{\mathfrak{S}} B$.

(iv) $p \Vdash_{\mathfrak{S}} (C[z, \vec{x}] \to A)$ iff. $p, C[z, \vec{x}] \Vdash_{\mathfrak{S}} A$ for all variables $\vec{x} = x_0, ..., x_m$ and z.

(v) If $p \Vdash_{\mathfrak{S}} (A \to B)$ and $q \Vdash_{\mathfrak{S}} A$ then $p, q \Vdash_{\mathfrak{S}} B$.

(vi) The following are equivalent (if x is a variable not occurring in the formula $\mathrm{Tr}_{\mathfrak{S}}[p,u]$):

a) $p \Vdash_{\mathfrak{S}} (\forall x \in z) D$,

b) $\forall x (p \Vdash_{\mathfrak{S}} (x \in z \to D))$ and

c) $(\forall x \in z)(p \Vdash_{\mathfrak{S}} D)$.

Proof. The proof works exactly as in Avigad [1]. The first assertion follows directly from Definition 2.21. By the same definition $C[z, \vec{x}] \Vdash_{\mathfrak{S}} C[z, \vec{x}]$ is equivalent to $C[z, \vec{x}] \to C[z, \vec{x}]$.

The direction from left to right of the third assertion follows from the fact that the formula $\mathrm{Tr}_{\mathfrak{S}}[p,a] \to (A \wedge B)$ implies $\mathrm{Tr}_{\mathfrak{S}}[p,a] \to A$ as well as $\mathrm{Tr}_{\mathfrak{S}}[p,a] \to B$. For the converse direction assume $\mathrm{Tr}_{\mathfrak{S}}[p,a] \to A$ and $\mathrm{Tr}_{\mathfrak{S}}[p,b] \to B$. Then it's provable by the previous lemma in IKP_0^- that $\mathrm{Tr}_{\mathfrak{S}}[p,c] \to (A \wedge B)$ if c is $a \cup b$.

By Lemma 1.23 IKP_0^- proves that

$$\mathrm{Tr}_{\mathfrak{S}}[p,a] \to (C[z, \vec{x}] \to A) \text{ is equivalent to } (\mathrm{Tr}_{\mathfrak{S}}[p,a] \wedge C[z, \vec{x}]) \to A.$$

By Definition 2.21 the latter is equivalent to

$$\mathrm{Tr}_{\mathfrak{S}}[p \cup \{\langle i{+}n, z, x_0, ..., x_m \rangle\}, a] \to A$$

if $C[z_0, \vec{y}]$ is the $(i{+}1)$-th formula of $\mathfrak{S}$. Therefore the fourth assertion holds.

By the previous lemma and the first assertion IKP_0^- proves that

$$\mathrm{Tr}_{\mathfrak{S}}[p,a] \to (A \to B) \text{ and } \mathrm{Tr}_{\mathfrak{S}}[q,b] \to A \text{ imply together } \mathrm{Tr}_{\mathfrak{S}}[p \cup q, a \cup b] \to B.$$

Hence the fifth assertion holds.

In the sixth assertion we have by Lemma 1.23 (xiv) and (xv) that a) implies intuitionistically b) and b) implies intuitionistically c), respectively. In the following we work informally within IKP_0^- and prove that c) implies a). So assume

$$(\forall x \in z)\exists u(\mathrm{Tr}_{\mathfrak{S}}[p,u] \to D).$$

By Δ_0 collection it follows that there is a w such that

$$(\forall x \in z)(\exists u \in w)(\mathrm{Tr}_{\mathfrak{S}}[p,u] \to D).$$

If v is $\cup w$, we have that $u \in w$ implies $u \subseteq v$ and hence by the previous lemma

$$(\forall x \in z)(\mathrm{Tr}_{\mathfrak{S}}[p,v] \to D),$$

which is, since x does not occur in $\mathrm{Tr}_{\mathfrak{S}}[p,v]$, by Lemma 1.23 (viii) intuitionistically equivalent to $\mathrm{Tr}_{\mathfrak{S}}[p,v] \to (\forall x \in z)D$. $\square$

Now we are ready to introduce our forcing relations.

Definition 2.24 ($\Vdash_{\mathfrak{S}}$)**.** Let $\mathfrak{S}$ be a finite sequence $D_0[z_0,\vec{y}], ..., D_n[z_0,\vec{y}]$ of negative Δ_0 formulas with at most the variables z_0, $\vec{y}$ free. For an arbitrary $\mathcal{L}$ formula A the $\mathcal{L}$ formula $p \Vdash_{\mathfrak{S}} A$ is defined inductively as follows:

(i) If A is atomic, then $p \Vdash_{\mathfrak{S}} A$ is the formula $p \Vvdash_{\mathfrak{S}} A$.

(ii) If A is $B \wedge C$, then $p \Vdash_{\mathfrak{S}} A$ is the formula $(p \Vdash_{\mathfrak{S}} B) \wedge (p \Vdash_{\mathfrak{S}} C)$.

(iii) If A is $B \vee C$, then $p \Vdash_{\mathfrak{S}} A$ is the formula $(p \Vdash_{\mathfrak{S}} B) \vee (p \Vdash_{\mathfrak{S}} C)$.

(iv) If A is $B \to C$, then $p \Vdash_{\mathfrak{S}} A$ is the formula

$$\forall q(p \subseteq q \to ((q \Vdash_{\mathfrak{S}} B) \to (q \Vdash_{\mathfrak{S}} C))).$$

(v) If A is $\forall x B[x]$ then $p \Vdash_{\mathfrak{S}} A$ is the formula $\forall x_0(p \Vdash_{\mathfrak{S}} B[x_0])$ where x_0 does not appear in $\mathrm{Tr}_{\mathfrak{S}}[p,u]$ nor in $B[x]$.

(vi) If A is $\exists x B[x]$ then $p \Vdash_{\mathfrak{S}} A$ is the formula $\exists x_0(p \Vdash_{\mathfrak{S}} B[x_0])$ where x_0 does not appear in $\mathrm{Tr}_{\mathfrak{S}}[p,u]$ nor in $B[x]$.

The abbreviations $p, \forall z_0 B[z_0,\vec{x}] \Vdash_{\mathfrak{S}} A$; $p, B[z,\vec{x}] \Vdash_{\mathfrak{S}} A$ (for B negative Δ_0); $p,q \Vdash_{\mathfrak{S}} A$ and $\Vdash_{\mathfrak{S}} A$ are defined as for $\Vvdash_{\mathfrak{S}}$ in Definition 2.21.

The next lemma corresponds to Proposition 2.4 in Avigad [1] and its first assertion can be proved as ibidem by induction on the length of the proof of A. We will give another argument below (which is also indicated in [1]).

Lemma 2.25. *For any sequence $\mathfrak{S}$ of negative Δ_0 formulas and all $\mathcal{L}$ formulas A, B and $C_0, ..., C_n$ we have:*

(i) If the theory $\{C_0, ..., C_n\}$ proves A intuitionistically, then the theory IKP_0^- + $\{p \Vdash_{\mathfrak{S}} C_0, ..., p \Vdash_{\mathfrak{S}} C_n\}$ proves $p \Vdash_{\mathfrak{S}} A$ intuitionistically.

(ii) If B is an intuitionistic consequence of A, then IKP_0^- proves

$$p \Vdash_{\mathfrak{S}} A \quad \textit{implies} \quad p \Vdash_{\mathfrak{S}} B.$$

Proof. Consider the following Kripke structure (for the definition of Kripke semantics see for example section 5.3 in [33]): all supersets of p are our possible worlds and they are partially ordered by $\preceq$, where $q_0 \preceq q_1$ holds iff. $q_0 \subseteq q_1$; the universes of each world are in each case the class of all sets; and we define that A is true in the world q, iff. $q \Vdash_{\mathfrak{S}} A$. By the definition of $\Vdash_{\mathfrak{S}}$ and Lemma 2.23 (i), IKP_0^- proves indeed that our structure fulfills all requirements of a Kripke structure. Since we can prove intuitionistically that intuitionistic logic is sound w.r.t. Kripke semantics, we get the first assertion.
The second assertion is a direct consequence of the first one. □

In the next definition we distinguish formulas which have useful properties in association with a given sequence of negative Δ_0 formulas, as we will see.

Definition 2.26 (Prominent formulas)**.** We call an $\mathcal{L}$ formula $A[z_0, \vec{y}]$ *prominent for* $\mathfrak{S}$ if it is a negative Δ_0 formula such that every of its sub-formulas occur in the sequence $\mathfrak{S}$ of formulas with at most the variables $z_0, \vec{y}$ free.

Notice that the formula $A[z_0, \vec{y}]$ in the previous definition occurs itself in $\mathfrak{S}$ and contains therefore no other free variables than the indicated ones.

Lemma 2.27. *Let $A[z_0, \vec{y}]$ be a prominent formula for $\mathfrak{S}$. Then IKP_0^- proves for arbitrary variables $\vec{x} = x_0, ..., x_m$ and z that*

$$p \Vdash_{\mathfrak{S}} A[z, \vec{x}] \textit{ is equivalent to } p \Vvdash_{\mathfrak{S}} A[z, \vec{x}].$$

Proof. The proof is as the proof of Lemma 5.6 in Avigad [1] by induction on the complexity of A. If $A[z, \vec{x}]$ is atomic, the assertion follows by definition. If $A[z, \vec{x}]$ is a formula of the form $B[z, \vec{x}] \wedge C[z, \vec{x}]$ then the assertion follows by Lemma 2.23 (iii) and the induction hypothesis.
If $A[z, \vec{x}]$ is of the form $B[z, \vec{x}] \to C[z, \vec{x}]$ we know by the induction hypothesis that $p \Vdash_{\mathfrak{S}} A[z, \vec{x}]$ is equivalent to

$$\forall q(p \subseteq q \to ((q \Vvdash_{\mathfrak{S}} B[z, \vec{x}]) \to (q \Vvdash_{\mathfrak{S}} C[z, \vec{x}]))). \tag{2.3}$$

We assume for one direction that $p \Vdash_{\mathfrak{S}} A[z, \vec{x}]$ holds. We have clearly $p, B[z, \vec{x}] \Vvdash_{\mathfrak{S}} B[z, \vec{x}]$ (this notation makes sense since $B[z_0, \vec{y}]$ is a sub-formula of the prominent formula $A[z_0, \vec{y}]$) and therefore it follows by (2.3) that

$$p, B[z, \vec{x}] \Vvdash_{\mathfrak{S}} C[z, \vec{x}].$$

We get by Lemma 2.23 (iv)

$$p \Vdash\!\vdash_{\mathfrak{S}} (B[z,\vec{x}] \to C[z,\vec{x}]).$$

For the other direction assume $p \Vdash\!\vdash_{\mathfrak{S}} (B[z,\vec{x}] \to C[z,\vec{x}])$. By Lemma 2.23 (v) we have for an arbitrary q with $q \Vdash\!\vdash_{\mathfrak{S}} B[z,\vec{x}]$ that $p,q \Vdash\!\vdash_{\mathfrak{S}} C[z,\vec{x}]$. Since $q = p \cup q$ if $q \supseteq p$ we get (2.3) and therefore $p \Vdash_{\mathfrak{S}} A[z,\vec{x}]$.

If A is of the form $(\forall v \in a)B[v,z,\vec{x}]$, then $p \Vdash_{\mathfrak{S}} A[z,\vec{x}]$ is the formula

$$\forall v_0 (p \Vdash_{\mathfrak{S}} (v_0 \in a \to B[v_0,z,\vec{x}]))$$

(where v_0 does not occur in $\mathrm{Tr}_{\mathfrak{S}}[p,u]$) which is by induction hypothesis equivalent to

$$\forall v_0 (p \Vdash\!\vdash_{\mathfrak{S}} (v_0 \in a \to B[v_0,z,\vec{x}])).$$

Since v_0 does not appear in $\mathrm{Tr}_{\mathfrak{S}}[p,u]$, this is by Lemma 2.23 (vi) equivalent to $p \Vdash\!\vdash_{\mathfrak{S}} (\forall v_0 \in a)B[v_0,z,\vec{x}]$ and to $p \Vdash\!\vdash_{\mathfrak{S}} A[z,\vec{x}]$. □

The next lemma corresponds to the Lemmas 5.12 and 5.13 in Avigad [1]. We prove it as ibidem.

Lemma 2.28. *Let A be a negative Δ_0 formula, $B[x]$ an arbitrary Δ_0 formula of $\mathcal{L}$ in which y does not occur, $B'[x,y]$ an arbitrary $\mathcal{L}$ formula and $C[x,y]$ any negative $\mathcal{L}$ formula such that every of its atomic sub-formulas D, $x \in y$, as well as $x \in y \to D$ occur in the sequence $\mathfrak{S}$ (modulo renaming variables). Then IKP_0^- proves:*

(i) $A \to \exists x B[x]$ *implies* $\exists y (A \to (\exists x \in y) B[x])$,

(ii) $p \Vdash_{\mathfrak{S}} (x \in y \to B'[x,y])$ *implies* $x \in y \to (p \Vdash_{\mathfrak{S}} B'[x,y])$ *and*

(iii) $x_0 \in x_1 \to (p \Vdash_{\mathfrak{S}} C[x_0,x_1])$ *is equivalent to* $p \Vdash_{\mathfrak{S}} (x_0 \in x_1 \to C[x_0,x_1])$ *for all variables* x_0, x_1 *not occurring in* $\mathrm{Tr}_{\mathfrak{S}}[p,u]$.

Proof. For the first assertion we assume $A \to \exists x B[x]$. There is a set $\emptyset$ containing no elements (we apply Δ_0^- separation to any set and the formula $\bot$) and therefore by pairing and Δ_0^- separation a set $\{\emptyset\}$ containing only the element $\emptyset$ (we apply Δ_0^- separation to a set which contains $\emptyset$ and the formula $(\forall w \in x)\bot$). In the following we assume that w does not appear in A nor in $B[x]$. Again by Δ_0^- separation, there is a set a such that

$$\forall w (w \in a \leftrightarrow (w \in \{\emptyset\} \land A)).$$

It follows that $w \in a$ implies A and therefore we have by our assumption $w \in a \to \exists x B[x]$. It follows $\forall w (w \in a \to \exists x B[x])$ and by Δ_0 collection

$$\exists y \forall w (w \in a \to (\exists x \in y) B[x]).$$

Since A implies $\emptyset \in a$ we can conclude

$$\exists y(A \to (\exists x \in y)B[x]).$$

For the second assertion assume $p \Vdash_{\mathfrak{S}} x \in y \to B'[x,y]$ which is by definition

$$\forall q(p \subseteq q \to ((q \Vdash_{\mathfrak{S}} x \in y) \to (q \Vdash_{\mathfrak{S}} B'[x,y]))).$$

Since $x \in y$ implies $p \Vdash_{\mathfrak{S}} x \in y$ it follows $x \in y \to (p \Vdash_{\mathfrak{S}} B'[x,y])$.

The direction from right to left of the third assertion follows from the second one. The direction from left to right of the third assertion is proved by induction on the complexity of C. When C is atomic we assume

$$x_0 \in x_1 \to (p \Vdash_{\mathfrak{S}} C[x_0,x_1])$$

which is by definition $x_0 \in x_1 \to (p \Vvdash_{\mathfrak{S}} C[x_0,x_1])$. The latter is

$$x_0 \in x_1 \to \exists u(\mathrm{Tr}_{\mathfrak{S}}[p,u] \to C[x_0,x_1]).$$

By the first assertion there is a w such that

$$x_0 \in x_1 \to (\exists u \in w)(\mathrm{Tr}_{\mathfrak{S}}[p,u] \to C[x_0,x_1]).$$

If v is a set containing at least all elements of $\cup w$ it follows by Lemma 2.22 $x_0 \in x_1 \to (\mathrm{Tr}_{\mathfrak{S}}[p,v] \to C[x_0,x_1])$ which is equivalent to

$$\mathrm{Tr}_{\mathfrak{S}}[p,v] \to (x_0 \in x_1 \to C[x_0,x_1])$$

by Lemma 1.23. Hence we have $p \Vvdash_{\mathfrak{S}} (x_0 \in x_1 \to C[x_0,x_1])$ which is by Lemma 2.27 equivalent to $p \Vdash_{\mathfrak{S}} (x_0 \in x_1 \to C[x_0,x_1])$.

If $C[x_0,x_1]$ is of the form $D_0 \wedge D_1$, then $x_0 \in x_1 \to (p \Vdash_{\mathfrak{S}} C[x_0,x_1])$ is

$$x_0 \in x_1 \to ((p \Vdash_{\mathfrak{S}} D_0) \wedge (p \Vdash_{\mathfrak{S}} D_1))$$

which is by Lemma 1.23 equivalent to

$$(x_0 \in x_1 \to (p \Vdash_{\mathfrak{S}} D_0)) \wedge (x_0 \in x_1 \to (p \Vdash_{\mathfrak{S}} D_1)).$$

This implies by induction hypothesis

$$(p \Vdash_{\mathfrak{S}} (x_0 \in x_1 \to D_0)) \wedge (p \Vdash_{\mathfrak{S}} (x_0 \in x_1 \to D_1)).$$

So we have $p \Vdash_{\mathfrak{S}} ((x_0 \in x_1 \to D_0) \wedge (x_0 \in x_1 \to D_1))$ and by the Lemmas 1.23 and 2.25 $p \Vdash_{\mathfrak{S}} (x_0 \in x_1 \to C[x_0,x_1])$.

If $C[x_0,x_1]$ is of the form $D_0 \to D_1$, then $x_0 \in x_1 \to (p \Vdash_{\mathfrak{S}} C[x_0,x_1])$ is

$$x_0 \in x_1 \to \forall q(p \subseteq q \to ((q \Vdash_{\mathfrak{S}} D_0) \to (q \Vdash_{\mathfrak{S}} D_1)))$$

which is by Lemma 1.23 equivalent to

$$\forall q(x_0 \in x_1 \to (p \subseteq q \to ((q \Vdash_{\mathfrak{G}} D_0) \to (q \Vdash_{\mathfrak{G}} D_1))))$$

and this to

$$\forall q(p \subseteq q \to (x_0 \in x_1 \to ((q \Vdash_{\mathfrak{G}} D_0) \to (q \Vdash_{\mathfrak{G}} D_1))))$$

and to

$$\forall q(p \subseteq q \to ((x_0 \in x_1 \to (q \Vdash_{\mathfrak{G}} D_0)) \to (x_0 \in x_1 \to (q \Vdash_{\mathfrak{G}} D_1)))).$$

By induction hypothesis and the direction from right to left it follows

$$\forall q(p \subseteq q \to ((q \Vdash_{\mathfrak{G}} (x_0 \in x_1 \to D_0)) \to (q \Vdash_{\mathfrak{G}} (x_0 \in x_1 \to D_1))))$$

and that is $p \Vdash_{\mathfrak{G}} ((x_0 \in x_1 \to D_0)) \to (x_0 \in x_1 \to D_1))$. Hence

$$p \Vdash_{\mathfrak{G}} x_0 \in x_1 \to C[x_0, x_1]$$

by the Lemmas 1.23 and 2.25.

If $C[x_0, x_1]$ is of the form $\forall z D[z]$, then $x_0 \in x_1 \to (p \Vdash_{\mathfrak{G}} C[x_0, x_1])$ is

$$x_0 \in x_1 \to \forall x_2(p \Vdash_{\mathfrak{G}} D[x_2])$$

(where x_2 does not occur in $\mathrm{Tr}_{\mathfrak{G}}[p, u]$) which is equivalent to

$$\forall x_2(x_0 \in x_1 \to (p \Vdash_{\mathfrak{G}} D[x_2]))$$

by Lemma 1.23. By induction hypothesis it follows

$$\forall x_2(p \Vdash_{\mathfrak{G}} (x_0 \in x_1 \to D[x_2])).$$

That is $p \Vdash_{\mathfrak{G}} \forall z(x_0 \in x_1 \to D[z])$ and by the Lemmas 1.23 and 2.25 we get finally

$$p \Vdash_{\mathfrak{G}} (x_0 \in x_1 \to C[x_0, x_1]). \qquad \square$$

The proof of the next lemma is as the proofs of the Lemmas 5.7 and 5.13 in Avigad [1].

Lemma 2.29. *Let $A[z_0, \vec{y}]$ be a prominent formula for $\mathfrak{G}$, B any negative $\mathcal{L}$ formula and C an arbitrary $\mathcal{L}$ formula. Furthermore let $\vec{x} = x_0, ..., x_m$ and z be arbitrary variables. Then IKP_0^- proves:*

(i) $\Vdash_{\mathfrak{G}} \exists x A[z, \vec{x}]$ *is equivalent to* $\exists x A[z, \vec{x}]$ *for any* $x \in \{x_0, .., x_m, z\}$*,*

(ii) $\forall z_0 A[z_0, \vec{x}] \Vdash_{\mathfrak{G}} \forall z_0 A[z_0, \vec{x}]$*,*

(iii) $p \Vdash_{\mathfrak{S}} \neg\forall z A[z,\vec{x}]$ *implies* $p \Vdash_{\mathfrak{S}} \exists y \neg(\forall z \in y)A[z,\vec{x}]$*, if also the formula* $\neg(\forall x \in y)A[z_0,\vec{y}]$ *is prominent for* $\mathfrak{S}$*,*

(iv) $p \Vdash_{\mathfrak{S}} (\forall x \in y)C[x,y]$ *implies* $(\forall x_0 \in y)(p \Vdash_{\mathfrak{S}} C[x_0,y])$ *where* x_0 *does not occur in* $\mathrm{Tr}_{\mathfrak{S}}[p,u]$ *and* y *is arbitrary,*

(v) $p \Vdash_{\mathfrak{S}} (\forall x \in y)\exists z A$ *implies* $p \Vdash_{\mathfrak{S}} \exists w(\forall x \in y)\neg(\forall z \in w)\neg A$ *if also* $(\forall x \in y)\neg(\forall z \in w)\neg A$ *is prominent for* $\mathfrak{S}$ *and*

(vi) $(\forall x_0 \in x_1)(p \Vdash_{\mathfrak{S}} B[x_0,x_1])$ *is equivalent to* $p \Vdash_{\mathfrak{S}} (\forall x \in x_1)B[x,x_1]$ *for all variables* x_0, x_1 *not occurring in* $\mathrm{Tr}_{\mathfrak{S}}[p,u]$ *if every atomic subformula* D *of* B*,* $x \in y$*, as well as* $x \in y \to D$ *occur in the sequence* $\mathfrak{S}$ *(modulo renaming variables).*

Proof. The first assertion follows from the definitions of $\Vdash_{\mathfrak{S}}$ and $\Vvdash_{\mathfrak{S}}$ and Lemma 2.27.

For the second assertion notice that

$$\forall z_0 A[z_0,\vec{x}] \Vdash_{\mathfrak{S}} \forall z_0 A[z_0,\vec{x}]$$

is $\forall x_0(\forall z_0 A[z_0,\vec{x}] \Vdash_{\mathfrak{S}} A[x_0,\vec{x}])$ which is by Lemma 2.27 equivalent to

$$\forall x_0(\forall z_0 A[z_0,\vec{x}] \Vvdash_{\mathfrak{S}} A[x_0,\vec{x}]).$$

This is by definition equivalent to $\forall x_0 \exists u((\forall z_0 \in u)A[z_0,\vec{x}] \to A[x_0,\vec{x}])$ which is provable in IKP_0^- (for each x_0 let u be $\{x_0\}$).

For the third assertion assume $p \Vdash_{\mathfrak{S}} \neg\forall z A[z,\vec{x}]$ which is by definition

$$\forall q(p \subseteq q \to ((q \Vdash_{\mathfrak{S}} \forall z A[z,\vec{x}]) \to (q \Vdash_{\mathfrak{S}} \bot))).$$

By the second assertion we have $p, \forall z_0 A[z_0,\vec{x}] \Vdash_{\mathfrak{S}} \forall z_0 A[z_0,\vec{x}]$ and therefore $p, \forall z_0 A[z_0,\vec{x}] \Vdash_{\mathfrak{S}} \bot$. Since $\bot$ is atomic this is by definition equivalent to

$$\exists u((\mathrm{Tr}_{\mathfrak{S}}[p,u] \wedge (\forall z_0 \in u)A[z_0,\vec{x}]) \to \bot).$$

By Lemma 1.23 this is equivalent to $\exists u(\mathrm{Tr}_{\mathfrak{S}}[p,u] \to ((\forall z \in u)A[z,\vec{x}] \to \bot))$. It follows (take $x_0 = u$) $\exists x_0 \exists u(\mathrm{Tr}_{\mathfrak{S}}[p,u] \to \neg(\forall z \in x_0)A[z,\vec{x}])$ which is nothing else than

$$\exists x_0(p \Vvdash_{\mathfrak{S}} \neg(\forall z \in x_0)A[z,\vec{x}]).$$

By Lemma 2.27 and the definition of $\Vdash_{\mathfrak{S}}$, this is equivalent to

$$p \Vdash_{\mathfrak{S}} \exists y \neg(\forall z \in y)A[z,\vec{x}]).$$

For the fourth assertion assume $p \Vdash_{\mathfrak{S}} (\forall x \in y)C[x,y]$ which is by definition

$$\forall x_0(p \Vdash_{\mathfrak{S}} (x_0 \in y \to C[x_0,y])).$$

By the second assertion of the previous lemma it follows

$$\forall x_0(x_0 \in y \to p \Vdash_{\mathfrak{S}} C[x_0, y])$$

which is $(\forall x_0 \in y)(p \Vdash_{\mathfrak{S}} C[x_0])$.

For the fifth assertion assume $p \Vdash_{\mathfrak{S}} (\forall x \in y)\exists z A[x, y]$. Therefore we have by the fourth assertion $(\forall x' \in y)(p \Vdash_{\mathfrak{S}} \exists z A[x', z])$ and this is

$$(\forall x' \in y)\exists x_0(p \Vdash_{\mathfrak{S}} A[x', x_0])$$

by definition (x' and x_0 do not appear in $\mathrm{Tr}_{\mathfrak{S}}[p, u]$). By Lemma 2.27 this is equivalent to

$$(\forall x' \in y)\exists x_0(p \Vvdash_{\mathfrak{S}} A[x', x_0])$$

which is by definition $(\forall x' \in y)\exists x_0 \exists u(\mathrm{Tr}_{\mathfrak{S}}[p, u] \to A[x', x_0])$. If we let w be $\{u, x_0\}$ we get $(\forall x' \in y)\exists w(\exists x_0, u \in w)(\mathrm{Tr}_{\mathfrak{S}}[p, u] \to A[x', x_0])$ and therefore by Δ_0 collection

$$\exists v(\forall x' \in y)(\exists w \in v)(\exists x_0, u \in w)(\mathrm{Tr}_{\mathfrak{S}}[p, u] \to A[x', x_0]).$$

And if we let $v' = \cup v$ and $v'' = \cup\cup v$ then $w \subseteq v'$ and $u \subseteq v''$ for every $w \in v$ and $u \in w$. Therefore it follows by Lemma 2.22

$$\exists v', v''(\forall x' \in y)(\exists x_0 \in v')(\mathrm{Tr}_{\mathfrak{S}}[p, v''] \to A[x', x_0]).$$

By Lemma 1.23 this implies

$$\exists v', v''(\forall x' \in y)(\mathrm{Tr}_{\mathfrak{S}}[p, v''] \to (\exists x_0 \in v')A[x', x_0])$$

and therefore by the same lemma

$$\exists v', v''(\forall x' \in y)(\mathrm{Tr}_{\mathfrak{S}}[p, v''] \to \neg(\forall x_0 \in v')\neg A[x', x_0]).$$

Again by the same lemma, this is equivalent to

$$\exists v', v''(\mathrm{Tr}_{\mathfrak{S}}[p, v''] \to (\forall x' \in y)\neg(\forall x_0 \in v')\neg A[x', x_0]).$$

The latter is by definition and renaming of the bounded variables x' and x_0 the same as

$$\exists v'(p \Vvdash_{\mathfrak{S}} (\forall x \in y)\neg(\forall z \in v')\neg A[x, z])$$

and this is by Lemma 2.27 and the definition of $\Vdash_{\mathfrak{S}}$ equivalent to

$$p \Vdash_{\mathfrak{S}} \exists w(\forall x \in y)\neg(\forall z \in w)\neg A[x, z].$$

The sixth assertion follows by Definition 2.24 (v) from the third assertion of the previous lemma. □

The next three lemmas correspond to lemma 5.8 in Avigad [1] and are proved similar as ibidem.

Lemma 2.30. *Let $\exists xA$ be an instance of pairing, union or Δ_0^- separation. Then IKP_0^- proves $\Vdash_{\mathfrak{S}} \exists xA$ if A is prominent for $\mathfrak{S}$.*

Proof. Since all of these axioms are also available in IKP_0^-, $\Vdash_{\mathfrak{S}} \exists xA$ follows by the first assertion of Lemma 2.29. □

The next lemma states that also instances of Δ_0 collection$^\sharp$ and $(MP)_{res}$ are forced if we chose a suitable sequence $\mathfrak{S}$. It is a direct consequence of the definition of $\Vdash_{\mathfrak{S}}$ and the fifth and third assertion of Lemma 2.29.

Lemma 2.31 ($\Vdash_{\mathfrak{S}}$ Δ_0 collection$^\sharp$ and $\Vdash_{\mathfrak{S}}$ $(MP)_{res}$)**.** *If the formulas*

$$(\forall x \in a)\neg(\forall y \in z)\neg A[x, y]$$

and $\neg(\forall x \in y)B[x]$ are prominent for $\mathfrak{S}$, then IKP_0^- proves

(i) $\Vdash_{\mathfrak{S}} (\forall x \in a)\exists yA[x, y] \to \exists z(\forall x \in a)\neg(\forall y \in z)\neg A[x, y]$ *and*

(ii) $\Vdash_{\mathfrak{S}} \neg\forall xB[x] \to \exists y\neg(\forall x \in y)B[x]$.

Lemma 2.32 ($\Vdash_{\mathfrak{S}}$ $\in$-induction$^-$)**.** *Let A be any negative $\mathcal{L}$ formula such that $y \in x$, every atomic sub-formula B of A, as well as $y \in x \to B$ occur in the sequence $\mathfrak{S}$. Then IKP^- proves*

$$\Vdash_{\mathfrak{S}} \forall x((\forall y \in x)A[y] \to A[x]) \to \forall xA[x].$$

Analogous for IKP_ω^- and $\in$-induction$^-$ restricted to ω.

Proof. Assume $p \Vdash_{\mathfrak{S}} \forall x((\forall y \in x)A[y] \to A[x])$, which is by definition

$$\forall x_0\forall q(p \subseteq q \to ((q \Vdash_{\mathfrak{S}} (\forall y \in x_0)A[y]) \to (q \Vdash_{\mathfrak{S}} A[x_0]))),$$

where x_0 does not occur in $\mathrm{Tr}_{\mathfrak{S}}[p, u]$. This implies

$$\forall x_0((p \Vdash_{\mathfrak{S}} (\forall y \in x_0)A[y]) \to (p \Vdash_{\mathfrak{S}} A[x_0]))$$

which is by the sixth assertion of Lemma 2.29 equivalent to

$$\forall x_0((\forall x_1 \in x_0)(p \Vdash_{\mathfrak{S}} A[x_1]) \to (p \Vdash_{\mathfrak{S}} A[x_0]))$$

where x_1 does not occur in $\mathrm{Tr}_{\mathfrak{S}}[p, u]$. Applying $\in$-induction this leads us to $\forall x_0(p \Vdash_{\mathfrak{S}} A[x_0])$ which is by definition $p \Vdash_{\mathfrak{S}} \forall xA[x]$. All in all we have proved for an arbitrary p that

$$p \Vdash_{\mathfrak{S}} \forall x((\forall y \in x)A[y] \to A[x]) \text{ implies } p \Vdash_{\mathfrak{S}} \forall xA[x]$$

which is by Definition 2.24 what we want.

The proof that analogous results hold for IKP_ω^- and $\in$-induction$^-$ restricted to ω is analogous. □

The next lemma corresponds to Theorem 5.15 in Avigad [1]. In its formulation we refer to Lemma 1.24, which tells us that for every Δ_0 formula A there exists a negative Δ_0 formula intuitionistically equivalent to A^N.

Lemma 2.33 ($\Vdash_{\mathfrak{S}}$ N-infinity)**.** *If* $A[x]$ *is the formula*

$$(\exists y \in x)\mathrm{zero}[y] \wedge (\forall y \in x)(\exists z \in x)\mathrm{succ}[y, z])$$

and some negative Δ_0 *formula equivalent to* $A^N[x]$ *is prominent for* $\mathfrak{S}$*, then* IKP_0^- *proves* $\Vdash_{\mathfrak{S}} \exists x A^N[x]$.

Proof. Let $B[x]$ be a negative Δ_0 formula which is equivalent to $A^N[x]$ which is prominent for $\mathfrak{S}$. Since N-infinity is available in IKP_0^-, this theory proves $\exists x B[x]$. By the first assertion of Lemma 2.29 this implies $\Vdash_{\mathfrak{S}} \exists x B[x]$ and therefore by Lemma 2.25 $\Vdash_{\mathfrak{S}} \exists x A^N[x]$. □

In the next definition we define a class of formulas, for which, as we will see, $\mathsf{IKP}^\sharp + (MP_{res})$ is conservative over IKP^-.

Definition 2.34 ($\mathcal{D}_{res}$)**.** The set $\mathcal{D}_{res}$ of $\mathcal{L}$ formulas is inductively defined as follows:

(i) Every negative Δ_0 formula is in $\mathcal{D}_{res}$.

(ii) If A and B are in $\mathcal{D}_{res}$, then also $A \wedge B$ is in $\mathcal{D}_{res}$.

(iii) If A and B are in $\mathcal{D}_{res}$, then also $A \vee B$ is in $\mathcal{D}_{res}$.

(iv) If A is in $\mathcal{D}_{res}$, then also $\forall x A$ is in $\mathcal{D}_{res}$.

(v) If A is in $\mathcal{D}_{res}$, then also $\exists x A$ is in $\mathcal{D}_{res}$.

Lemma 2.35. *If* A *is in* $\mathcal{D}_{res}$ *and every of its negative* Δ_0 *sub-formulas is prominent for* $\mathfrak{S}$*, then the theory* IKP_0^- *proves that* $\Vdash_{\mathfrak{S}} A$ *is equivalent to* A.

Proof. The proof is on the complexity of A. If A is a negative Δ_0 formula, the assertion follows from Lemma 2.29 (i). If A is more complex, it follows directly from the definition of $\Vdash_{\mathfrak{S}}$ and the induction hypothesis. □

Lemma 2.36. *Let* $\mathcal{A} \subseteq \mathcal{D}_{res}$ *be a set of* $\mathcal{L}$ *formulas. If the* $\mathcal{L}$ *formula* A *is provable in* $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{A}$*, then there exists a finite sequence* $\mathfrak{T}$ *of negative* Δ_0 *formulas such that for every finite sequence* $\mathfrak{S}$ *of negative* Δ_0 *formulas which contains at least all formulas of* $\mathfrak{T}$ *the theory* $\mathsf{IKP}^- + \mathcal{A}$ *proves* $\Vdash_{\mathfrak{S}}$ A.

Analogous assertions hold for the versions of $\mathsf{IKP}^\sharp + (MP_{res})$ *and* IKP^- *with restricted induction principles.*

Proof. Assume that A is provable in $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{A}$ and let

$$B_0, ..., B_n, A$$

be a proof of A in a Hilbert-style system. Let $\mathfrak{T}$ be a finite sequence which contains enough formulas such that we can apply the Lemmas 2.30-2.33 to all instances of axioms of $\mathsf{IKP}^\sharp + (MP_{res})$ occurring in $B_0, ..., B_n, A$ and such that every negative Δ_0 sub-formula of any formula in $\mathcal{A} \cap \{B_0, ..., B_n, A\}$ is prominent for $\mathfrak{T}$. I.e. $\mathfrak{T}$ and every super-sequence $\mathfrak{S}$ of $\mathfrak{T}$ contain enough formulas such that IKP^- proves $\Vdash_{\mathfrak{T}} C$ and $\Vdash_{\mathfrak{S}} C$ for all axioms of $\mathsf{IKP}^\sharp + (MP_{res})$ occurring in $B_0, ..., B_n, A$, and furthermore, by the previous lemma, such that $\mathsf{IKP}^- + \mathcal{A}$ proves $\Vdash_{\mathfrak{T}} D$ and $\Vdash_{\mathfrak{S}} D$ for all D in $\mathcal{A} \cap \{B_0, ..., B_n, A\}$. By Lemma 2.25 we can conclude that IKP^- proves $\Vdash_{\mathfrak{S}} A$. $\square$

Theorem 2.37. *Let $\mathcal{A} \subseteq \mathcal{D}_{res}$ be a set of $\mathcal{L}$ formulas. If the $\mathcal{L}$ formula A is in $\mathcal{D}_{res}$ and provable in $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{A}$, then it is also provable in $\mathsf{IKP}^- + \mathcal{A}$.*

Analogous assertions hold for the versions of $\mathsf{IKP}^\sharp + (MP_{res})$ and IKP^- with restricted induction principles.

PROOF. Let A be in $\mathcal{D}_{res}$ and provable in $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{A}$. By Lemma 2.36 there exists a finite sequence $\mathfrak{T}$ of negative Δ_0 formulas such that $\mathsf{IKP}^- + \mathcal{A}$ proves $\Vdash_{\mathfrak{S}} A$ for every finite super-sequence $\mathfrak{S}$ of $\mathfrak{T}$. Let $\mathfrak{S}$ be such a sequence which contains besides the formulas of $\mathfrak{T}$ also all sub-formulas of A which are Δ_0 and negative. We can conclude by Lemma 2.36 that IKP^- proves A. $\square$

3 Interpreting Pure Set Theories in Operational Set Theories

3.1 Interpreting KPS in OST + Inac

In this small section we will see that KPS can be interpreted in $\mathsf{OST} + \mathsf{Inac}$ in the easiest possible way: $\mathsf{OST} + \mathsf{Inac}$ proves every formula which is provable in KPS.

The following lemma can be proved as Theorem 7 in Feferman [14] or Theorem 6 in Jäger [18].

Lemma 3.1. *Every formula of $\mathcal{L}_\in$ which is provable in KP is also provable in OST.*

We prove the next lemma and theorem as Theorem 22 is proved in Jäger and Zumbrunnen [26].

Lemma 3.2. *The theory OST^- prove that* $\mathrm{Org}[\kappa] \to \mathrm{Frg}[\kappa]$.

PROOF. Assume $\mathrm{Org}[\kappa]$. Let ξ be an arbitrary ordinal less than κ and f a set-theoretic function with $\mathrm{Dom}[f, \xi]$ and $\mathrm{Ran}_\subseteq[f, \kappa]$. By Proposition 1.36 we have $\mathrm{op}(f)\downarrow$ as well as $\mathrm{op}(f, x) = f'x$ for all $x \in \xi$. That is $\mathrm{op}(f) : \xi \to \kappa$. Because κ is operationally regular, it follows that there is a $\eta < \kappa$ with $\mathrm{op}(f) : \xi \to \eta$. Therefore we have $\mathrm{Ran}_\subseteq[f, \eta]$ and κ is also functionally regular. $\square$

Theorem 3.3. *Every formula of $\mathcal{L}_\in$ which is provable in KPS is also provable in $\mathsf{OST} + \mathsf{Inac}$.*

PROOF. By Lemma 3.1 it is enough to prove the axiom (SLim) within $\mathsf{OST} + \mathsf{Inac}$. So let α be an arbitrary ordinal. By (Inac) there is an operationally regular ordinal $\beta > \alpha$. We know by the previous lemma that this β is also functionally regular. Therefore (SLim) holds. $\square$

3.2 Interpreting KPSd in OST + Inac

In this section we will see how we can deal with the relation symbol Ad in the theory OST using only the language $\mathcal{L}_\in^\circ$ (i.e. a language without the relation symbol Ad). We will define a special version of the constructible hierarchy within OST and see that $\mathsf{OST} + \mathsf{Inac}$ proves that it is a model of KPSd. Since the theory KP is contained in OST (c.f. Lemma 3.1) we can use many theorems of Barwise [2], but formulated for OST instead of for KP.

Before we introduce the constructible hierarchy we proof the next lemma and some facts about cardinal numbers which we will use later.

Lemma 3.4. *Let $\vec{u}$ be the sequence of variables $u_0, ..., u_{n-1}$ and $A[\vec{u}]$ a Δ_1 formula of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free. There exists a closed $\mathcal{L}^\circ$ term t_A such that* OST *proves the formula*

$$t_A\downarrow \wedge (t_A : \mathbf{V}^n \to \mathbf{B}) \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \mathsf{t}).$$

Proof. Let $A[\vec{x}]$ be a Δ_1 formula of $\mathcal{L}^\circ$ with at most the variables $\vec{x}$ free. Then there are variables $\vec{y} = y_0, ...y_{m-1}$ and Δ_0 formulas $B[\vec{x}, \vec{y}]$ and $C[\vec{x}, \vec{y}]$ with at most the variables $\vec{x}$ and $\vec{y}$ free such that OST proves $A[\vec{x}] \leftrightarrow \forall \vec{y} B[\vec{x}, \vec{y}]$ and $A[\vec{x}] \leftrightarrow \exists \vec{y} C[\vec{x}, \vec{y}]$. It follows that OST proves $\forall \vec{y} B[\vec{x}, \vec{y}] \leftrightarrow \exists \vec{y} C[\vec{x}, \vec{y}]$ and therefore also $\neg(\exists \vec{y}(\neg B[\vec{x}, \vec{y}]) \wedge \exists \vec{y} C[\vec{x}, \vec{y}])$ and $\exists \vec{y}(\neg B[\vec{x}, \vec{y}]) \vee \exists \vec{y} C[\vec{x}, \vec{y}]$ and thus also $\exists \vec{y}(\neg B[\vec{x}, \vec{y}] \vee C[\vec{x}, \vec{y}])$. We define the formula $D[\vec{x}, z]$ as

$$\begin{aligned}\mathrm{Tup}_{m+1}(z) \wedge (\ (\neg B[\vec{x}, (z)_1, ..., (z)_m] \wedge (z)_0 = \mathsf{f})& \\ \vee (C[\vec{x}, (z)_1, ..., (z)_m] \wedge (z)_0 = \mathsf{t})\ ).&\end{aligned}$$

Of course $D[\vec{x}, z]$ is Δ_0 and so there is a closed term t_D as described in Lemma 1.31. Now let's define the term

$$t_A := \lambda \vec{x}.\mathrm{p}_0(\mathbb{C}(\lambda z.t_D(\vec{x}, z))).$$

If one notice that an $(m+1)$-tuple $\langle x, y_0, ..., y_{m-1}\rangle$ is coded as the ordered pair $\langle x, \langle y_0, \langle ... \langle y_{m-2}, y_{m-1}\rangle ...\rangle\rangle\rangle$, it's easy to check that t_A fulfills the asserted properties. $\square$

Because we can prove the axiom of choice in OST (c.f. Proposition 1.32), it is possible to talk reasonably about the cardinality of sets within this theory. We call an ordinal α a *cardinal number*, if there is no bijective function from α to an ordinal $\beta < \alpha$. We will write On for the class of all ordinal numbers.

First we want to prove in OST that every set has certain cardinality.

Lemma 3.5. OST *proves that every set a has certain cardinality. That is, there is a unique cardinal κ such that there exists a bijective set theoretic function f from a to κ.*

We call the cardinal κ the cardinality of a and write $|a|$ for it.

Proof of Lemma 3.5. First, prove that for every ordinal α there is a cardinal κ such that there exists a bijective set theoretic function f_0 from

α to κ. It's easy to see, that OST proves that α is a cardinal if $\alpha < \omega$ or $\alpha = \omega$. So the assumption is true for this ordinals and we prove by transfinite induction that it holds also for all other ordinals. If it holds for $\beta \geq \omega$, that is, there is a bijection g from β to a cardinal κ, and $\alpha = \beta+1$, then we define f_0 (by operational separation and Lemma 1.31) as the set

$$\{\langle \beta, g'0\rangle\} \cup \{\langle \gamma, g'(\gamma+1)\rangle \in \omega \times \kappa : \gamma < \omega\} \cup \{\langle \gamma, g'\gamma\rangle \in g : \omega \leq \gamma < \beta\}.$$

If α is a limit ordinal and the assumption holds for all $\beta < \alpha$, then we have two cases. The first one is that α is a cardinal and the assumption holds trivially. Else, by definition of the cardinals, there must be a bijection g from α to an ordinal $\beta < \alpha$. But by the induction hypothesis, there is a bijection h from β to a cardinal κ. Then OST proves that if f_0 is the composition of h and g, it is a bijection from α to κ.

Since the axiom of choice is available in OST, it proves that there exists for every set a an ordinal α and a bijective set theoretic function g from a to α (it is well known that the axiom of choice implies that every set can be well ordered, c.f. for instance Jech [27], Krivine [28] or Kunen [29]). OST proves that the composition of f_0 and g is a bijection from a to κ.

To see that the cardinality κ of a set α is unique is easy: Assume there are two cardinals κ_1 and κ_2 with bijections f_1 and f_2 from a. Then the composition of f_1 and f_2^{-1} (the inverse function of f_2) is a bijection between κ_1 and κ_2 and by the definition of the cardinals we get $\kappa_1 = \kappa_2$. $\square$

Corollary 3.6. OST *proves for any two sets a an b that one (or both) of the following two cases is fulfilled.*

(i) There is an injection from a to b and $|a| \leq |b|$.

(ii) There is a surjection from a onto b and $|a| \geq |b|$.

We will prove the next lemma similar as one does it within ZFC (for instance in Krivine [28]).

Lemma 3.7. OST *proves for every cardinal $\kappa \geq \omega$ that $|\kappa \times \kappa| = \kappa$.*

For proving this lemma we define an operation which is an order isomorphism between On × On and On.

Definition 3.8 ($\leq_\times$). Let $A_{\leq_\times}[\alpha, \beta, \gamma, \delta]$ be the Δ_0 formula

$$\begin{aligned}\max(\alpha, \beta) < \max(\gamma, \delta) \;\vee\; &(\max(\alpha, \beta) = \max(\gamma, \delta) \wedge \alpha < \gamma)\\ &\vee\; (\max(\alpha, \beta) = \max(\gamma, \delta) \wedge \alpha = \gamma \wedge \beta \leq \delta).\end{aligned}$$

We will write $\langle \alpha, \beta\rangle \leq_\times \langle \gamma, \delta\rangle$ for the formula $A_{\leq_\times}[\alpha, \beta, \gamma, \delta]$ as well as $a <_\times b$ for $a \leq_\times b \wedge a \neq b$. Further we write $a_{\alpha,\beta}$ for the set $\{\langle \eta, \xi\rangle : \langle \eta, \xi\rangle <_\times \langle \alpha, \beta\rangle\}$.

It's easy to see that OST proves that $\leq_\times$ is a well order relation on On$\times$On. Further OST proves that $a_{\alpha,\beta}$ is in fact a set, since, if $\gamma = \max(\alpha, \beta)$, all its elements are also elements of $(\gamma{+}1) \times (\gamma{+}1)$, because

$$\langle \eta, \xi \rangle <_\times \langle \alpha, \beta \rangle \to \eta < \gamma{+}1 \wedge \xi < \gamma{+}1.$$

Let Bij$[f]$ be a Δ_0 formula which expresses that f is a bijective (set theoretic) function and $B[\alpha, \beta, b]$ the Δ_0 formula

$$\mathrm{Tup}_2[b] \ \wedge \ \mathrm{Ord}[(b)_1] \ \wedge \ \mathrm{Bij}[(b)_0] \ \wedge \ \mathrm{Dom}[(b)_0, a_{\alpha,\beta}] \ \wedge \ \mathrm{Ran}[(b)_0, (b)_1] \\ \wedge \ (\forall x, y \in a_{\alpha,\beta})(x <_\times y \leftrightarrow (b)_0' x < (b)_0' y).$$

So $B[\alpha, \beta, b]$ means b is an ordered pair and its first component is an isomorphism between $a_{\alpha,\beta}$ and its second component, which is an ordinal. Since $\leq_\times$ is a well order also on $a_{\alpha,\beta}$, there is for any ordinals α, β one b such that $B[\alpha, \beta, b]$ holds (since every well ordering is isomorphic to one ordinal). Let t_B be the term defined in Lemma 1.31.

Definition 3.9 ($t_{\leq_\times}$)**.** The term $t_{\leq_\times}$ is defined as $\lambda\alpha\beta.\mathrm{p}_0(\mathbb{C}(\lambda b.t_B(\alpha, \beta, b)))$.

Now we have that $t_{\leq_\times}(\alpha, \beta)$ is the order type of $\langle \alpha, \beta \rangle$ with respect to the well ordering $\leq_\times$. It's evident that $t_{\leq_\times}(\alpha, \beta) < t_{\leq_\times}(\gamma, \delta)$ iff. $\langle \alpha, \beta \rangle <_\times \langle \gamma, \delta \rangle$. Thus $t_{\leq_\times}$ is a one-to-one order homomorphism between On $\times$ On and On.

Proof of Lemma 3.7. We prove in OST by transfinite induction on $\kappa \geq \omega$ the formula

$$\text{`}\kappa \text{ is not a cardinal number'} \ \vee \ |\kappa \times \kappa| = \kappa.$$

It's well known that there is a bijection between $\omega \times \omega$ and ω (for instance a Cantor pairing function); of course its existence can be proved in OST. It's easy to see that OST proves: if κ is $\beta{+}1 > \omega$, then it is not a cardinal.

Now assume Lim$[\kappa]$ and the assumption holds for all ordinals less than κ. That is, if κ is a cardinal, then $|\lambda \times \lambda| = \lambda$ for every cardinal $\lambda \geq \omega$ less than κ. We'll show that $t_{\leq_\times} : \kappa \times \kappa \to \kappa$ (then fun$(t_{\leq_\times}, \kappa \times \kappa)$ is the required injection). By the definition of cardinals, we have that $t_{\leq_\times}\langle \eta, \xi \rangle \in \kappa$ iff $|t_{\leq_\times}\langle \eta, \xi \rangle| < \kappa$, so we prove that $|t_{\leq_\times}\langle \eta, \xi \rangle| < \kappa$ for all ordinals $\eta, \xi < \kappa$. Because every ordinal contains exactly all ordinals less than itself and $t_{\leq_\times}$ is order preserving, we can prove in OST for any $\eta, \xi < \kappa$, that if $a_{\eta,\xi}$ is defined as above then $\mathbb{R}(t_{\leq_\times}, a_{\eta,\xi}) = t_{\leq_\times}\langle \eta, \xi \rangle$ (and thus $|a_{\eta,\xi}| = |t_{\leq_\times}\langle \eta, \xi \rangle|$). Further if $\gamma := \max(\eta, \xi)$, we get that $a_{\eta,\xi}$ is a subset of $(\gamma{+}1) \times (\gamma{+}1)$ because $\langle \eta_0, \xi_0 \rangle <_\times \langle \eta, \xi \rangle \to \eta_0 < \gamma{+}1 \wedge \xi_0 < \gamma{+}1$. Hence $|a_{\eta,\xi}| \leq |(\gamma{+}1) \times (\gamma{+}1)|$. But $|\gamma| < \kappa$ and thus $|\gamma{+}1| < \kappa$ and by the induction hypothesis it follows that $|(\gamma{+}1) \times (\gamma{+}1)| = |\gamma{+}1|$ or $|\gamma{+}1| < \omega$. So $|(\gamma{+}1) \times (\gamma{+}1)| < \kappa$ and thus $|t_{\leq_\times}\langle \eta, \xi \rangle| < \kappa$, which finishes the proof. □

Now we are ready to introduce the constructible hierarchy. In the following we write ${\mathcal{L}_\in}^f$ and ${\mathcal{L}_\in^\circ}^f$ for $\mathcal{L}_\in$ and $\mathcal{L}_\in^\circ$, respectively, extended by finitely many function symbols. We introduce the notion of Σ function symbol w.r.t. OST.

Definition 3.10 (Σ function symbol)**.** We call the function symbol F of $\mathcal{L}_\in^f$ a Σ *function symbol* if there is a Σ formula $A[x_0, ..., x_n, y]$ of $\mathcal{L}_\in$ such that OST proves

$$\forall x_0, ..., x_n \exists ! y A[x_0, ..., y],$$

and F is defined by the a new axiom

$$\text{(F)} \qquad F(x_0, ..., x_n) = y \leftrightarrow A[x_0, ..., y].$$

By Lemma 5.4 of Chapter I in Part A of Barwise [2], we can treat Σ function symbols as atomic symbols of the language $\mathcal{L}_\in$. We will therefore consider formulas A of $\mathcal{L}_\in^f$ containing Σ function symbols $F_0, ..., F_n$ as $\mathcal{L}_\in$ formulas and say that OST proves A (or write $\mathsf{OST} \vdash A$) instead of OST + (the definition-axioms of $F_0, ..., F_n$) proves A.

We introduce the constructible hierarchy as Barwise in [2]. The proof of the next theorem can be found in in the fifth and sixth section of Chapter II in Part A of Barwise [2].

Theorem 3.11. *In* OST *there can be introduced binary Σ function symbols $\mathscr{F}_1$, ..., $\mathscr{F}_N$ and unary Σ function symbols $\mathscr{D}$ and L such that:*

(i) $\mathsf{OST} \vdash \mathscr{D}(a) = a \cup \{\mathscr{F}_i(x, y) : x, y \in a \text{ and } 1 \leq i \leq N\}$,

(ii) $\mathsf{OST} \vdash \mathscr{F}_1(x, y) = \{x, y\}$,

(iii) $\mathsf{OST} \vdash \mathscr{F}_2(x, y) = \bigcup x$,

(iv) $\mathsf{OST} \vdash \mathrm{Tran}[a] \to \mathrm{Tran}[\mathscr{D}(a)]$,

(v) For each Δ_0 formula $A[\vec{x}, y]$ with at most the variables $\vec{x} = x_1, ... x_n$ and y free there is a term $\mathscr{F}$ (n-ary) built from $\mathscr{F}_1$, ..., $\mathscr{F}_N$ such that

$$\mathsf{OST} \vdash \mathscr{F}(x_1, ..., x_n, a) = \{y \in a : A[\vec{x}, y]\},$$

(vi) $\mathsf{OST} \vdash L(0) = 0$,

(vii) $\mathsf{OST} \vdash L(\alpha + 1) = \mathscr{D}(L(\alpha) \cup \{L(\alpha)\})$,

(viii) $\mathsf{OST} \vdash L(\alpha) = \bigcup_{\beta < a} L(\beta)$ *if* $\mathrm{Lim}[\alpha]$.

To simplify matters we will use the constructible hierarchy not only as defined above. We therefore need a Σ function symbol TC such that TC(a) is the transitive closure of the set a (for the proof of its existence see Barwise [2]). The defined new Σ function symbol exists by Σ recursion (see also Barwise [2]).

Definition 3.12 (L^{Ad})**.** We write L^{Ad} for a Σ function symbol such that

(i) $\mathsf{OST} \vdash L^{\mathsf{Ad}}(0) = 0$,

(ii) OST proves that $L^{\mathsf{Ad}}(\alpha + 1)$ is the set

$$\mathscr{D}(L^{\mathsf{Ad}}(\alpha) \cup \{L^{\mathsf{Ad}}(\alpha)\}) \cup \mathrm{TC}(\{\{\langle \beta, b \rangle : \beta \leq \alpha \wedge b = L^{\mathsf{Ad}}(\beta)\}\}),$$

(iii) $\mathsf{OST} \vdash L^{\mathsf{Ad}}(\alpha) = \bigcup_{\beta < \alpha} L^{\mathsf{Ad}}(\beta)$ if $\mathrm{Lim}[\alpha]$.

For this alternative definition of the constructible hierarchy we now that

$$\{\langle \beta, b \rangle : \beta \leq \alpha \wedge b = L^{\mathsf{Ad}}(\beta)\} \in L^{\mathsf{Ad}}(\alpha + 1)$$

for all ordinals α. We will use this property later in the proof of the Lemmas 3.21 and 3.23.

We'll write L_α for $L(\alpha)$. We say that a set a is constructible if there is an α such that OST proves $a \in L_\alpha$. If $A[x, \alpha]$ is a Σ formula of $\mathcal{L}_\in$ which expresses $x = L_\alpha$, then the Π formula $\forall y(y \neq x \rightarrow \neg A[y, \alpha])$ of $\mathcal{L}_\in$ expresses the same. Thus there are Δ formulas which express that $x = L_\alpha$ and $x \in L_\alpha$, respectively. So we can write $x = L_\alpha$ and $x \in L_\alpha$, respectively, as abbreviation for this formulas. If $\vec{x} = x_1, ..., x_n$ we use the abbreviation $\vec{x} \in L_\alpha$ for $x_1 \in L_\alpha \wedge ... \wedge x_n \in L_\alpha$. And we write $x \in \mathbf{L}$ for the formula $\exists \alpha(x \in L_\alpha)$. We use the same notations also for L^{Ad} and $\mathbf{L}^{\mathsf{Ad}}$.

The proof of the following lemma is straightforward by transfinite induction.

Lemma 3.13. OST *proves for all ordinals α, β the following facts about the constructible hierarchy.*

(i) $\mathrm{Tran}[L_\alpha]$,

(ii) $L_\alpha \in L_\beta \leftrightarrow \alpha < \beta$,

(iii) $L_\alpha \subseteq L_\beta \leftrightarrow \alpha \leq \beta$,

(iv) $x, y \in L_\alpha \rightarrow \{x, y\} \in L_{\alpha+1}$.

The same assertions hold also if we replace L by L^{Ad}.

Similar assertions as stated in the next lemma are proved for instance also in Barwise [2].

Lemma 3.14. *Let $A[\vec{x}, u]$ be a Δ_0 formula of $\mathcal{L}_\in$ with at most the variables $\vec{x} = x_1, ...x_n$ and y free.* OST *proves for every ordinal α that*

(i) if $a \in L_\alpha$ and $\vec{x} \in L_\alpha$ then there exists a finite ordinal $m \in \omega$ such that $\{y \in a : A[\vec{x}, y]\} \in L_{\alpha+m} \subseteq L_{\alpha+\omega}$ and

(ii) there is an ordinal $m < \omega$ such that $\alpha \in L_{\alpha+m} \subseteq L_{\alpha+\omega}$.

The same assertions holds also if we replace L by L^{Ad}.

Proof. The first assertion follows from Theorem 3.11 (v).

We show the second assertion by induction on α. If $\alpha = 0$, then $\alpha \in L_{\alpha+1}$. If $\alpha = \beta + 1$ and the assertion holds for β, then there is an $m_\beta < \omega$ with $\beta \in L_{\beta+m_\beta}$. Thus

$$\{\beta\} = \mathscr{F}_1(\beta, \beta) \in L_{\beta+m_\beta+1}, \quad \{\beta, \{\beta\}\} = \mathscr{F}_1(\beta, \{\beta\}) \in L_{\beta+m_\beta+2} \quad \text{and}$$

$$\alpha = \beta \cup \{\beta\} = \mathscr{F}_2(\{\beta, \{\beta\}\}, 0) \in L_{\beta+m_\beta+3}.$$

Finally let α be a limit ordinal and assume that there is for every $\beta < \alpha$ an $m_\beta < \omega$ with $\beta \in L_{\beta+m_\beta}$. Then $\beta + m_\beta < \alpha$ and, by the definition of L_α, $\beta \in L_\alpha$ for every $\beta < \alpha$. Thus $\alpha \subseteq L_\alpha$. By the first assertion and the previous lemma there is an $m < \omega$ such that $c := \{y \in L_\alpha : \mathrm{Ord}[y]\} \in L_{\alpha+1+m}$. Because L_α is transitive, c is an ordinal γ and we have $\alpha \leq \gamma$. Since thus $\alpha \in \gamma$ or $\alpha = \gamma$ and also $L_{\alpha+1+m}$ is transitive, we have $\alpha \in L_{\alpha+1+m}$.

The assertions for L^{Ad} are proved the same way. □

Lemma 3.15. OST *proves that there is for every ordinal $\alpha \geq \omega$ a bijection between α and L_α and that there is a bijection between α and L^{Ad}_α.*

Proof. We prove by transfinite induction on $\alpha \geq \omega$ that $|\alpha| = |L_\alpha|$. It's easy to see that L_n is finite for each $n < \omega$. So L_ω is a countable union of finite sets which is countable (the well known proofs are also available in OST), i.e. there is a bijection between ω and L_ω.

Now assume $\omega \leq \alpha = \beta+1$ and there is a bijection f between β and L_β. By the definition of $\mathscr{D}$, Corollary 3.6, Lemma 3.7 and the fact that the cardinality of L_β is $|L_\beta \cup \{L_\beta\}| \geq \omega$ we have

$$\begin{aligned}|L_\alpha| = |\mathscr{D}(L_\beta \cup \{L_\beta\})| &\leq |(N+1) \times |L_\beta| \times |L_\beta|| \\ &= |(N+1) \times |L_\beta|| \leq ||L_\beta| \times |L_\beta|| = |L_\beta| = |\beta|.\end{aligned}$$

Because $L_\beta \subseteq L_\alpha$ we have also $|\beta| = |L_\beta| \leq |L_\alpha|$ and so $|L_\alpha| = |\beta|$. Since $|\alpha| = |\beta|$ we have altogether $|\alpha| = |L_\alpha|$.

If α is a limit ordinal and $|L_\beta| = |\beta|$ for all $\beta < \alpha$, then there is for each such β a bijection from L_β to β. Let $A[f, \beta]$ be a Δ formula which expresses that f is a bijection from L_β to β and t_A the corresponding term as defined in Lemma 3.4. We set $g := \mathrm{fun}(\lambda\beta.\mathbb{C}(\lambda f.t_A(f, \beta)), \alpha)$. Then g is a set theoretic function with domain α and for each $\beta < \alpha$, $g'\beta$ is a bijection from L_β to β. Let a be the set $\{\langle b, \beta\rangle \in L_\alpha \times \alpha : b \in L_\beta\}$ (which exists by operational separation and Lemma 3.4). We can define (by operational separation and Lemma 3.4) the set theoretic function

$$h := \{\langle\langle b, \beta_1\rangle, \langle\gamma, \beta_2\rangle\rangle \in a \times (\alpha \times \alpha) : \gamma = (g'\beta_1)'b \wedge \beta_1 = \beta_2\}.$$

It's easy to check that h is injective with domain a and the range of h is a subset of $(\alpha \times \alpha)$, thus $|a| \leq |\alpha \times \alpha|$. Because $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$, it's also easy to check that $|L_\alpha| \leq |a|$ (just look at the surjection $\langle b, \beta\rangle \mapsto b$). Hence $|L_\alpha| \leq |\alpha \times \alpha| = ||\alpha| \times |\alpha||$ and therefore by Lemma 3.7 $|L_\alpha| \leq |\alpha|$. We can define within OST the set theoretic function $\beta \mapsto L_\beta$ from α to L_α, which is by Lemma 3.13 injective. Thus $|L_\alpha| \geq |\alpha|$.

The prove for L^{Ad} is very similar. □

Definition 3.16 (Order of constructible sets)**.** We write $\mathrm{od}^{\mathsf{Ad}}(x) = \alpha$ for the $\mathcal{L}_\in$ formula $(\forall\beta < \alpha)(x \notin L^{\mathsf{Ad}}_\beta) \wedge x \in L^{\mathsf{Ad}}_\alpha$ and we say x *has the order* α (w.r.t. L^{Ad}) if it holds.

The formula $\mathrm{od}^{\mathsf{Ad}}(x) = \alpha$ is clearly Δ. For later use we define for any ordinal κ a similar formula as $\mathrm{Org}[\alpha]$ for which it is easy to see that it is a Δ formula of $\mathcal{L}_\in$.

Definition 3.17 (Formula org)**.** For any variable κ of $\mathcal{L}_\in$ we write $\mathrm{org}[\kappa]$ as abbreviation for of formula

$$\mathrm{Lim}[\kappa] \wedge (\forall a \in L^{\mathsf{Ad}}_\kappa)(\forall f \in L^{\mathsf{Ad}}_{\kappa+\omega})(\, \mathrm{Fun}[f] \wedge \mathrm{Dom}[f, a] \wedge \mathrm{Ran}_\subseteq[f, \kappa] \\ \rightarrow (\exists\beta < \kappa)\mathrm{Ran}_\subseteq[f, \beta]\,).$$

The proof of the following lemma is straight forward.

Lemma 3.18. OST *proves for every ordinal* α *that* $\mathrm{Org}[\alpha] \rightarrow \mathrm{Lim}[\alpha]$*.*

Lemma 3.19. OST *proves the formula* $\mathrm{Org}[\kappa] \rightarrow \mathrm{org}[\kappa]$*.*

Proof. Let κ be an ordinal with $\mathrm{Org}[\kappa]$, $a \in L^{\mathsf{Ad}}_\kappa$ and $f \in L^{\mathsf{Ad}}_{\kappa+\omega}$ such that $\mathrm{Fun}[f]$, $\mathrm{Dom}[f, a]$ and $\mathrm{Ran}_\subseteq[f, \kappa]$. We have to show that there is a $\beta < \kappa$ with $\mathrm{Ran}_\subseteq[f, \beta]$. If $a = \emptyset$, then this is of course true, so let's assume $a \neq \emptyset$. By Lemma 3.18 we have $\mathrm{Lim}[\kappa]$ and hence there is an ordinal $\alpha < \kappa$ such that $a \in L^{\mathsf{Ad}}_\alpha$, and so $a \subseteq L^{\mathsf{Ad}}_\alpha$, because L^{Ad}_α is transitive. Now we want to show that there is a surjective function g from α to a. In the case that $\alpha < \omega$ we have that L^{Ad}_α is finite and thus also a is finite. It's easy to see that we can choose $\alpha < \omega$ large enough such that there is a surjective function g from α to a. In the other case that $\alpha \geq \omega$ we know by Lemma 3.15 that there is a bijection g_0 from α to L^{Ad}_α. Let x_0 be an element of a and

$$g := \{\langle x, y\rangle \in g_0 : y \in a\} \cup \{\langle x, x_0\rangle : \neg(\exists y \in a)(\langle x, y\rangle \in g_0)\}.$$

OST proves the existence of g since it is a subset of $\alpha \times a$ which can be defined by a Δ_0 formula. It's easy to check that g is a surjection form α to a. Let h be the set $\{\langle x, y\rangle \in \alpha \times \kappa : (\exists z \in a)(g'x = z \wedge f'z = y)\}$ (it's clear that OST proves its existence), so h is the composition of the set theoretic functions g and f (usually written as $f \circ g$). Thus $\mathrm{op}(h) : \alpha \to \kappa$ and by the definition of the formula $\mathrm{Org}[\kappa]$ there must be a $\beta < \kappa$ such that $\mathrm{op}(h) : \alpha \to \beta$. Because g is surjective it follows that h and f has the same ranges and thus $\mathrm{Ran}_{\subseteq}[f, \beta]$. □

Now we are ready to translate formulas of $\mathcal{L}_{\mathsf{Ad}}$ to formulas of $\mathcal{L}_{\in}^{\circ}$.

Definition 3.20. Let A be any formula of $\mathcal{L}_{\mathsf{Ad}}$. We get the $\mathcal{L}_{\in}^{\circ}$ formula $\mathring{A}$ by simultaneous replacing any occurrence of

(i) $\forall x$ by $(\forall x \in \mathbf{L}^{\mathsf{Ad}})$,

(ii) $\exists x$ by $(\exists x \in \mathbf{L}^{\mathsf{Ad}})$ and

(iii) $\mathsf{Ad}(x)$ by $\exists \kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge x = L_{\kappa}^{\mathsf{Ad}})$

for every variable x in A.

Lemma 3.21. *Let $A[\vec{x}, u]$ be a Δ_0 formula of $\mathcal{L}_{\mathsf{Ad}}$ with at most the variables $\vec{x} = x_0, ..., x_n$ and y free.* OST *proves for every ordinal α that if $a \in L_{\alpha}^{\mathsf{Ad}}$ and $\vec{x} \in L_{\alpha}^{\mathsf{Ad}}$, then there exists a finite ordinal $m < \omega$ such that*

$$\{y \in a : \mathring{A}[\vec{x}, y]\} \in L_{\alpha+m}^{\mathsf{Ad}} \subseteq L_{\alpha+\omega}^{\mathsf{Ad}}.$$

Proof. It is enough to look at the atomic Δ_0 formula $A[u] := \mathsf{Ad}(u)$ of $\mathcal{L}_{\mathsf{Ad}}$: by Lemma 3.14 the rest goes through by induction on the length of the formula $A[\vec{x}, y]$.

Let $a_{L^{\mathsf{Ad}}} := \{\langle \beta, b\rangle : \beta < \alpha \wedge b = L_{\beta}^{\mathsf{Ad}}\} \in L_{\alpha+1}^{\mathsf{Ad}}$. And let $B[y]$ be the Δ_0 formula expressing that there is a $\kappa < \alpha$ with $\kappa + \omega < \alpha$, $\kappa > \omega$ and such that $\langle \kappa + \omega, b\rangle \in a_{L^{\mathsf{Ad}}}$ implies

$$\begin{aligned}(\mathrm{Lim}[\kappa] \wedge (\forall x \in y)(\forall f \in b)(\mathrm{Fun}[f] \wedge \mathrm{Dom}[f, x] \\ \wedge \mathrm{Ran}_{\subseteq}[f, \kappa] \to (\exists \beta < \kappa)(\mathrm{Ran}_{\subseteq}[f, \beta])),\end{aligned}$$

where $\beta = \kappa + \omega$ is the Δ_0 formula

$$(\forall \gamma \in \beta)(\mathrm{Lim}[\gamma] \to \gamma \leq \kappa) \wedge \mathrm{Lim}[\beta] \wedge \beta > \kappa.$$

Then $a_0 := \{y \in a : \mathring{A}[y] \wedge \forall \kappa(y = L_{\kappa}^{\mathsf{Ad}} \to \kappa + \omega < \alpha)\} = \{y \in a : B[y]\}$ is an element of $L_{\alpha+m}^{\mathsf{Ad}}$ for some $m < \omega$ by Lemma 3.14. If there is no $\kappa < \alpha$ such that $\mathrm{org}[\kappa]$ and $\kappa + \omega \geq \alpha$, then $\{y \in a : \mathring{A}[y]\} = a_0 \in L_{\alpha+m}^{\mathsf{Ad}}$. If there is a $\kappa < \alpha$ such that $\mathrm{org}[\kappa]$ and $\kappa + \omega \geq \alpha$, then it is unique since it has to be a limit ordinal. Therefore and by the definition of L^{Ad} we have $\{y \in a : \mathring{A}[y]\} = a_0 \cup \{L_{\kappa}^{\mathsf{Ad}}\} \in L_{\alpha+m+3}^{\mathsf{Ad}}$. □

Lemma 3.22. *Let A be the first axiom of the axioms $\mathcal{AD}$. Then* OST *proves $\mathring{A}$.*

Proof. The formula $\mathring{A}$ is the $\mathcal{L}^{\circ}_{\in}$ formula

$$\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge a = L^{\mathsf{Ad}}_{\kappa}) \to (\omega \in a \wedge \mathrm{Tran}[a]).$$

So let $\kappa > \omega$ be an ordinal with $\mathrm{org}[\kappa]$. Then $\mathrm{Lim}[\kappa]$ and thus $\kappa \geq \omega + \omega$ and by Lemma 3.14 $\omega \in L^{\mathsf{Ad}}_{\omega+\omega} \subseteq L^{\mathsf{Ad}}_{\kappa}$. $\mathrm{Tran}[L^{\mathsf{Ad}}_{\kappa}]$ holds by Lemma 3.13. $\square$

Lemma 3.23. *Let A be an instance of the axiom schema (ii) of $\mathcal{AD}$. Then* OST *proves $\mathring{A}$.*

Proof. A is the formula $\mathsf{Ad}(a) \to (\forall\vec{b} \in a)B^a[\vec{b}]$ and $B[\vec{u}]$ in each case an instance of a particular axiom.

First of all, let B be the pairing axiom. Then $\mathring{A}$ is the formula

$$\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge a = L^{\mathsf{Ad}}_{\kappa}) \to (\forall u, v \in a)(\exists x \in a)(u \in x \wedge v \in x).$$

That OST proves this formula follows from the fact that $\mathrm{org}[\kappa]$ implies $\mathrm{Lim}[\kappa]$, the definition of L^{Ad}_{κ} for a limit ordinal κ and Lemma 3.13.

If $B[u]$ is an instance of Δ_0 separation, then $\mathring{A}$ is the formula

$$\begin{aligned}
&\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge a = L^{\mathsf{Ad}}_{\kappa}) \to \\
&\qquad (\forall\vec{z} \in a)(\forall b \in a)(\exists x \in a)(\ (\forall y \in x)(y \in a \wedge C^a[\vec{z}, y]) \\
&\qquad\qquad\qquad\qquad \wedge (\forall y \in a)(C^a[\vec{z}, y] \to y \in x)\ ),
\end{aligned}$$

where $C^a[\vec{z}, y]$ is the translation of a Δ_0 formula of $\mathcal{L}_{\mathsf{Ad}}$ with at most the variables $\vec{z} = z_1, ..., z_n$ and y free. Let $b, \vec{z} \in L^{\mathsf{Ad}}_{\kappa}$ and $\mathrm{org}[\kappa]$. So κ is a limit ordinal and so there must be an $\alpha < \kappa$ such that $\vec{z}, b \in L^{\mathsf{Ad}}_{\alpha}$. By Lemma 3.21 there is an $m < \omega$ so that $x := \{y \in b : C^a[\vec{z}, y]\} \in L^{\mathsf{Ad}}_{\alpha+m}$. Since κ is a limit ordinal, $\alpha + m < \kappa$ and thus $x \in L^{\mathsf{Ad}}_{\kappa}$. It follows that OST proves $\mathring{A}$.

Now let $B[u]$ be an instance of Δ_0 collection. Then $\mathring{A}$ is equivalent to the formula

$$\begin{aligned}
&\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge a = L^{\mathsf{Ad}}_{\kappa}) \to \\
&\qquad\qquad (\ (\forall b \in a)((\forall x \in b)(\exists y \in a)C^a[x, y] \to \\
&\qquad\qquad\qquad\qquad (\exists z \in a)(\forall x \in b)(\exists y \in z)C^a[x, y]\ ),
\end{aligned}$$

where $C^a[u, v]$ is the translation of a Δ_0 formula of $\mathcal{L}_{\mathsf{Ad}}$. Assume $\mathrm{org}[\kappa]$, $a = L^{\mathsf{Ad}}_{\kappa}$, $b \in a$ and $(\forall x \in b)(\exists y \in a)C^a[x, y]$. Since there is a Δ_0 formula of $\mathcal{L}_{\in}$ which expresses that certain set is an ordered pair and $a \in L^{\mathsf{Ad}}_{\kappa+1}$, there is by the definition of the constructible hierarchy and Lemma 3.14 an $m_1 < \omega$ such that $b \times a \in L^{\mathsf{Ad}}_{\kappa+m_1}$. Thus by Lemma 3.21 there is an $m_2 < \omega$ such that

$$a_0 := \{\langle x, y\rangle \in b \times a : C^a[x, y]\} \in L^{\mathsf{Ad}}_{\kappa+m_1+m_2}.$$

Let $a_{L^{\mathsf{Ad}}}$ be the set $\{\langle\beta, L^{\mathsf{Ad}}_\beta\rangle : \beta < \kappa\}$ which is by the definition of L^{Ad} an element of $L^{\mathsf{Ad}}_{\kappa+1}$. Further let $\widetilde{\mathrm{od}}(z) = \alpha$ be the Δ_0 formula

$$(\forall\beta < \alpha)(\neg\exists x \in a_{L^{\mathsf{Ad}}}((x)_1 = \beta \wedge z \in (x)_2)) \wedge (\exists x \in a_{L^{\mathsf{Ad}}})((x)_1 = \alpha \wedge z \in (x)_2).$$

Then for every $z \in L^{\mathsf{Ad}}_\kappa$ we have $\mathrm{od}^{\mathsf{Ad}}(z) = \alpha \leftrightarrow \widetilde{\mathrm{od}}(z) = \alpha$. Let $D[x, \delta]$ be the Δ_0 formula

$$(\exists y \in a)(\langle x, y\rangle \in a_0 \wedge (\forall z \in a)(\langle x, z\rangle \in a_0 \to \widetilde{\mathrm{od}}(y) \leq \widetilde{\mathrm{od}}(z)) \wedge \delta = \widetilde{\mathrm{od}}(y)).$$

That is, for every $x \in b$, $D[x, \delta]$ holds iff δ is the least possible order of an element y of a such that $C^a[x, y]$. Because all parameters of $D[x, \delta]$ as well as $b \times \kappa$ are elements of $L^{\mathsf{Ad}}_{\kappa+m_3}$ for some $m_3 < \omega$, there must be by Lemma 3.14 an $m_4 < \omega$ such that

$$f := \{\langle x, \delta\rangle \in b \times \kappa : D[x, \delta]\} \in L^{\mathsf{Ad}}_{\kappa+m_4} \subseteq L^{\mathsf{Ad}}_{\kappa+\omega}.$$

It's easy to see that we have $\mathrm{Fun}[f]$, $\mathrm{Dom}[f, b]$ and $\mathrm{Ran}_\subseteq[f, \kappa]$. Because $\mathrm{org}[\kappa]$ there must be a $\beta < \kappa$ such that $\mathrm{Ran}_\subseteq[f, \beta]$. Because of the definition of f and the fact that $\delta < \beta$ implies $L^{\mathsf{Ad}}_\delta \subseteq L^{\mathsf{Ad}}_\beta$, we can conclude that if $z = L^{\mathsf{Ad}}_\beta$, we have $z \in a$ and $(\forall x \in b)(\exists y \in z)C^a[x, y]$. Hence OST proves the formula $\mathring{A}$.

In the last case B is an instance of (Tran) and $\mathring{A}$ is

$$\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge a = L^{\mathsf{Ad}}_\kappa) \to (\forall x \in a)(\exists y \in a)(x \subseteq y \wedge \mathrm{Tran}[y]).$$

Since L^{Ad}_α is transitive for every α, OST proves this formula as well. □

Lemma 3.24. *Let A be the third axiom of the axioms $\mathcal{AD}$. Then* OST *proves $\mathring{A}$.*

Proof. The formula $\mathring{A}$ is the $\mathcal{L}^\circ_\in$ formula

$$\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge a = L^{\mathsf{Ad}}_\kappa) \wedge \exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge b = L^{\mathsf{Ad}}_\kappa)$$
$$\to (a \in b \vee a = b \vee b \in a).$$

By Lemma 3.13 OST proves this formula. □

Lemma 3.25. *Let A be an instance of an axiom of* KP *with at most the variables $\vec{x}$ free. Then* OST *proves $\vec{x} \in \mathbf{L}^{\mathsf{Ad}} \to \mathring{A}$.*

Proof. If A is an instance of extensionality, pairing, union, infinity or $\in$-induction, the assertion holds clearly. If A is an instance of Δ_0 separation, the assertion follows directly from Lemma 3.21.

The remaining case is if A is an instance of Δ_0 collection. Let $C[x, y, \vec{z}]$ be a Δ_0 formula of $\mathcal{L}_{\mathsf{Ad}}$ with at most the indicated variables free and $D[x, y, \vec{z}]$ the $\mathcal{L}_{\in}^{\circ}$ formula $\mathring{C}[x, y, \vec{z}]$ but every occurrence of $\exists\kappa(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge x = L_{\kappa}^{\mathsf{Ad}})$ replaced by $\mathrm{ad}[x] := (\exists\kappa < \mathrm{od}^{\mathsf{Ad}}(x))(\kappa > \omega \wedge \mathrm{org}[\kappa] \wedge x = L_{\kappa}^{\mathsf{Ad}})$, which is equivalent to a Δ formula. OST proves that if $x \in \mathbf{L}^{\mathsf{Ad}}$, then the translation of the $\mathcal{L}_{\mathsf{Ad}}$ formula $\mathsf{Ad}(x)$ is equivalent to the formula $\mathrm{ad}[x]$. Thus we can show by induction on the length of $C[x, y, \vec{z}]$, that if $x, y, \vec{z} \in \mathbf{L}^{\mathsf{Ad}}$ then $\mathring{C}[x, y, \vec{z}]$ is equivalent to $D[x, y, \vec{z}]$. Thus we can proof that OST proves $\vec{x} \in \mathbf{L}^{\mathsf{Ad}} \to \mathring{A}$ if A is an instance of Δ_0 collection by Lemma 3.4 using operational choice and operational replacement as in the proof (the case of Δ_0 collection) of Theorem 6 in Jäger [18]. $\square$

We will see in the proof of the next theorem that L_{κ}^{Ad} corresponds to a strong admissible set in OST (i.e. OST proves $\mathring{\mathrm{Sd}}[L_{\kappa}^{\mathsf{Ad}}]$), if κ is an operationally regular ordinal.

Lemma 3.26. *Let A be the axiom* (SdLim). *Then* $\mathsf{OST} + (\mathsf{Inac})$ *proves* $\mathring{A}$.

Proof. The $^\circ$-translation of the formula $\mathrm{Sd}[b]$ is the formula

$$\mathring{\mathsf{Ad}}(b) \ \wedge\ (\forall x \in b)(\forall f \in \mathbf{L}^{\mathsf{Ad}})(\ \mathrm{Fun}[f] \wedge \mathrm{Dom}[f, x] \wedge \mathrm{Ran}_{\subseteq}[f, b] \\ \to (\exists y \in b)\mathrm{Ran}_{\subseteq}[f, y]\ ).$$

We have to prove

$$(\forall a \in \mathbf{L}^{\mathsf{Ad}})(\exists b \in \mathbf{L}^{\mathsf{Ad}})(a \in b \wedge \mathring{\mathrm{Sd}}[b])).$$

So let $a \in \mathbf{L}^{\mathsf{Ad}}$. Then there is an ordinal α with $a \in L_{\alpha}^{\mathsf{Ad}}$. By (Inac) there is an ordinal $\beta > \alpha$ with $\beta > \omega$ and $\mathrm{Org}[\beta]$. We set $b := L_{\beta}^{\mathsf{Ad}}$ and, since by Lemma 3.19 we have also $\mathrm{org}[\beta]$, it follows that $a \in b$ and $\mathring{\mathsf{Ad}}(b)$. Now assume $f \in \mathbf{L}^{\mathsf{Ad}}$, $\mathrm{Fun}[f]$, the domain of f is an element of b and $\mathrm{Ran}_{\subseteq}[f, b]$. Let $\tilde{f}$ be the function with the same domain as f such that $\tilde{f}(x) = \mathrm{od}^{\mathsf{Ad}}(f(x))$ (it can be defined similar as the function f in the proof of Lemma 3.23). Since β is a limit ordinal, it follows $\mathrm{Ran}_{\subseteq}[\tilde{f}, \beta]$. One can show as in the proof of Lemma 3.19, that there is a $\gamma < \beta$ with $\mathrm{Ran}_{\subseteq}[\tilde{f}, \gamma]$. Therefore we have $\mathrm{Ran}_{\subseteq}[f, L_{\gamma}^{\mathsf{Ad}}]$ and $L_{\gamma}^{\mathsf{Ad}} \in b$. $\square$

Summing up the Lemmas 3.22-3.26, we get the following theorem.

Theorem 3.27. *The theory* KPSd *can be embedded into* $\mathsf{OST} + \mathsf{Inac}$*; i.e. we have for all $\mathcal{L}_{\mathsf{Ad}}$ formulas A with at most the variables $\vec{x}$ free:*

If KPSd *proves A then* $\mathsf{OST} + \mathsf{Inac}$ *proves* $\vec{x} \in \mathbf{L}^{\mathsf{Ad}} \to \mathring{A}$.

Later we will use the following corollary.

Corollary 3.28. *Every* Σ *sentence of* $\mathcal{L}_\in$ *which is provable in* KPSd *is also provable in* OST + Inac.

Proof. Assume that A is a Σ sentence of $\mathcal{L}_\in$ which is provable in KPSd. Since the relation symbol Ad does not occur in A, $\mathring{A}$ is the formula $A^{\mathbf{L}^{\mathsf{Ad}}}$. And since A has no free variables, $A^{\mathbf{L}^{\mathsf{Ad}}}$ is provable in OST + Inac by the previous theorem. By Σ persistency (c.f. for instance Lemma 4.2 of Chapter I in Part A of Barwise [2]) also A is provable in OST + Inac. $\square$

3.3 Interpreting KP in OST$^-$ via IKP$^-$

In the first part of this section we will embed IKP$^-$ (and some extensions) into OST$^-$ (and some extensions) by a well-known realisation method. In the second part we will put this result together with the main results of the sections 2.2, 2.3 and 2.4. All in all we will obtain so an interpretation of KP in OST$^-$. In the whole section, the formulations of all definitions, remarks, lemmas, theorems and proofs as well as some comments are taken from Sato and Zumbrunnen [31].

3.3.1 Realisation of IKP$^-$ in OST$^-$

First of all we define realisability of formulas of $\mathcal{L}$. This is done similar as for instance in Feferman [10] or [12].

Definition 3.29 (Realiser $\mathfrak{r}$)**.** For each $\mathcal{L}$ formula A the $\mathcal{L}^\circ$ formula $f \mathrel{\mathfrak{r}} A$, f *realises* A or f is a *realiser* of A, is inductively defined as follows:

(i) If A is atomic, then $f \mathrel{\mathfrak{r}} A$ is the formula A.

(ii) If A is the formula $B \wedge C$, then $f \mathrel{\mathfrak{r}} A$ is the formula

$$\exists g, h(f = \langle g, h\rangle \wedge (g \mathrel{\mathfrak{r}} B) \wedge (h \mathrel{\mathfrak{r}} C)).$$

(iii) If A is the formula $B \vee C$, then $f \mathrel{\mathfrak{r}} A$ is the formula

$$\exists g((f = \langle \overline{0}, g\rangle \wedge g \mathrel{\mathfrak{r}} B) \vee (f = \langle \overline{1}, g\rangle \wedge g \mathrel{\mathfrak{r}} C)).$$

(iv) If A is the formula $B \to C$, then $f \mathrel{\mathfrak{r}} A$ is the formula

$$\forall g(g \mathrel{\mathfrak{r}} B \to f(g) \mathrel{\mathfrak{r}} C).$$

(v) If A is the formula $\exists x B[x]$, then $f \mathrel{\mathfrak{r}} A$ is the formula

$$\exists x \exists g(f = \langle x, g\rangle \wedge g \mathrel{\mathfrak{r}} B[x]).$$

(vi) If A is the formula $\forall x B[x]$, then $f \mathrel{\mathfrak{r}} A$ is the formula $\forall x(f(x) \mathrel{\mathfrak{r}} B[x])$.

We say that the formula $A[\vec{x}]$ with at most the free variables $\vec{x}$ is *realisable in* some theory $\mathcal{T}$, if $\mathcal{T}$ proves the formula $\exists f \forall \vec{x}(f(\vec{x}) \mathrel{\mathfrak{r}} A[\vec{x}])$. We just say that a formula is *realisable* if it is realisable in OST_0^-.

We have introduced the notion of realisability only for formulas of $\mathcal{L}$ because we want to realise all axioms of IKP^-. However, we will use the notation $f \mathrel{\mathfrak{r}} A[t]$ also if t is an $\mathcal{L}^\circ$ term but not an $\mathcal{L}$ term. When we do so, we write $f \mathrel{\mathfrak{r}} A[t]$ for the formula $(f \mathrel{\mathfrak{r}} A[v])[t/v]$.

For proving the closure of realisability under the rules of inference of intuitionistic logic we will need the following lemma which can be proved by a straightforward induction on on the length of the formula A.

Lemma 3.30. *If A is an $\mathcal{L}$ formula then the free variables of $f \mathrel{\mathfrak{r}} A$ are the variable f and the free variables of A.*

We introduce now for each negative $\mathcal{L}$ formula A a term which delivers us a realiser of A if and only if A holds.

Definition 3.31 (Term $r_{A,\vec{x}}$)**.** We assign to each finite sequence $\vec{x} = x_0, ..., x_n$ of variables and each negative $\mathcal{L}$ formula $A[\vec{x}]$, in which at most the variables $\vec{x}$ occur freely, an $\mathcal{L}^\circ$ term $r_{A,\vec{x}}$ inductively defined as follows:

(i) If $A[\vec{x}]$ is atomic then $r_{A,\vec{x}}$ is the term $\lambda\vec{x}.\overline{0}$.

(ii) If $A[\vec{x}]$ is the formula $B[\vec{x}] \wedge C[\vec{x}]$ then $r_{A,\vec{x}}$ is the term

$$\lambda\vec{x}.\mathrm{p}(r_{B,\vec{x}}(\vec{x}), r_{C,\vec{x}}(\vec{x})).$$

(iii) If $A[\vec{x}]$ is the formula $B[\vec{x}] \to C[\vec{x}]$ then $r_{A,\vec{x}}$ is the term $\lambda\vec{x}, g.r_{C,\vec{x}}(\vec{x})$, where g is a variable not occurring in $r_{C,\vec{x}}$.

(iv) If $A[\vec{x}]$ is the formula $\forall y B[x_0, ..., x_i, y, x_{i+1}, ..., x_n]$ then $r_{A,\vec{x}}$ is the term

$$\lambda x_0...x_n y.r_{B,x_0,...,x_i,y,x_{i+1},...,x_n}(x_0, ..., x_i, y, x_{i+1}, ..., x_n).$$

Lemma 3.32. *We have for each finite sequence $\vec{x} = x_0, ..., x_n$ of variables and each negative $\mathcal{L}$ formula $A[\vec{x}]$ in which at most the variables $\vec{x}$ occur freely:*

(i) $r_{A,\vec{x}}$ is a closed term.

(ii) OST_0^- *proves that if there is a realiser of* A*, then* $r_{A,\vec{x}}(\vec{x})$ *is a realiser of* A*.*

(iii) OST_0^- *proves* $\forall \vec{x}(A[\vec{x}] \leftrightarrow r_{A,\vec{x}}(\vec{x}) \mathrel{\mathfrak{r}} A[\vec{x}])$.

Proof. The proof is by induction on the length of the formula A. For the case with $\rightarrow$ in the third assertion we need the second one. $\square$

The next lemma follows directly by the second and third assertion of the lemma above.

Lemma 3.33. *If* A *is a negative realisable* $\mathcal{L}$ *formula, then* OST_0^- *proves* A*.*

Before we prove the realisability of all non-logical axioms of IKP^-, we treat its logical axioms and rules.

Lemma 3.34. *All axioms of intuitionistic logic are realisable and realisability (in any theory which contains* OST_0^-*) is closed under its rules of inference.*

Proof. We can assume that the equality axioms for $=$ are formulated using only negative formulas. Since these axioms are available in OST_0^-, they are therefore realisable by Lemma 3.32.

Since λ-abstraction is available, the proof of the realisability of all the propositional axioms of a Hilbert-style calculus is straightforward in view of Lemma 1.37 and Proposition 1.35. For instance

$$\lambda fgh.\mathrm{iszero}(\mathrm{p}_0(h), f(\mathrm{p}_1(h)), g(\mathrm{p}_1(h)))$$

realises

$$(A \rightarrow B) \rightarrow ((C \rightarrow B) \rightarrow ((A \vee C) \rightarrow B)).$$

Further $\lambda f.f(y)$ realises $\forall x A[x] \rightarrow A[y]$ and $\lambda f.\mathrm{p}(y, f)$ realises $A[y] \rightarrow \exists x A[x]$ if y is not bounded in A. For proving the closure of realisability under the quantifier rules we assume in both cases

$$\forall x, \vec{y}(f(x, \vec{y}) \mathrel{\mathfrak{r}} (A[x, \vec{y}] \rightarrow B[x, \vec{y}])) \tag{3.1}$$

that is, we have for all $x, \vec{y}$

$$\forall g((g \mathrel{\mathfrak{r}} A[x, \vec{y}]) \rightarrow ((f(x, \vec{y}))(g) \mathrel{\mathfrak{r}} B[x, \vec{y}])). \tag{3.2}$$

For the $\forall$-rule we assume in addition that x does not occur freely in $A[x, \vec{y}]$, for which we write $A[\vec{y}]$, and get by Lemma 3.30

$$\forall g((g \mathrel{\mathfrak{r}} A[\vec{y}]) \rightarrow \forall x((f(x, \vec{y}))(g) \mathrel{\mathfrak{r}} B[x, \vec{y}]))$$

because the rule is also available in classical logic. Therefore

$$\forall \vec{y}(\lambda gx.f(x, \vec{y}, g) \mathrel{\mathfrak{r}} (A[\vec{y}] \rightarrow \forall x B[x, \vec{y}])).$$

For the $\exists$-rule we deduce from (3.2) for all $x, \vec{y}$

$$\forall g(\mathrm{p}_0(g) = x \wedge (\mathrm{p}_1(g) \mathrel{\mathfrak{r}} A[x,\vec{y}]) \to ((f(\mathrm{p}_0(g),\vec{y}))(\mathrm{p}_1(g)) \mathrel{\mathfrak{r}} B[\mathrm{p}_0(g),\vec{y}]))$$

and assume in addition that x does not occur freely in $B[x,\vec{y}]$, for which we write $B[\vec{y}]$. We get by Lemma 3.30 for all $\vec{y}$

$$\forall g(\exists x(\mathrm{p}_0(g) = x \wedge (\mathrm{p}_1(g) \mathrel{\mathfrak{r}} A[x,\vec{y}])) \to ((f(\mathrm{p}_0(g),\vec{y}))(\mathrm{p}_1(g)) \mathrel{\mathfrak{r}} B[\vec{y}])$$

because also this rule is available in classical logic. Therefore

$$\forall \vec{y}(\lambda g.f(\mathrm{p}_0(g),\vec{y},\mathrm{p}_1(g)) \mathrel{\mathfrak{r}} (\exists x A[x,\vec{y}] \to B[\vec{y}])).$$

That realisability is closed under modus ponens is obvious. □

In the next steps, we prove the realisability of all non-logical axioms of IKP^-.

Lemma 3.35 (Pairing)**.** *The formula $\exists x(a \in x \wedge b \in x)$ is realisable.*

Proof. Let $B[x,a,b]$ be the negative formula $(a \in x \wedge b \in x)$. Then we have by Lemma 3.32 that $r_{B,x,a,b}(\mathbb{D}(a,b),a,b) \mathrel{\mathfrak{r}} B[\mathbb{D}(a,b),a,b])$. If we set

$$f = \lambda ab.\mathrm{p}(\mathbb{D}(a,b), r_{B,x,a,b}(\mathbb{D}(a,b),a,b)),$$

we have therefore for every a, b

$$f(a,b) = \langle \mathbb{D}(a,b), r_{B,x,a,b}(\mathbb{D}(a,b),a,b)\rangle$$

and

$$r_{B,x,a,b}(\mathbb{D}(a,b),a,b) \mathrel{\mathfrak{r}} B[\mathbb{D}(a,b),a,b].$$

That is, $f(a,b)$ realises the formula $\exists x(a \in x \wedge b \in x)$. □

Lemma 3.36 (Union)**.** *The formula $\exists x(\forall y \in a)(\forall z \in y)(z \in x)$ is realisable.*

Proof. Let $B[a,x]$ be the negative formula $(\forall y \in a)(\forall z \in y)(z \in x)$. Then Lemma 3.32 implies $r_{B,a,x}(a,\mathbb{U}(a)) \mathrel{\mathfrak{r}} B[a,\mathbb{U}(a)]$. Similar as in the last proof, $\lambda a.\mathrm{p}(\mathbb{U}(a), r_{B,a,x}(a,\mathbb{U}(a)))$ applied to a is a realiser of $\exists x B[a,x]$. □

Lemma 3.37 (Δ_0^- separation)**.** *For all negative Δ_0 formulas $A[y,a,\vec{v}]$ of $\mathcal{L}$ in which x does not occur and with at most the variables $y, a, \vec{v}$ free, the formula*

$$\exists x((\forall y \in x)(y \in a \wedge A[y,a,\vec{v}]) \wedge (\forall y \in a)(A[y,a,\vec{v}] \to y \in x))$$

is realisable.

Proof. Let $B[x, a, \vec{v}]$ be the negative formula

$$(\forall y \in x)(y \in a \wedge A[y, a, \vec{v}]) \wedge (\forall y \in a)(A[y, a, \vec{v}] \to y \in x).$$

Let t_A be the term defined in Lemma 1.31. Then Lemma 3.32 implies

$$r_{B,x,a,\vec{v}}(\mathbb{S}(\lambda y.t_A(y, a, \vec{v}), a), a, \vec{v}) \ \mathfrak{r} \ B[\mathbb{S}(\lambda y.t_A(y, a, \vec{v}), a), a, \vec{v}].$$

As in the last two proofs,

$$\lambda a, \vec{v}.\mathrm{p}(\mathbb{S}(\lambda y.t_A(y, a, \vec{v}), a), r_{B,x,a,\vec{v}}(\mathbb{S}(\lambda y.t_A(y, a, \vec{v}), a), a, \vec{v}))$$

applied to $a, \vec{v}$ is the realiser we are searching for. $\square$

The next task is the realisability of all instances of Δ_0 collection. We will prove that the collection schema is even realisable for arbitrary formulas, not only for Δ_0 formulas:

Lemma 3.38 (Collection). *For all formulas $A[x, y, a, \vec{v}]$ of $\mathcal{L}$ in which z does not occur and with at most the variables $x, y, a, \vec{v}$ free, the formula*

$$(\forall x \in a)\exists y A[x, y, a, \vec{v}] \to \exists z(\forall x \in a)(\exists y \in z)A[x, y, a, \vec{v}]$$

is realisable.

Proof. Assume $f \ \mathfrak{r} \ (\forall x \in a)\exists y A[x, y, a, \vec{v}]$. That is to say

$$\forall x(\forall g((g \ \mathfrak{r} \ (x \in a)) \to \exists y \exists h((f(x))(g) = \langle y, h\rangle \wedge (h \ \mathfrak{r} \ A[x, y, a, \vec{v}])))).$$

If t is the term $\lambda xg.\mathrm{p}(\mathrm{p}_0(f(x, g)), \mathrm{p}(\overline{0}, \mathrm{p}_1(f(x, g))))$ and $B[x, y, z, a, \vec{v}]$ is the formula $y \in z \wedge A[x, y, a, \vec{v}]$ we get,

$$\begin{aligned}&\forall x(\forall g(\ (g \ \mathfrak{r} \ (x \in a)) \to \\ &\qquad \exists y \exists h(t(x, g) = \langle y, h\rangle \wedge (h \ \mathfrak{r} \ B[x, y, \mathbb{R}(\lambda x.\mathrm{p}_0(f(x, g)), a), a, \vec{v}])) \ )).\end{aligned}$$

since $\mathrm{p}_0(f(x, g)) \in \mathbb{R}(\lambda x.\mathrm{p}_0(f(x, g)), a)$ if $x \in a$, everything and therefore $\overline{0}$ realises $y \in z$ if it holds and z does not occur in $A[x, y, a, \vec{v}]$. And since $g \ \mathfrak{r} \ (x \in a)$ is the formula $x \in a$ and therefore $\overline{0}$ realises $x \in a$ if it holds, we have also

$$\begin{aligned}&\forall x(\forall g((g \ \mathfrak{r} \ (x \in a)) \to \\ &\qquad\qquad \exists y \exists h(\ (\lambda xg.t(x, \overline{0}))(x, g) = \langle y, h\rangle \\ &\qquad\qquad\qquad \wedge (h \ \mathfrak{r} \ B[x, y, \mathbb{R}(\lambda x.\mathrm{p}_0(f(x, \overline{0})), a), a, \vec{v}]) \ ))).\end{aligned}$$

It follows that

$$\mathrm{p}(\mathbb{R}(\lambda x.\mathrm{p}_0(f(x, \overline{0})), a), \lambda xg.t(x, \overline{0})) \ \mathfrak{r} \ \exists z(\forall x{\in}a)\exists y B[x, y, z, a, \vec{v}].$$

We can conclude that if $s := \lambda a, \vec{v}, f.\mathrm{p}(\mathbb{R}(\lambda x.\mathrm{p}_0(f(x,\overline{0})), a), \lambda xg.t(x,\overline{0}))$ we have for all $a, \vec{v}$

$$s(a,\vec{v}) \ \mathfrak{r} \ ((\forall x \in a)\exists y A[x,y,a,\vec{v}] \to \exists z(\forall x \in a)(\exists y \in z)A[x,y,a,\vec{v}])$$

since $\exists z(\forall x{\in}a)\exists y B[x,y,a,\vec{v}]$ is nothing else than

$$\exists z(\forall x{\in}a)(\exists y \in z)A[x,y,a,\vec{v}].$$

□

Lemma 3.39 ($\in$-induction)**.** *For arbitrary $\mathcal{L}$ formulas $A[x,\vec{v}]$ with at most the variables $x, \vec{v}$ free the formula*

$$\forall x((\forall y \in x)A[y,\vec{v}] \to A[x,\vec{v}]) \to \forall x A[x,\vec{v}]$$

is realisable in OST^-*.*

Analogous assertions hold for restricted induction principles and OST^- *with only the corresponding principles.*

Proof. Let t be the term $\lambda hgx.g(x, \lambda yf.h(g,y))$ and s the term $\mathrm{fix}(t)$. From the recursion theorem it follows $s{\downarrow}$ and $s(g) \simeq \lambda x.g(x, \lambda yf.s(g,y))$. Now we fix a g and assume that it realises the antecedent

$$\forall x((\forall y \in x)A[y,\vec{v}] \to A[x,\vec{v}]),$$

i.e.

$$g \ \mathfrak{r} \ \forall x(\forall y(y \in x \to A[y,\vec{v}]) \to A[x,\vec{v}]).$$

This is by definition

$$\forall x(\forall h((h \ \mathfrak{r} \ \forall y(y \in x \to A[y,\vec{v}])) \to ((g(x))(h) \ \mathfrak{r} \ A[x,\vec{v}]))).$$

In the following we prove by $\in$-induction that $\forall x(s(g,x) \ \mathfrak{r} \ A[x,\vec{v}])$. For that purpose we fix an x and assume $(\forall y \in x)(s(g,y) \ \mathfrak{r} \ A[y,\vec{v}])$ which is equivalent to

$$\forall y(\forall f(y \in x \to ((\lambda f.s(g,y))(f) \ \mathfrak{r} \ A[y,\vec{v}]))).$$

The latter means $\forall y(\lambda f.s(g,y) \ \mathfrak{r} \ (y \in x \to A[y,\vec{v}]))$ and it follows

$$\lambda yf.s(g,y) \ \mathfrak{r} \ \forall y(y \in x \to A[y,\vec{v}]).$$

By the assumption about g we get therefore $g(x, \lambda yf.s(g,y)) \ \mathfrak{r} \ A[x,\vec{v}]$, and we can conclude $s(g,x) \ \mathfrak{r} \ A[x,\vec{v}]$ since $s(g,x) \simeq g(x, \lambda yf.s(g,y))$. We have proved now that $s(g,x) \ \mathfrak{r} \ A[x,\vec{v}]$ follows from $(\forall y \in x)(s(g,y) \ \mathfrak{r} \ A[y,\vec{v}])$, therefore the former holds by $\in$-induction for every x. All in all we get that for all $\vec{v}$ the term $(\lambda\vec{v}.s)(\vec{v})$ is the realiser we are searching for.

That analogous assertions hold for restricted induction principles and OST^- with only the corresponding principles can be proved analogously.

□

Lemma 3.40 (N-infinity)**.** *The formula*

$$\exists x((\exists y \in x)\mathrm{zero}[y] \wedge (\forall y \in x)(\exists z \in x)\mathrm{succ}[y,z])^N$$

is realisable.

Proof. Let $B[x]$ be the negative formula

$$((\exists y \in x)\mathrm{zero}[y] \wedge (\forall y \in x)(\exists z \in x)\mathrm{succ}[y,z])^N.$$

Then Lemma 3.32 implies $r_{B,x}(\omega)\ \mathfrak{r}\ B[\omega]$. Similar as in the proofs of the Lemmas 3.35 to 3.37, $\mathrm{p}(\omega, r_{B,x}(\omega))$ is a realiser of $\exists x B[x]$. $\Box$

In the next definition we introduce a special notation of Skolemisation. We will use it in order to generalise our realisation result to extensions of IKP^- and OST^-.

Definition 3.41. If A is an $\mathcal{L}$ formula of the form

$$\forall x_0 \exists y_0 ... \forall x_n \exists y_n B[x_0, ... x_n, y_0, ... y_n],$$

where B is a negative formula of $\mathcal{L}$, we write $A^{\mathrm{s}}[f_0, ..., f_n]$ for its *operational Skolemisation*, the $\mathcal{L}^\circ$ formula

$$\forall x_0, ..., x_n B[x_0, ..., x_n, f_0(x_0), f_1(x_0, x_1), ..., f_n(x_0, ..., x_n)]$$

and $A^{\exists \mathrm{s}}$ for the formula $\exists f_0, ..., f_n A^{\mathrm{s}}[f_0, ..., f_n]$.

If $\mathcal{A}$ is a set of formulas of the form described above, we write $\mathcal{A}^{\exists \mathrm{s}}$ for the set $\{A^{\exists \mathrm{s}} : A \in \mathcal{A}\}$.

Lemma 3.42. *Let* $\vec{x} = x_0, ... x_n$*,* $\vec{y} = y_0, ... y_n$ *and* $\vec{z} = z_0, ..., z_m$*, and* $A[\vec{z}]$ *be an* $\mathcal{L}$ *formula of the form* $\forall x_0 \exists y_0 ... \forall x_n \exists y_n B[\vec{x}, \vec{y}, \vec{z}]$ *with at most* $\vec{z}$ *free where* B *is a negative formula of* $\mathcal{L}$*. Then there exists an* $\mathcal{L}^\circ$ *term* t *such that* OST_0^- *proves*

$$A^{\mathrm{s}}[f_0, ..., f_n, \vec{z}] \to t(f_0, ..., f_n){\downarrow} \wedge \forall \vec{z}(t(f_0, ..., f_n, \vec{z})\ \mathfrak{r}\ A[\vec{z}]).$$

Proof. Assume $A^{\mathrm{s}}[f_0, ..., f_n, \vec{z}]$, i.e.

$$\forall \vec{x} B[\vec{x}, f_0(x_0), f_1(x_0, x_1), ..., f_n(x_0, ..., x_n), \vec{z}].$$

Therefore we have by Lemma 3.32 that

$$r_{B,\vec{x},\vec{y},\vec{z}}(\vec{x}, f_0(x_0), f_1(x_0, x_1), ..., f_n(x_0, ..., x_n), \vec{z})$$

realises

$$B[\vec{x}, f_0(x_0), f_1(x_0, x_1), ..., f_n(x_0, ..., x_n), \vec{z}]$$

for all $\vec{x}$. Now let s be the term

$$\lambda x_0.\mathrm{p}(f_0(x_0), \lambda.x_1.\mathrm{p}(f_1(x_0,x_1), \ldots \\ \lambda x_n.\mathrm{p}(f_n(x_0,\ldots,x_n), r_{B,\vec{x},\vec{y},\vec{z}}(\vec{x}, f_0(x_0), f_1(x_0,x_1), \ldots, f_n(x_0,\ldots,x_n), \vec{z})) \\ \ldots))$$

and check that it realises $\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n B[\vec{x}, \vec{y}, \vec{z}]$. Therefore $\lambda f_0, \ldots, f_n, \vec{z}.s$ is the term we are searching for. □

If $\mathcal{A}$ is a set of formulas of the form $\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n B$, where B is negative, then the previous lemma implies that every formula in $\mathcal{A}$ is realisable in $\mathsf{OST}_0^- + \mathcal{A}^{\exists s}$. Therefore the following theorem is a consequence of the Lemmas 3.33-3.42.

Theorem 3.43. *Let $\mathcal{A}$ be a set of formulas of the form $\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n B$, where B is a negative $\mathcal{L}$ formula. If A is a negative $\mathcal{L}$ formula provable in $\mathsf{IKP}^- + \mathcal{A}$, then A is also provable in $\mathsf{OST}^- + \mathcal{A}^{\exists s}$.*

Analogous assertions hold for the versions of IKP^- and OST^- with restricted induction principles.

Remark 4. Notice that all arguments used to prove the Lemmas 3.33 to 3.42 are also available in intuitionistic logic. Therefore the previous theorem holds also for the weakening of OST^- based on intuitionistic logic. Similar variants of such intuitionistic systems were considered in Cantini [5] and Cantini and Crosilla [6, 7, 8].

3.3.2 Merging some Preceding Results

Before we can combine the last theorem of the previous subsection with the main results of the sections 2.2, 2.3 and 2.4, there are still some lemmas to prove.

Lemma 3.44. *Let $\mathcal{T}$ be a theory containing KP^{int}, R an $(n+1)$-ary relation symbol of $\mathcal{L}$ which is suitable for $*$-translation and x the sequence of variables $x_0, \ldots, x_{n-1}$. If $A[a,b]$ is the formula*

$$\mathrm{Bis}[a] \wedge b \subseteq \mathrm{field}(a) \wedge (\forall \vec{z} \in \mathrm{field}(a))(\exists y' \in \mathrm{field}(a)) R(\vec{z}, y'),$$

then $\mathcal{T}$ proves

$$(\forall \vec{x}, y \in \vec{b})(\ \forall a (A[a,b] \to (\exists y' \in \mathrm{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))) \\ \leftrightarrow \exists y'(y \sim y' \wedge R(\vec{x}, y'))\ ).$$

PROOF. For the direction from left to right assume $\vec{x}, y \in \vec{b}$ and

$$\forall a(\mathrm{Bis}[a] \wedge b \subseteq \mathrm{field}(a) \wedge (\forall \vec{z} \in \mathrm{field}(a))(\exists y' \in \mathrm{field}(a))R(\vec{z}, y') \\ \to (\exists y' \in \mathrm{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))).$$

Let a be a set with $\mathrm{Bis}[a]$ and $b \subseteq \mathrm{field}(a)$ (such a set exists by Lemma 4.3 in Avigad [1]). Because R is suitable for $*$-translation, $\mathcal{T}$ proves by Definition 2.11 (ii) that

$$(\forall \vec{z} \in \mathrm{field}(a))\exists y' R(\vec{z}, y'),$$

and by Δ_0 collection that there is a c with

$$(\forall \vec{z} \in \mathrm{field}(a))(\exists y' \in c)R(\vec{z}, y').$$

Then one can prove in $\mathcal{T}$ that there is an a' such that $\mathrm{field}(a) \cup c \subseteq \mathrm{field}(a')$. (c.f. again Lemma 4.3 in Avigad [1]). It follows

$$\mathrm{Bis}[a'] \wedge b \subseteq \mathrm{field}(a') \wedge (\forall \vec{z} \in \mathrm{field}(a))(\exists y' \in \mathrm{field}(a'))R(\vec{z}, y'),$$

and therefore $(\exists y' \in \mathrm{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))$ by our assumption. Then

$$(\exists y' \in \mathrm{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))$$

follows by the definition of $\sim$.

The direction from right to left follows by Lemma 2.10 and from Definition 2.11 (i). $\square$

Lemma 3.45. *There is a closed term r such that OST^-_ω proves for every operation f and every set a that*

$$\textit{If } \mathsf{OST}^-_\omega \textit{ proves } \underbrace{f(f(\ldots f}_{n\ times}(a)))\!\downarrow \textit{ for each natural number } n, \\ \textit{then it proves } r(f,a) = \cup\{a, f(a), f(f(a)), \ldots\}.$$

PROOF. Let $A[n]$ be the Δ_0 formula $n = \emptyset$ and t_A the corresponding term due to Lemma 1.31. Furthermore let t_0 be the term $\lambda gnfa.a$, t_1 the term $\lambda gnfa.f(g(\mathbb{U}(n), a))$ and s the term $\lambda gnfa.(\mathrm{ite}_A(t_0, t_1, n))(g, n, f, a)$. Since we consider the empty set as $0 \in \omega$ and $\cup n$ corresponds to $n-1$ for all non-empty $n \in \omega$, OST^-_ω proves by Lemma 1.30 and the axioms for $\mathbb{D}$ and $\mathbb{U}$ and $\in$-induction on ω that for every $n \in \omega$:

$$(\mathrm{fix}(s))(n, f, a) = s(\mathrm{fix}(s), n, f, a) = \begin{cases} a & \text{if } n = 0 \text{ and} \\ f(\mathrm{fix}(s)(n-1, f, a)) & \text{if } n > 0, \end{cases}$$

if we have for every natural number m that

$$\underbrace{f(f(...f}_{m \text{ times}}(a)))\downarrow.$$

Now we set $r := \lambda fa.\mathbb{U}(\mathbb{R}(\lambda n.(\mathrm{fix}(s))(n,f,a),\omega))$. It is easy to check that OST^-_ω proves that $r(f,a)$ corresponds to the set $\cup\{a, f(a), f(f(a)), ...\}$. $\square$

Lemma 3.46. *Let R be an $(n+1)$-ary relation symbol of $\mathcal{L}$ and x the sequence of variables $x_0, ..., x_{n-1}$. If $A[a,b]$ is the formula*

$$\mathrm{Bis}[a] \wedge b \subseteq \mathrm{field}(a) \wedge (\forall \vec{z} \in \mathrm{field}(a))(\exists y' \in \mathrm{field}(a))R(\vec{z}, y'),$$

then $\mathsf{OST}^- + \exists f \forall \vec{x} R(\vec{x}, f(\vec{x}))$ proves the formulas

(i) $(\forall y, z \in b)(\ \forall a(A[a,b] \to y \sim_a z) \ \leftrightarrow\ y = z\ )$ *and*

(ii) $(\forall \vec{x}, y \in b)(\ \forall a(A[a,b] \to$
$(\exists y' \in \mathrm{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))) \leftrightarrow R(\vec{x}, y)\ ).$

Proof. If $y = z$, $\mathrm{Bis}[a]$ and $y, z \in \mathrm{field}(a)$ holds, then $y \sim_a z$ follows by $\in$-induction (because $\mathrm{Bis}[a]$ implies $x \sim_a x$ by $\in$-induction). That establishes the direction from right to left of the first assertion.

The direction from right to left of the second assertion follows by the same direction of the first assertion.

For the other direction of (i) and (ii), let r be the term defined in the proof of the previous lemma. Assume that $\forall \vec{x} R(\vec{x}, f(\vec{x}))$ and let a be

$$\mathbb{R}(\lambda x.\mathbb{D}(x,x), r(\lambda y.\mathbb{U}(\mathbb{D}(\mathbb{U}(y), f(y))), b))$$

which is by Lemma 3.45 the set

$$\{\{x\} : x \in \cup\{b, \cup b \cup f(b), \cup(\cup b \cup f(b)) \cup f(\cup b \cup f(b)), ..\}\}.$$

Then it follows by $\in$-induction and extensionality that $A[a,b]$ and by extensionality that $y \sim_a z$ is equivalent to $y = z$ for all $y, z \in b$. That establishes the direction from left to right of (i). And it follows that

$$(\exists y' \in \mathrm{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))) \text{ implies } R(\vec{x}, y),$$

which establishes the same direction of (ii). $\square$

Lemma 3.47. *Let $\mathcal{R}$ be a set of relation symbols, $\mathcal{T}$ a theory containing KP^{int}, such that all relation symbols of $\mathcal{R}$ are suitable for $*$-translation with respect to $\mathcal{T}$, and $\mathcal{S}$ a theory containing OST^- as well as an axiom of the form $\exists f \forall \vec{x} R(\vec{x}, f(\vec{x}))$ for each relation symbol R of $\mathcal{R}$.*

(i) For every Δ_0 formula B of $\mathcal{L}$ containing only relation symbols of $\mathcal{R}$ there exists an $\mathcal{L}$ formula B' of the form $\forall x C$ such that: C is Δ_0, $\mathcal{T}$ proves $B^* \leftrightarrow B'$ and $\mathcal{S}$ proves $B' \leftrightarrow B$.

(ii) For every $\mathcal{L}$ formula F of the form $\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} B$, where B is Δ_0 and contains only relation symbols of $\mathcal{R}$, there exists an $\mathcal{L}$ formula F'' of the form $\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} \forall z D$ such that: D is Δ_0, D is negative, $F'' \to F''^N$ is intuitionistically valid, $\mathcal{T}$ proves $F^* \leftrightarrow F''$ and $\mathcal{S}$ proves $B \to \forall z D$.

Proof. We assume in the proof of both assertions that B contains only one relation symbol R of arity $(n+1)$ of $\mathcal{R}$. The proof would be very similar but more tedious with more relation symbols.

Let B be a Δ_0 formula of $\mathcal{L}$. Let B_0 be the Δ_0 formula which we get from B^* if we first replace every occurrence of $(\forall x \in^* y)A^*$ by $(\forall x \in y)A^*$, every occurrence of $(\exists x \in^* y)A^*$ by $(\exists x \in y)A^*$, every occurrence

$$\text{of } \exists y'(y \sim y' \wedge R(\vec{x}, y')) \text{ by } (\exists y' \in \text{field}(a))(y \sim_a y' \wedge R(\vec{x}, y'))$$

and then in the end replace every occurrence of $y \sim z$ by $y \sim_a z$, where a is a variable not occurring in B^*. Then B_0 is Δ_0. Let $v_0, ..., v_k$ be all variables which occur freely in B_0 and $A'[a, b]$ the formula

$$\text{Bis}[a] \wedge b \subseteq \text{field}(a) \wedge (\forall \vec{z} \in \text{field}(a))(\exists y' \in \text{field}(a))R(\vec{z}, y').$$

We define now B' to be the formula

$$\forall a(\ A'[a, \{v_0, ..., v_k\}] \to B_0\ ).$$

Using the fact that Bis$[a]$ implies that field$[a]$ is transitive (this follows directly from the definitions of iso and field), the Lemmas 2.10, 2.13 and 3.44 we can proof by induction on the length of B that $\mathcal{T}$ proves $B^* \leftrightarrow B'$. And, using the previous lemma, it is straightforward to prove also by induction on the complexity of B that $\mathcal{S}$ proves $B' \leftrightarrow B$.

For the second assertion assume that F is the formula

$$\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} B$$

where B is Δ_0. Assume that $B' = \forall z C$ is as in the first assertion. Let D be a negative Δ_0 formula intuitionistically equivalent to C^N (the existence of D is assured by Lemma 1.24 (ii)) and set $F'' := \forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} \forall z D$. That F'' has all the stated properties follows from Lemma 1.23 and the first assertion, since F^* is the formula $\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} B^*$. □

The formulation of the next theorem is rather complicated. We will present later a similar assertion (c.f. Theorem 3.49) with an easier formulation. However, the proof of the next theorem can also be applied on variants of OST^-

based on intuitionistic logic and some extensions (c.f. Remark 5), whereas the proof of Theorem 3.49 can not.

Theorem 3.48. *Let $\mathcal{A}$ be a set of $\mathcal{L}$ formulas of the form*

$$\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} B,$$

where B is a Δ_0 formula, which are suitable for $*$-translation with respect to $\mathsf{KP}^{int} + \mathcal{A}^*$. Let $\mathcal{T}$ be a theory containing $\mathsf{OST}^- + \mathcal{A}^{\exists s}$ as well as an axiom of the form $\exists f \forall \vec{x} R(\vec{x}, f(\vec{x}))$ for each relation symbol R occurring in $\mathcal{A}$. If A is a Π_1 formula of $\mathcal{L}$ suitable for $*$-translation with respect to $\mathsf{KP}^{int} + \mathcal{A}^*$ and provable in $\mathsf{KP} + \mathcal{A}$, then A is provable in $\mathcal{T}$.

Proof. Assume A is Π_1, i.e. KP proves that it is equivalent to a formula of the form $\forall x B$ where B is Δ_0, and provable in $\mathsf{KP} + \mathcal{A}$. Therefore by Theorem 2.14 B^* is provable in $\mathsf{KP}^{int} + \mathcal{A}^*$. Now let $\mathcal{B}$ be the set $\{F'' : F \in \mathcal{A}\}$, where the F'' are as in the second assertion of the previous lemma. It follows that B^* is also provable in $\mathsf{KP}^{int} + \mathcal{B}$. Let $B' = \forall y C$ be as in the first assertion of the previous lemma where C is a Δ_0 formula of $\mathcal{L}$. So $\mathsf{KP}^{int} + \mathcal{B}$ proves also B' and $\forall y C^N$. Furthermore C^N is strongly negative and by Lemma 1.24 (ii) intuitionistically equivalent to some negative Δ_0 formula D, i.e. C^N is very weak Σ_1 and $\forall y C^N$ is in $\mathcal{C}_{res}$. So, by the properties of $\mathcal{B}$ and Theorem 2.19, $\forall y C^N$ is also provable in $\mathsf{IKP}^\sharp + (MP_{res}) + \mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{D}_{res}$ and since $\forall y C^N$ is intuitionistically equivalent to $\forall y D$ and the latter is in $\mathcal{D}_{res}$, both formulas are by Theorem 2.37 provable in $\mathsf{IKP}^- + \mathcal{B}$. By Theorem 3.43 $\mathsf{OST}^- + \mathcal{B}^{\exists s}$ proves therefore $\forall y C^N$ which is classically equivalent to B'. Now let $\mathcal{S}$ be the theory $\mathsf{OST}^- + \mathcal{B}^{\exists s}$ with in addition an axiom of the form $\exists f \forall \vec{x} R(\vec{x}, f(\vec{x}))$ for each relation symbol R occurring in $\mathcal{A}$. By the first assertion of the previous lemma $\mathcal{S}$ proves B and also A. By the second assertion of the previous lemma and the definition of $\mathcal{B}$, $\mathcal{T}$ proves every formula in $\mathcal{S}$ and therefore also A. $\Box$

Remark 5. By Remark 4 we can prove as in the previous proof: *Let $\mathcal{A}$ be a set of $\mathcal{L}$ formulas of the form $\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} B$, where B is a Δ_0 formula, which are suitable for $*$-translation with respect to $\mathsf{KP}^{int} + \mathcal{A}^*$. Let $\mathcal{T}$ be a theory based on intuitionistic logic, containing all non-logical axioms of $\mathsf{OST}^- + \mathcal{A}^{\exists s}$ as well as an axiom of the form $\exists f \forall \vec{x} R(\vec{x}, f(\vec{x}))$ for each relation symbol R occurring in $\mathcal{A}$. If B is a formula of $\mathcal{L}$ of the form $\forall x B$ (where B is Δ_0) suitable for $*$-translation with respect to $\mathsf{KP}^{int} + \mathcal{A}^*$ and provable in $\mathsf{KP} + \mathcal{A}$, then $\forall x B'^N$ (where B' is the formula of Lemma 3.47) is provable in $\mathcal{T}$.*

We introduce the following axiom, a double-negation-interpreted version of extensionality, for the next proof:

(N-Ext) $$((a = b) \leftrightarrow (\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a))^N.$$

In the proof of the next theorem, which is a similar assertion as the previous theorem, it is not necessary to make the detour via KP^{int}.

Theorem 3.49. *Let $\mathcal{A}$ be a set of $\mathcal{L}$ formulas of the form*

$$\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} B,$$

where B is a Δ_0 formula. If A is a Π_1 formula of $\mathcal{L}$ provable in $\mathsf{KP} + \mathcal{A}$, then A is provable in $\mathsf{OST}^- + \mathcal{A}^{\exists s}$.

Proof. Let A be a Π_1 formula, i.e. it has the form $\forall x B$ where B is Δ_0, which is provable in $\mathsf{KP} + \mathcal{A}$. Let $\mathcal{B}$ be the set

$$\{\forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} C^N : C \text{ is } \Delta_0 \text{ and } \forall x_0 \exists y_0 ... \forall x_n \exists y_n \forall x_{n+1} C \in \mathcal{A}\}.$$

By Lemma 1.23 (ix) every formula of $\mathcal{B}$ implies its double-negation interpretation intuitionistically. Furthermore (N-Ext) is the double-negation interpretation of axiom (7) of KP. Therefore and by Lemma 2.15

$$\mathsf{IKP}^\sharp + (MP_{res}) + (N\text{-Ext}) + \mathcal{B}$$

proves all double-negation interpreted axioms of $\mathsf{KP} + \mathcal{A}$. Since $\forall x B$ is provable in the latter theory, $\forall x B^N$ is provable in $\mathsf{IKP}^\sharp + (MP_{res}) + (N\text{-Ext}) + \mathcal{B}$. Because $\forall x B^N$, (N-Ext) and all formulas of $\mathcal{B}$ are in $\mathcal{D}_{res}$, Theorem 2.37 implies that also $\mathsf{IKP}^- + (N\text{-Ext}) + \mathcal{B}$ proves $\forall x B^N$. And since N-Ext and $\forall x B^N$ are clearly negative formulas, Theorem 3.43 implies that $\forall x B^N$ is also provable in $\mathsf{OST}^- + (N\text{-Ext}^{\exists s}) + \mathcal{B}^{\exists s}$. Because $N\text{-Ext}^{\exists s}$ is identical to N-Ext and therefore provable in OST^- and because the formulas of $\mathcal{A}^{\exists s}$ imply the corresponding formulas of $\mathcal{B}^{\exists s}$ classically, also $\mathsf{OST}^- + \mathcal{A}^{\exists s}$ proves $\forall x B^N$ and therefore A. □

In the previous proof it is essential that OST^- proves N-Ext. This proof does therefore not work for an intuitionistic version of OST^-. On the other hand, in contrast to the proof of Theorem 3.48, it is not necessary in the previous proof to translate and retranslate formulas with respect to the $*$-translation. Therefore $\in$-induction is not necessary. By remark 3 and Lemma 1.23 we know that

$$\mathsf{IKP}_0^\sharp + (MP_{res}) + (N\text{-Ext}) \text{ as well as } \mathsf{IKP}_\omega^\sharp + (MP_{res}) + (N\text{-Ext})$$

prove the double-negation interpretation of each axiom of KP_0 and KP_ω, respectively. In the previous proof we therefore only refer to theorems and lemmas such that it is possible to generalise the assertion:

Theorem 3.50. *The assertion of Theorem 3.49 holds also if we replace* KP *and* OST^- *by their* 0- *and* ω*-versions.*

The next aim is to apply Theorem 3.49 to the theories $\mathsf{KP} + (\mathcal{P})$ and $\mathsf{OST}^- + (\mathbb{P})$. Since the theory $\mathsf{OST}^- + (\mathbb{P})$ does not contain any axioms about the relation $\mathcal{P}$, this can not be done directly. Therefore we translate formulas containing $\mathcal{P}$ to formulas of $\mathcal{L}^\circ_\in$.

Definition 3.51 (Formula $A_\mathbb{P}$)**.** If A is a formula of $\mathcal{L}_\mathcal{P}$, we write $A_\mathbb{P}$ for the $\mathcal{L}^\circ_\in$ formula which we get if we replace any occurrence of $\mathcal{P}(x, y)$ by $\mathbb{P}x = y$.

Lemma 3.52. *If A is an $\mathcal{L}^\circ$ formula provable in $\mathsf{OST}^- + (\mathcal{P}) + (\mathbb{P})$ then $A_\mathbb{P}$ is provable in $\mathsf{OST}^- + (\mathbb{P})$. The assertion also holds if we replace OST^- by its 0- or ω-version.*

That the previous lemma holds is obvious, since if B is the axiom $(\mathcal{P})$ then $B_\mathbb{P}$ is provable in $\mathsf{OST}^-_0 + (\mathbb{P})$.

Theorem 3.53. *We have for any Π_1 formula A of $\mathcal{L}_\in$, any Π_1 formula B of $\mathcal{L}_\mathcal{P}$ and any Π_1 formula C of $\mathcal{L}_{\mathsf{Ad}}$:*

(i) If A is provable in KP, then it is provable in OST^-.

(ii) If A is provable in $\mathsf{KP} + (\mathrm{Beta})$, then it is provable in $\mathsf{OST}^- + (\mathbb{B})$.

(iii) If B is provable in $\mathsf{KP} + (\mathcal{P})$, then $B_\mathbb{P}$ is provable in $\mathsf{OST}^- + (\mathbb{P})$.

(iv) If C is provable in KPI, then it is provable in $\mathsf{OST}^- + \mathcal{AD} + (\mathbb{A})$.

The assertions (i), (iii) and (iv) also hold if we replace KP, KPI and OST^- by their 0- or ω-versions.

Proof. All four assertions are (more or less directly) special cases of Theorem 3.49. In the first case $\mathcal{A}$ is the empty set.

For the second assertion we set $\mathcal{A} := \{\mathrm{Beta}'\}$, therefore $\mathcal{A}^{\exists \mathrm{s}}$ is the set which contains as only element the formula

$$\begin{aligned}\exists f, g \forall a, r(\ &\mathrm{Fun}[f(a,r)] \wedge \mathrm{Dom}[f(a,r), g(a,r)] \\ &\wedge g(a,r) \subseteq a \wedge \mathrm{Prog}[g(a,r), a, r] \\ &\wedge (\forall x \in g(a,r))((f(a,r))'x = \{(f(a,r))'y : y \in g(a,r) \wedge \langle y, x\rangle \in r\})\ ).\end{aligned}$$

If f is the operation $\mathbb{B}$ and g the operation $\lambda ar.\mathrm{dom}(\mathbb{B}(a,r))$, then the theory $\mathsf{OST}^- + (\mathbb{B})$ proves by the Propositions 1.35 and 1.36 that it has the properties stated in this formula. Therefore $\mathsf{OST}^- + (\mathbb{B})$ contains $\mathsf{OST}^- + \mathcal{A}^{\exists \mathrm{s}}$ and proves A by Theorem 3.49 and Proposition 1.22 if A is provable in $\mathsf{KP} + (\mathrm{Beta})$.

For the third assertion let A_0 be the $\mathcal{L}_\mathcal{P}$ formula $\forall x \exists y \mathcal{P}(x, y)$, A_1 the $\mathcal{L}_\mathcal{P}$ formula

$$\forall z(\mathcal{P}(x, y) \to (z \in y \leftrightarrow z \subseteq x))$$

and $\mathcal{A}$ the set $\{A_0, A_1\}$. Then $\mathsf{KP} + \mathcal{A}$ clearly proves that there can be at most one power set of a given set (by extensionality) and therefore it proves the axiom $(\mathcal{P})$. Furthermore $\mathsf{OST}^- + (\mathcal{P}) + (\mathbb{P})$ clearly proves $A_0^{\mathrm{s}}[\mathbb{P}]$ (that is the formula $\forall x \mathcal{P}(x, \mathbb{P}(x))$) and A_1^{s} (it stays the formula A_1). So $\mathsf{OST}^- + (\mathcal{P}) + (\mathbb{P})$ contains $\mathsf{OST}^- + \mathcal{A}^{\exists \mathrm{s}}$ and proves by Theorem 3.49 all Π_1 formulas of $\mathcal{L}$ which are provable in $\mathsf{KP} + (\mathcal{P})$. Finally, the third assertion follows from the previous lemma.

For the fourth case we set $\mathcal{A} := \mathcal{AD} \cup \{\mathrm{Lim}\}$ and assume that KPl proves C. Then, by Theorem 3.49, also $\mathsf{OST}^- + \mathcal{AD}^{\exists \mathrm{s}} + \mathsf{Lim}^{\exists \mathrm{s}}$ proves C. Because $\mathcal{AD}^{\exists \mathrm{s}} = \mathcal{AD}$ ($\mathcal{AD}$ contains only Δ_0 formulas) and the axiom about $\mathbb{A}$ clearly implies $\mathsf{Lim}^{\exists \mathrm{s}}$, also $\mathsf{OST}^- + \mathsf{Ad} + (\mathbb{A})$ proves C.

That we can generalise the assertions as stated follows by Theorem 3.50, since therefore all needed lemmas, propositions and theorems are also available for the relevant theories with restricted induction principles. $\square$

Remark 6. We can combine the extensions of KP and OST^- in the previous theorem freely and get analogous assertions.

4 Interpreting Operational Set Theories in Pure Set Theories

4.1 Interpreting $\mathsf{OST} + (\mathsf{Inac})$ in $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$

In this section we show how one can interpret the theory OST + (Inac) in the theory KPS + $(\mathbf{V}{=}\mathbf{L})$. We use essentially the same model construction which is presented in Jäger [18] for interpreting OST in KP + $(\mathbf{V}{=}\mathbf{L})$. All definitions, lemmas and theorems in this section are also presented in Jäger and Zumbrunnen [26]; we will use a lot of notations, phrases and proofs from ibidem.

We start with coding the used constants of $\mathcal{L}^{\circ}_{\in}$. In order to do this we fix pairwise different sets $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\mathsf{t}}$, $\widehat{\mathsf{f}}$, $\widehat{\mathsf{el}}$, $\widehat{\mathsf{non}}$, $\widehat{\mathsf{dis}}$, $\widehat{\mathsf{e}}$, $\widehat{\mathbb{D}}$, $\widehat{\mathbb{U}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$, $\widehat{\mathbb{C}}$, $\widehat{\mathbb{P}}$, $\widehat{\mathbb{B}}$ and $\widehat{\mathbb{A}}$ which are not ordered pairs nor triples.

We will code $\mathcal{L}^{\circ}_{\in}$ terms in which the function symbol $\circ$ appears as ordered pairs and triples. The terms $\mathsf{k}x$, $\mathsf{s}x$ and $\mathsf{s}xy$, for instance, will be represented by the sets $\langle \mathsf{k}, x\rangle$, $\langle \mathsf{s}, x\rangle$ and $\langle \mathsf{s}, x, y\rangle$, respectively. Whenever it is provable in OST that some term is equal to a shorter one, we will chose the shorter one as code. We will represent for example the term $\mathsf{k}xy$ by x and not by $\langle \mathsf{k}, x, y\rangle$.

Now let R be a ternary relation symbol of $\mathcal{L}$.

Definition 4.1. The formula $\mathfrak{A}[R, \alpha, a, b, c]$ is the disjunction of the following twenty-two formulas.

(1) $a = \widehat{\mathsf{k}} \wedge c = \langle \widehat{\mathsf{k}}, b\rangle$,

(2) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{k}} \wedge (a)_1 = c$,

(3) $a = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, b\rangle$,

(4) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, (a)_1, b\rangle$,

(5) $\mathrm{Tup}_3[a] \wedge (a)_0 = \widehat{\mathsf{s}} \wedge (\exists x, y \in L_\alpha)(R((a)_1, b, x) \wedge R((a)_2, b, y) \wedge R(x, y, c))$,

(6) $a = \widehat{\mathsf{el}} \wedge c = \langle \widehat{\mathsf{el}}, b\rangle$,

(7) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{el}} \wedge (a)_1 \in b \wedge c = \widehat{\mathsf{t}}$,

(8) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{el}} \wedge (a)_1 \notin b \wedge c = \widehat{\mathsf{f}}$,

(9) $a = \widehat{\mathsf{non}} \wedge b = \widehat{\mathsf{t}} \wedge c = \widehat{\mathsf{f}}$,

(10) $a = \widehat{\mathsf{non}} \wedge b = \widehat{\mathsf{f}} \wedge c = \widehat{\mathsf{t}}$,

(11) $a = \widehat{\mathsf{dis}} \wedge c = \langle \widehat{\mathsf{dis}}, b \rangle$,

(12) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{dis}} \wedge (a)_1 = \widehat{\mathsf{t}} \wedge c = \widehat{\mathsf{t}}$,

(13) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{dis}} \wedge (a)_1 = \widehat{\mathsf{f}} \wedge b = \widehat{\mathsf{t}} \wedge c = \widehat{\mathsf{t}}$,

(14) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{dis}} \wedge (a)_1 = \widehat{\mathsf{f}} \wedge b = \widehat{\mathsf{f}} \wedge c = \widehat{\mathsf{f}}$,

(15) $a = \widehat{\mathsf{e}} \wedge c = \langle \widehat{\mathsf{e}}, b \rangle$,

(16) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{e}} \wedge (\exists x \in b)(R((a)_1, x, \widehat{\mathsf{t}})) \wedge c = \widehat{\mathsf{t}}$,

(17) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{e}} \wedge (\forall x \in b)(R((a)_1, x, \widehat{\mathsf{f}})) \wedge c = \widehat{\mathsf{f}}$,

(18) $a = \widehat{\mathbb{D}} \wedge c = \langle \widehat{\mathbb{D}}, b \rangle$,

(19) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{D}} \wedge c = \{(a)_1, b\}$,

(20) $a = \widehat{\mathbb{U}} \wedge c = \cup b$,

(21) $a = \widehat{\mathbb{S}} \wedge c = \langle \widehat{\mathbb{S}}, b \rangle$,

(22) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{S}} \wedge (\forall x \in b)(R((a)_1, x, \widehat{\mathsf{t}}) \vee R((a)_1, x, \widehat{\mathsf{f}}))$
$\wedge (\forall x \in c)(x \in b \wedge R((a)_1, x, \widehat{\mathsf{t}})) \;\wedge\; (\forall x \in b)(x \notin c \to R((a)_1, x, \widehat{\mathsf{f}}))$,

(23) $a = \widehat{\mathbb{R}} \wedge c = \langle \widehat{\mathbb{R}}, b \rangle$,

(24) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{R}} \wedge (\forall x \in b)(\exists y \in c) R((a)_1, x, y)$
$\wedge (\forall y \in c)(\exists x \in b) R((a)_1, x, y)$,

(25) $a = \widehat{\mathbb{C}} \wedge R(b, c, \widehat{\mathsf{t}}) \wedge (\forall x \in L_\alpha)(x <_{\mathbf{L}} c \to \neg R(b, x, \widehat{\mathsf{t}}))$
$\wedge (\forall x \in L_\alpha) \neg R(\widehat{\mathbb{C}}, b, x)$.

Since we have chosen pairwise different codes $\widehat{c}$ for all constants c of $\mathcal{L}_\in^\circ$, it is easy to see that $\mathfrak{A}[R, \alpha, a, b, c]$ implies: exactly one of the clauses (1)-(25) holds for α, a, b and c. It is easy to see that $\mathfrak{A}[R, \alpha, a, b, c]$ is a Δ formula w.r.t. KP. Therefore we can apply Proposition 1.17 to it and the definition below is justified.

Definition 4.2. We write $B_{\mathfrak{A}}[\alpha, a, b, c]$ for the Σ formula of $\mathcal{L}_\in$ associated to the formula $\mathfrak{A}[R, \alpha, a, b, c]$ according to Proposition 1.17. Furthermore we write

$$B_{\mathfrak{A}}^{<\alpha}[a, b, c] \quad \text{for the formula} \quad (\exists \beta < \alpha) B_{\mathfrak{A}}[\beta, a, b, c] \quad \text{and}$$

$$Ap_{\mathfrak{A}}[a, b, c] \quad \text{for the formula} \quad \exists \alpha B_{\mathfrak{A}}[\alpha, a, b, c].$$

The formula $B_{\mathfrak{A}}[\alpha, a, b, c]$ is defined such that we have by Proposition 1.17

$$B_{\mathfrak{A}}[\alpha, a, b, c] \leftrightarrow (a, b, c \in L_\alpha \wedge \mathfrak{A}[(\exists \xi < \alpha) B_{\mathfrak{A}}[\xi, .], \alpha, a, b, c]). \qquad (\mathfrak{A})$$

The intended meaning of $B_{\mathfrak{A}}[\alpha, a, b, c]$ is that a applied to b equals c on level α. Therefore $Ap_{\mathfrak{A}}[a, b, c]$ means that a applied to b equals c (on any level). With the next two lemmas we show that $Ap_{\mathfrak{A}}[a, b, c]$ is functional in its third argument. Their proofs are exactly as in Jäger and Zumbrunnen [26].

Lemma 4.3. KP *proves that*

$$B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a] \wedge B_{\mathfrak{A}}[\beta, \widehat{\mathbb{C}}, f, b] \to \alpha = \beta \wedge a = b.$$

Proof. Assume $B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a]$, $B_{\mathfrak{A}}[\beta, \widehat{\mathbb{C}}, f, b]$ and, without loss of generality, $\alpha \leq \beta$. By $(\mathfrak{A})$ we have $\widehat{\mathbb{C}}, f, a \in L_\alpha \subseteq L_\beta$ and $b \in L_\beta$. Furthermore in view of clause (25) and again by $(\mathfrak{A})$ we have $(\forall x \in L_\beta)\neg B_{\mathfrak{A}}^{<\beta}[\widehat{\mathbb{C}}, f, x]$. Since we have assumed $B_{\mathfrak{A}}[\alpha, \widehat{\mathbb{C}}, f, a]$, this implies $\alpha = \beta$. By $(\mathfrak{A})$ and in view of clause (25) our assumptions also imply

$$B_{\mathfrak{A}}^{<\alpha}[f, a, \widehat{\mathsf{t}}] \wedge (\forall x \in L_\alpha)(x <_{\mathbf{L}} a \to \neg B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\mathsf{t}})] \text{ and}$$

$$B_{\mathfrak{A}}^{<\alpha}[f, b, \widehat{\mathsf{t}}] \wedge (\forall x \in L_\alpha)(x <_{\mathbf{L}} b \to \neg B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\mathsf{t}})].$$

Hence we also have $a = b$. □

Lemma 4.4. KP *proves that*

(i) $B_{\mathfrak{A}}^{<\alpha}[a, b, u] \wedge B_{\mathfrak{A}}^{<\alpha}[a, b, v] \to u = v$ *and*

(ii) $Ap_{\mathfrak{A}}[a, b, u] \wedge Ap_{\mathfrak{A}}[a, b, v] \to u = v$.

Proof. Since we can use the previous lemma, the proof of the first assertion is straightforward by transfinite induction on α. The second assertion follows directly from the first one. □

Now we are ready to introduce for each term t of $\mathcal{L}_\in$ a formula $[\![t]\!]_{\mathfrak{A}}[x]$ of $\mathcal{L}_\in$ expressing that the value of t is equal to x.

Definition 4.5 (Formula $[\![t]\!]_{\mathfrak{A}}$)**.** Let t be an $\mathcal{L}_\in^\circ$ term such that u does not occur in t. We define the $\mathcal{L}_\in$ formula $[\![t]\!]_{\mathfrak{A}}[u]$ inductively as follows:

(i) If t is a variable or the constant ω, then $[\![t]\!]_{\mathfrak{A}}[u]$ is the formula $(t = u)$.

(ii) If t is another constant, then $[\![t]\!]_{\mathfrak{A}}[u]$ is the formula $(\widehat{t} = u)$.

(iii) If t is the term (rs), then we set

$$[\![t]\!]_{\mathfrak{A}}[u] \ := \ \exists x \exists y([\![r]\!]_{\mathfrak{A}}[x] \wedge [\![s]\!]_{\mathfrak{A}}[y] \wedge Ap_{\mathfrak{A}}[x, y, u]).$$

Notice that $[\![t]\!]_{\mathfrak{A}}[x]$ is a Σ formula of $\mathcal{L}_\in$. Since we know now how to interpret term application, we are ready to translate any $\mathcal{L}_\in^\circ$ formula to a $\mathcal{L}_\in$ formula.

Definition 4.6 ($\star$-translation of $\mathcal{L}_\in^\circ$ formulas)**.** Let A be a formula of $\mathcal{L}_\in^\circ$. The $\mathcal{L}_\in$ formula $A^\star$ is inductively defined as follows:

(i) For the atomic formulas of $\mathcal{L}_\in^\circ$ we set

$$\begin{aligned} \bot^* &:= \bot, \\ (t\downarrow)^* &:= \exists x [\![t]\!]_{\mathfrak{A}}[x], \\ (s = t)^* &:= \exists x \exists y ([\![s]\!]_{\mathfrak{A}}[x] \wedge [\![t]\!]_{\mathfrak{A}}[y] \wedge x = y) \text{ and} \\ (s \in t)^* &:= \exists x \exists y ([\![s]\!]_{\mathfrak{A}}[x] \wedge [\![t]\!]_{\mathfrak{A}}[y] \wedge x \in y). \end{aligned}$$

(ii) If A is the formula $\neg B$, then $A^\star$ is $\neg B^\star$.

(iii) If A is the formula $(B \wedge C)$, $(B \vee C)$ or $(B \to C)$, then $A^\star$ is $(B^\star \wedge C^\star)$, $(B^\star \vee C^\star)$ or $(B^\star \to C^\star)$, respectively.

(iv) If A is the formula $\forall x B[x]$ or $\exists x B[x]$, then $A^\star$ is $\forall x B^\star[x]$ or $\exists x B^\star[x]$, respectively.

If A is an $\mathcal{L}_\in^\circ$ formula without any occurrence of the function symbol $\circ$ for term application (i.e. all terms in A are variables or constants), then we can proof by a straightforward induction on the length of A that A and $A^\star$ are logically equivalent.

We will proof now that we can interpret OST + (Inac) in KPS using this translation. The next lemma can be proved as Lemma 14 in Jäger [18]. The same proof is also elaborated in Zumbrunnen [34]. Notice that the proof there is for a formulation of OST without the axioms ($\mathbb{D}$) and ($\mathbb{U}$). However, since it is obvious that the $\star$-translations of these two axioms are provable in KP + ($\mathbf{V}$=$\mathbf{L}$), we can refer to this sources anyway.

Lemma 4.7. *If A is an axiom of* OST*, then* KP + ($\mathbf{V}$=$\mathbf{L}$) *proves $A^\star$.*

The proofs of the next lemma and the next theorem are very similar as in Jäger and Zumbrunnen [26].

Lemma 4.8. KP *proves that* $\mathrm{Frg}[\kappa] \to \mathrm{Org}^\star[\kappa]$.

PROOF. Let κ be a functionally regular ordinal. We have to prove that

$$(f : \alpha \to \kappa)^\star \to (\exists \beta < \kappa)(f : \alpha \to \beta)^\star$$

for all f and all $\alpha < \kappa$. So let us assume $\alpha < \kappa$ and $(f : \alpha \to \kappa)^\star$ which is clearly equivalent to

$$(\forall \xi < \alpha)(\exists \eta < \kappa) Ap_{\mathfrak{A}}[f, \xi, \eta].$$

This is a Σ formula. Therefore we can deduce by Σ reflection (c.f. Theorem 4.3 of Chapter I in Part A of Barwise [2]) that there is an a with

$$(\forall \xi < \alpha)(\exists \eta < \kappa) Ap^a_{\mathfrak{A}}[f, \xi, \eta].$$

Now we define by Δ_0 separation the set

$$g \ := \ \{\langle \xi, \eta \rangle \in \alpha \times \kappa : Ap^a_{\mathfrak{A}}[f, \xi, \eta]\}.$$

By Σ persistency (c.f. for instance Lemma 4.2 of Chapter I in Part A of Barwise [2]) and Lemma 4.4, g is a set-theoretic function from α to κ. Since κ is functionally regular, there exists a $\beta < \kappa$ with $\mathrm{Ran}_{\subseteq}[f, \beta]$. Hence we have

$$(\forall \xi < \alpha)(\exists \eta < \beta) Ap^a_{\mathfrak{A}}[f, \xi, \eta],$$

and $(f : \alpha \to \beta)^\star$ follows by Σ persistency. □

Theorem 4.9. *Let A be a formula of $\mathcal{L}^\circ_\in$. If* $\mathsf{OST} + (\mathsf{Inac})$ *proves A, then* $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ *proves $A^\star$.*

Proof. If an $\mathcal{L}^\circ_\in$ formula C can be derived in the logic of partial terms from an $\mathcal{L}^\circ_\in$ formula B, then $C^\star$ can be derived in standard first-order logic from $B^\star$. This can be proved by a straightforward induction on the length of the proof of C from B. Therefore it is enough to check that $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ proves all axioms of $\mathsf{OST} + (\mathsf{Inac})$. In view of Lemma 4.7 the only remaining axiom is (Inac). I.e. we have to prove the formula

$$\forall \xi \exists \eta (\xi < \eta \land \mathrm{Org}^\star[\eta])$$

within $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$, which is done easily: for every ordinal ξ there is by (SLim) a functionally regular ordinal $\eta > \xi$ for which $\mathrm{Org}^\star[\eta]$ holds by the previous lemma. □

Since $\mathcal{L}_\in$ formulas are logically equivalent to their $\star$-translations, we can now conclude by the Theorems 2.8, 3.3 and 4.9 that the theories $\mathsf{OST}{+}(\mathsf{Inac})$, KPS and $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ prove the same Σ sentences. Furthermore KPS is contained in KPSd by Proposition 1.19. And since the latter theory proves by Corollary 3.28 not more Σ sentences than $\mathsf{OST} + (\mathsf{Inac})$, we get the following corollary.

Corollary 4.10. *The theories* $\mathsf{OST} + (\mathsf{Inac})$, KPS, $\mathsf{KPS} + (\mathbf{V}{=}\mathbf{L})$ *and* KPSd *prove the same Σ sentences of the language $\mathcal{L}_\in$.*

4.2 Interpreting OST^- in KP

The method presented in this section is essentially the same as the one of the previous section. In this section we want to interpret operational set theories without the axiom for the choice operation ($\mathbb{C}$) in pure set theories. Therefore the inductive definition of the interpretation of term application will be simpler and we will not use the axiom (**V**=**L**). All definitions, lemmas, theorems and corollaries in this section are also presented in Sato and Zumbrunnen [31]; we will use a lot of notations, phrases and proofs from ibidem.

As in the last section we fix pairwise different sets $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\mathsf{t}}$, $\widehat{\mathsf{f}}$, $\widehat{\mathsf{el}}$, $\widehat{\mathsf{non}}$, $\widehat{\mathsf{dis}}$, $\widehat{\mathsf{e}}$, $\widehat{\mathbb{D}}$, $\widehat{\mathbb{U}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$, $\widehat{\mathbb{C}}$, $\widehat{\mathbb{P}}$, $\widehat{\mathbb{B}}$ and $\widehat{\mathbb{A}}$ which are not ordered pairs nor triples.

Let S be a quaternary relation symbol. We introduce in the next definition similar as in Definition 4.1 a Σ formulas on which we will apply the second version of Σ recursion (Proposition 1.18) instead of the first one (Proposition 1.17) as in the previous section. The formulas $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{E}$ will serve for the interpretation of term application w.r.t. OST^-, $\mathsf{OST}^- + (\mathbb{B})$, $\mathsf{OST}^- + (\mathbb{P})$ and $\mathsf{OST}^- + \mathcal{AD} + (\mathbb{A})$, respectively. In contrast to the formula $\mathfrak{A}$ defined in Definition 4.1, S will occur only positively in these formulas. Furthermore, the sub-formulas about the constructible hierarchy in $\mathfrak{A}$ are not used here. We will write $S^{<\alpha}(a, b, c)$ for the formula $(\exists \beta < \alpha) S(\beta, a, b, c)$.

Definition 4.11. The formula $\mathfrak{B}[S, \alpha, a, b, c]$ is the disjunction of the following formulas 1-24, $\mathfrak{C}[S, \alpha, a, b, c]$ the disjunction of the formulas 1-26, $\mathfrak{D}[S, \alpha, a, b, c]$ the disjunction of the formulas 1-24 and 27 and $\mathfrak{E}[S, \alpha, a, b, c]$ the disjunction of the formulas 1-24 and 28.

(1) $a = \widehat{\mathsf{k}} \wedge c = \langle \widehat{\mathsf{k}}, b \rangle$,

(2) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{k}} \wedge (a)_1 = c$,

(3) $a = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, b \rangle$,

(4) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, (a)_1, b \rangle$,

(5) $\mathrm{Tup}_3[a] \wedge (a)_0 = \widehat{\mathsf{s}} \wedge\ \exists x, y(S^{<\alpha}((a)_1, b, x) \wedge S^{<\alpha}((a)_2, b, y) \wedge S^{<\alpha}(x, y, c))$,

(6) $a = \widehat{\mathsf{el}} \wedge c = \langle \widehat{\mathsf{el}}, b \rangle$,

(7) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{el}} \wedge (a)_1 \in b \wedge c = \widehat{\mathsf{t}}$,

(8) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{el}} \wedge (a)_1 \notin b \wedge c = \widehat{\mathsf{f}}$,

(9) $a = \widehat{\mathsf{non}} \wedge b = \widehat{\mathsf{t}} \wedge c = \widehat{\mathsf{f}}$,

(10) $a = \widehat{\mathsf{non}} \wedge b = \widehat{\mathsf{f}} \wedge c = \widehat{\mathsf{t}}$,

(11) $a = \widehat{\mathsf{dis}} \wedge c = \langle \widehat{\mathsf{dis}}, b \rangle$,

(12) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{dis}} \wedge (a)_1 = \widehat{\mathsf{t}} \wedge c = \widehat{\mathsf{t}}$,

(13) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{dis}} \wedge (a)_1 = \widehat{\mathsf{f}} \wedge b = \widehat{\mathsf{t}} \wedge c = \widehat{\mathsf{t}}$,

(14) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{dis}} \wedge (a)_1 = \widehat{\mathsf{f}} \wedge b = \widehat{\mathsf{f}} \wedge c = \widehat{\mathsf{f}}$,

(15) $a = \widehat{\mathsf{e}} \wedge c = \langle \widehat{\mathsf{e}}, b \rangle$,

(16) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{e}} \wedge (\exists x \in b)(S^{<\alpha}((a)_1, x, \widehat{\mathsf{t}})) \wedge c = \widehat{\mathsf{t}}$,

(17) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathsf{e}} \wedge (\forall x \in b)(S^{<\alpha}((a)_1, x, \widehat{\mathsf{f}})) \wedge c = \widehat{\mathsf{f}}$,

(18) $a = \widehat{\mathbb{D}} \wedge c = \langle \widehat{\mathbb{D}}, b \rangle$,

(19) $\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{D}} \wedge c = \{(a)_1, b\}$,

(20) $a = \widehat{\mathbb{U}} \wedge c = \cup b$,

(21) $a = \widehat{\mathbb{S}} \wedge c = \langle \widehat{\mathbb{S}}, b \rangle$,

(22) $$\begin{aligned}\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{S}} &\wedge (\forall x \in b)(S^{<\alpha}((a)_1, x, \widehat{\mathsf{t}}) \vee S^{<\alpha}((a)_1, x, \widehat{\mathsf{f}})) \\ &\wedge (\forall x \in c)(x \in b \wedge S^{<\alpha}((a)_1, x, \widehat{\mathsf{t}})) \\ &\wedge (\forall x \in b)(x \notin c \to S^{<\alpha}((a)_1, x, \widehat{\mathsf{f}})),\end{aligned}$$

(23) $a = \widehat{\mathbb{R}} \wedge c = \langle \widehat{\mathbb{R}}, b \rangle$,

(24) $$\begin{aligned}\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{R}} &\wedge (\forall x \in b)(\exists y \in c)(S^{<\alpha}((a)_1, x, y)) \\ &\wedge (\forall y \in c)(\exists x \in b)(S^{<\alpha}((a)_1, x, y)),\end{aligned}$$

(25) $a = \widehat{\mathbb{B}} \wedge c = \langle \widehat{\mathbb{B}}, b \rangle$,

(26) $$\begin{aligned}\mathrm{Tup}_2[a] \wedge (a)_0 = \widehat{\mathbb{B}} \wedge \exists x(\ & x \subseteq (a)_1 \wedge (\forall y \in x)\mathrm{WP}[y, (a)_1, b] \\ &\wedge (\forall y \in (a)_1)(\mathrm{WP}'[y, (a)_1, b] \to y \in x) \wedge \mathrm{Fun}[c] \wedge \mathrm{Dom}[c, x] \\ &\wedge (\forall y \in x)(c'y = \{c'z : z \in x \wedge \langle z, y \rangle \in b\})\ ),\end{aligned}$$

(27) $a = \widehat{\mathbb{P}} \wedge \mathcal{P}(b, c)$,

(28) $a = \widehat{\mathbb{A}} \wedge b \in c \wedge \mathsf{Ad}(c) \wedge (\forall x \in c)(\neg b \in x \vee \neg \mathsf{Ad}(x))$.

Since S occurs only positively in all the clauses and since all of them are Σ formulas, we can apply Proposition 1.18 to them. Therefore, the next definition, analogously to Definition 4.2, is justified.

Definition 4.12. We write $B_{\mathfrak{B}}[\alpha, a, b, c]$ for the Σ formula of $\mathcal{L}_\in$ associated to the formula $\mathfrak{B}[S, \alpha, a, b, c]$ according to Proposition 1.18. Furthermore we write

$$Ap_{\mathfrak{B}}[a, b, c] \quad \text{for the formula} \quad \exists \alpha B_{\mathfrak{B}}[\alpha, a, b, c].$$

The formulas $B_{\mathfrak{C}}$, $B_{\mathfrak{D}}$ and $B_{\mathfrak{E}}$ as well as $Ap_{\mathfrak{C}}$, $Ap_{\mathfrak{D}}$ and $Ap_{\mathfrak{E}}$ are defined analogously.

Notice that $B_{\mathfrak{D}}$ and $Ap_{\mathfrak{D}}$ are $\mathcal{L}_{\mathcal{P}}$ formulas and that $B_{\mathfrak{E}}$ and $Ap_{\mathfrak{E}}$ are $\mathcal{L}_{\mathsf{Ad}}$ formulas. The next lemma and its proof is taken from Sato and Zumbrunnen [31].

Lemma 4.13. KP + (Beta) *proves*

(i) $\exists y Ap_{\mathfrak{C}}[\langle\widehat{\mathbb{B}}, a\rangle, b, y]$ *and*

(ii) $Ap_{\mathfrak{C}}[\langle\widehat{\mathbb{B}}, a\rangle, b, u] \wedge Ap_{\mathfrak{C}}[\langle\widehat{\mathbb{B}}, a\rangle, b, v] \to u = v.$

Proof. $Ap_{\mathfrak{C}}[\langle\widehat{\mathbb{B}}, a\rangle, b, u]$ codes nothing else than that there is an x such that $x = \{y \in a : \mathrm{WP}[y, a, b]\}$ and u is a function with domain x with $(\forall y \in x)(u'y = \{u'z : z \in x \wedge \langle z, y\rangle \in b\}$. Therefore both assertions follow from (iii) and (v) of Lemma 1.21. □

The next lemma is the analogous one to Lemma 4.4. If we can use the previous lemma, the proof of its first assertion is straightforward by induction on $\max(\alpha, \beta)$. Its second assertion is an immediate consequence of the first one.

Lemma 4.14. *The theory* KP *proves*

(i) $B_{\mathfrak{B}}[\alpha, a, b, u] \wedge B_{\mathfrak{B}}[\beta, a, b, v] \to u = v,$

(ii) $Ap_{\mathfrak{B}}[a, b, u] \wedge Ap_{\mathfrak{B}}[a, b, v] \to u = v.$

The analog assertions for the formulas $B_{\mathfrak{C}}$ *and* $Ap_{\mathfrak{C}}$ *can be proved in* KP + (Beta). *The analog assertions for the formulas* $B_{\mathfrak{D}}$ *and* $Ap_{\mathfrak{D}}$ *can be proved in* KP + $(\mathcal{P})$. *And the analog assertions for the formulas* $B_{\mathfrak{E}}$ *and* $Ap_{\mathfrak{E}}$ *can be proved in* KPI.

The next definition is analogous to Definition 4.5.

Definition 4.15 ($[\![t]\!]_{\mathfrak{B}}$, $[\![t]\!]_{\mathfrak{C}}$, $[\![t]\!]_{\mathfrak{D}}$ and $[\![t]\!]_{\mathfrak{E}}$)**.** Let t be an $\mathcal{L}_{\in}^{\circ}$ term with u not occurring in t. The $\mathcal{L}_{\in}$ formulas $[\![t]\!]_{\mathfrak{B}}[u]$ and $[\![t]\!]_{\mathfrak{C}}[u]$, the $\mathcal{L}_{\mathcal{P}}$ formula $[\![t]\!]_{\mathfrak{D}}[u]$ as well as the $\mathcal{L}_{\mathsf{Ad}}$ formula $[\![t]\!]_{\mathfrak{E}}[u]$ are analogously defined as the formula $[\![t]\!]_{\mathfrak{A}}[u]$ in Definition 4.5, but with respect to the formulas $Ap_{\mathfrak{B}}$, $Ap_{\mathfrak{C}}$, $Ap_{\mathfrak{D}}$ and $Ap_{\mathfrak{E}}$, respectively, instead of the formula $Ap_{\mathfrak{A}}$.

In the following we write $\mathcal{L}_{\mathsf{Ad}}^{\circ}$ for the language $\mathcal{L}_{\in}^{\circ}$ with the additional relation symbol Ad.

As in Definition 4.6, we introduce translations of formulas with respect to the new formulas for interpreting term application.

Definition 4.16 (More $\star$-translations of $\mathcal{L}_{\in}^{\circ}$ formulas)**.** Let A be a formula of $\mathcal{L}_{\in}^{\circ}$. The $\mathcal{L}_{\in}$ formulas $A^{\star -}$ and $A^{\star\mathbb{B}}$ as well as the $\mathcal{L}_{\mathcal{P}}$ formula $A^{\star\mathbb{P}}$ are analogously defined as the formula $A^{\star}$ in Definition 4.6, but with respect to the formulas $[\![t]\!]_{\mathfrak{B}}$, $[\![t]\!]_{\mathfrak{C}}$, and $[\![t]\!]_{\mathfrak{D}}$, respectively, instead of the formula $[\![t]\!]_{\mathfrak{A}}$.

Let B be a formula of $\mathcal{L}_{\mathsf{Ad}}^{\circ}$. The $\mathcal{L}_{\mathsf{Ad}}$ formula $B^{\star\mathbb{A}}$ is analogously defined as the formula $A^{\star}$ in Definition 4.6, but

(i) with the additional case

$$\mathsf{Ad}(t)^{\star\mathbb{A}} \;:=\; \exists x(\llbracket t \rrbracket_{\mathfrak{C}}[x] \wedge \mathsf{Ad}(x))$$

for atomic formulas of the form $\mathsf{Ad}(t)$,

(ii) and with respect to the formula $\llbracket t \rrbracket_{\mathfrak{C}}$ instead of the formula $\llbracket t \rrbracket_{\mathfrak{A}}$.

The next theorem and its proof as well as the next corollary are taken from Sato and Zumbrunnen [31].

Theorem 4.17. *Let A be an arbitrary formula of $\mathcal{L}_{\in}^{\circ}$ and B an arbitrary formula of $\mathcal{L}_{\mathsf{Ad}}^{\circ}$.*

(i) If OST^- *proves A, then* KP *proves $A^{\star -}$.*

(ii) If $\mathsf{OST}^- + (\mathbb{B})$ *proves A, then* $\mathsf{KP} + (\mathrm{Beta})$ *proves $A^{\star\mathbb{B}}$.*

(iii) If $\mathsf{OST}^- + (\mathbb{P})$ *proves A, then* $\mathsf{KP} + (\mathcal{P})$ *proves $A^{\star\mathbb{P}}$.*

(iv) If $\mathsf{OST}^- + \mathcal{AD} + (\mathbb{A})$ *proves B, then* KPI *proves $B^{\star\mathbb{A}}$.*

Proof. The assertions follows because the particular $\star$-translated axioms are provable in the respective theories. That the translations of the axioms of the logic of partial terms are provable is straightforward if we can use the previous lemma. That also the $\star$-translated axioms about the operations $\mathbb{D}$ and $\mathbb{U}$ are provable in the respective theories is obvious. We can prove that all four possible $\star$-translations of all the other axioms of OST^- are provable in KP, $\mathsf{KP}+(\mathrm{Beta})$, $\mathsf{KP}+(\mathcal{P})$ and KPI, respectively, as in the proof of Lemma 14 in Jäger [18]. If A is the axiom $(\mathbb{B})$, then $A^{\star\mathbb{B}}$ is provable in $\mathsf{KP} + (\mathrm{Beta})$ by the Lemmas 1.21 and 4.13. And if A is the axiom $(\mathbb{P})$, then $A^{\star\mathbb{P}}$ is obviously provable in $\mathsf{KP} + (\mathcal{P})$. Also if A is an axiom of $\mathcal{AD}$ (they are equivalent to their $\star\mathbb{A}$-translations) or the axiom $(\mathbb{A})$, then $A^{\star\mathbb{A}}$ is obviously provable in KPI. □

Since the $\star$-translations of an $\mathcal{L}_{\in}$ formula A are always equivalent to A, we can compound the previous theorem with Theorem 3.53 and we get:

Corollary 4.18. *The following theories prove in each case the same Π_1 formulas of $\mathcal{L}_{\in}$:*

(i) KP *and* OST^-.

(ii) $\mathsf{KP} + (\mathrm{Beta})$ *and* $\mathsf{OST}^- + (\mathbb{B})$.

(iii) $\mathsf{KP} + (\mathcal{P})$ *and* $\mathsf{OST}^- + (\mathbb{P})$.

(iv) KPI *and* $\mathsf{OST}^- + \mathcal{AD} + (\mathbb{A})$.

Also the last remark is taken from Sato and Zumbrunnen [31]:

Remark 7. It is easy to see, that the third line can be generalised in the following sense: Let $\mathbb{Q}$ be a new constant symbol which we add to $\mathcal{L}^\circ$, Q a binary relation symbol of $\mathcal{L}$, and $\mathcal{Q}$ some set of Π_1 formulas which imply that for each x there is at most one y such that $Q(x, y)$. Then we can introduce a translation $A^{\star\mathbb{Q}}$ of every formula A of $\mathcal{L}^\circ_\in$ (analogously as we did for $\mathbb{P}$) such that $\mathsf{OST}^- + \mathcal{Q} + (\forall x Q(x, \mathbb{Q}(x)))$ proves the Π_1 formula A of $\mathcal{L}_\in$ if and only if $\mathsf{KP} + \mathcal{Q} + (\forall x \exists y Q(x, y))$ proves $A^{\star\mathbb{Q}}$.

5 Operational Set Theory and Explicit Mathematics

In some sense, an operation in explicit mathematics corresponds to the choice operation $\mathbb{C}$ in operational set theory, if it assigns to each name of some non-empty type an instance of an individual of this type. We will see in the first part of this small chapter that such an operation can, provably in EET, not exist. We will work informally within EET in this first part.

In the second part of this chapter we discuss how we could formulate an axiom in operational set theory which is analogous to the axiom about the inverse image operation inv in explicit mathematics. We will see that the suggested formulations lead directly to inconsistencies.

We will write $\mathrm{GCO}[f]$ for the formula

$$\forall x(\Re(x) \land \exists y(y \,\dot{\in}\, x) \to f(x) \,\dot{\in}\, x).$$

We call an individual f for which $\mathrm{GCO}[f]$ holds a *global choice operator*. The axiom

$$(\mathsf{GCO}) \qquad\qquad \exists f \mathrm{GCO}[f],$$

states that there is a global choice operator. Notice that this axiom should not be mixed up with the axiom schema (AC) nor the axiom (AC_{V}) introduced in Feferman [10] (on page 111). As we will see, adding (GCO) to EET leads to a contradiction. But first we introduce the formula $\mathrm{GO}[f]$ given by

$$\begin{aligned} \forall x(\ &\Re(x) \land \forall z(z \,\dot{\in}\, x \to \exists y_0, y_1(z = (y_0, y_1))) \\ &\qquad \land \forall y_0, y_1, y_2((y_0, y_1) \,\dot{\in}\, x \land (y_0, y_2) \,\dot{\in}\, x \to y_1 = y_2) \\ &\qquad\qquad \to f(x)\downarrow \land \forall y_0, y_1((y_0, y_1) \,\dot{\in}\, x \to f(x, y_0) = y_1)\ ). \end{aligned}$$

That is, $\mathrm{GO}[f]$ states that whenever x names a type which represents the graph of some function, then $f(x)$ is the operation which yields the same values as this function. In other words, f translates graphs of functions to operations.

Lemma 5.1. *The theory* EET *proves that there is no individual* f *with* $\mathrm{GO}[f]$.

PROOF. Assume GO$[f]$. Let a be a name of the type which contains all ordered pairs x with

$$((\mathsf{p}_0x)(\mathsf{p}_0x) = 1 \land \mathsf{p}_1x = 0) \ \lor \ (((\mathsf{p}_0x)(\mathsf{p}_0x) \neq 1 \lor (\mathsf{p}_0x)(\mathsf{p}_0x)\uparrow) \land \mathsf{p}_1x = 1),$$

which exists by elementary comprehension (Proposition 1.39). That is, a names the graph of the function which assigns 0 to every x with $xx = 1$ and 1 to all other x. From our assumption on f it follows therefore that $(f(a))(x)\downarrow$ and

$$(f(a))(x) = \begin{cases} 0 & \text{if } xx = 1, \\ 1 & \text{else,} \end{cases}$$

for all x. Hence, if we let r be the term $f(a)$ we have

$$(f(a))(r) = 0 \text{ iff. } rr = 1 \text{ iff. } (f(a))(r) = 1.$$

Since $(f(a))(r)$ has to be 0 or 1 and $0 \neq 1$ (by the fifth applicative axiom of EET), this is a contradiction and therefore an f with GO$[f]$ can not exist. □

Now it is easy to proof the next theorem.

Theorem 5.2. *The theory* $\mathsf{EET} + (\mathsf{GCO})$ *is inconsistent.*

PROOF. Assume (GCO) and let f be an individual with GCO$[f]$. By elementary comprehension (Proposition 1.39) there is a closed term t such that we have for every x and a with $\Re(a)$:

$$\Re(t(a,x)) \land \forall y(y \,\dot{\in}\, t(a,x) \leftrightarrow (x,y) \,\dot{\in}\, a).$$

That is, if a names a type which represents a graph of a function and x is in the domain of a, then $t(a,x)$ names the type $\{y\}$, if y is the unique individual such that $(x,y) \in a$. It follows, if s is the term $\lambda ax.f(t(a,x))$ then EET proves $s\downarrow$ and GO$[s]$, because f is a global choice operation. Hence, by the previous lemma, $\mathsf{EET} + (\mathsf{GCO})$ is inconsistent. □

The proof of the previous theorem makes clear that EET even disproves the existence of an operation, which assigns to each name of some type of the form $\{y\}$ the element y.

If we formulate the axiom about the operation inv of explicit mathematics in operational set theory by replacing the notion of 'name of a type' by the notion of 'set', we get an axiom stating that there is an operation f such that

$$f(a,g)\downarrow \land \forall x(x \in f(a,g) \leftrightarrow fx \,\dot{\in}\, a).$$

Such an axiom could be easily used for proving the existence of a set of all sets and to get a contradiction.

Another way to formulate an inverse image axiom in operational set theory is by identifying types or classes by operations which represents their characteristic functions. A total operation f with $(f : \mathbf{V} \to \mathbf{B})$ corresponds in this sense to the class $\{x : fx = \mathsf{t}\}$. We will see that an inverse image axiom with respect to this interpretations of classes also leads to inconsistent theories. An operation f in operational set theory corresponds to the inverse image operation inv of explicit mathematics with respect to characteristic function operations if we have for all g and h

$$(g : \mathbf{V} \to \mathbf{B}) \ \to \ (f(g, h) : \mathbf{V} \to \mathbf{B}) \wedge \forall x(f(g, h, x) = \mathsf{t} \leftrightarrow g(hx) = \mathsf{t}).$$

We write $\mathrm{Inv}[f]$ if this is the case. The axiom (Inv), given by

$$(\mathsf{Inv}) \qquad \exists f \mathrm{Inv}[f],$$

states that there is an inverse image operation.

Theorem 5.3. *The theory* $\mathsf{OST}_0^- + (\mathsf{Inv})$ *is inconsistent.*

Proof. We proof in OST_0^- that there is no operation f with $\mathrm{Inv}[f]$. Assume that f is an operation with $\mathrm{Inv}[f]$ and let t be the term $\lambda gx.f(\lambda y.\mathsf{t}, g, x)$. Then we have by the properties of f and the strictness axioms of the logic of partial terms that $(t : \mathbf{V}^2 \to \mathbf{B})$ and

$$t(g, x) = \begin{cases} \mathsf{t} & \text{if } gx{\downarrow}, \\ \mathsf{f} & \text{if } \neg gx{\downarrow}. \end{cases}$$

for every g and x. Now let $A[y]$ be the formula $\neg y = \mathsf{t}$, t_A the corresponding term defined in Lemma 1.31 and $B[y]$ the formula $y = \mathsf{t}$. Furthermore let s be the term

$$\lambda gx.(\ \mathrm{ite}_B(\lambda gx.t_A(gx), \lambda gx.\mathsf{f}, t(g, x))\ )(g, x).$$

Then, by the Lemmas 1.29, 1.31, and 1.38, OST_0^- proves that for all g, x we have

$$s(g, x) = \begin{cases} (\lambda gx.t_A(gx))(g, x) = \mathsf{t} & \text{if } gx{\downarrow} \text{ and } \neg gx = \mathsf{t}, \\ (\lambda gx.t_A(gx))(g, x) = \mathsf{f} & \text{if } gx{\downarrow} \text{ and } gx = \mathsf{t}, \\ (\lambda gx.\mathsf{f})(g, x) = \mathsf{f} & \text{if } \neg gx{\downarrow}. \end{cases}$$

So, if we let r be the term $\lambda x.s(x, x)$, OST_0^- proves $r{\downarrow}$, $(r : \mathbf{V} \to \mathbf{B})$ and particularly also $rr{\downarrow}$. By the properties of s it follows therefore that

$$rr = \mathsf{t} \text{ iff. } s(r, r) = \mathsf{t} \text{ iff. } \neg rr = \mathsf{t}.$$

Hence we have proved in OST_0^- that the existence of an operation f with $\mathrm{Inv}[f]$ leads to a contradiction. $\square$

We can also formulate an inverse image axiom with respect to characteristic operations in explicit mathematics. We will see that this can also lead to inconsistent systems.

An operation f in explicit mathematics corresponds to the inverse image operation inv but with respect to characteristic function operations if we have for all g and h

$$\begin{aligned}&\forall x(g(x) = 0 \lor g(x) = 1) \ \rightarrow \\ &\qquad \forall x(f(g,h,x) = 0 \lor f(g,h,x) = 1) \land \forall x(f(g,h,x) = 1 \leftrightarrow g(hx) = 1).\end{aligned}$$

We write $\mathrm{Inv_E}[f]$ if this is the case. The axiom $(\mathrm{Inv}_{\mathsf{E}})$, given by

$$(\mathrm{Inv}_{\mathsf{E}}) \qquad\qquad \exists f \mathrm{Inv_E}[f],$$

is the analogous axiom to the axiom (Inv). In order to prove an inconsistency analogously as in the proof of the previous theorem, we have to add the well known axiom $(\mathsf{d_V})$ for definition by cases on the whole universe, given by

$$(\mathsf{d_V}) \qquad\qquad (a = b \rightarrow \mathsf{d_V}xyab = x) \land (a \neq b \rightarrow \mathsf{d_V}xyab = y),$$

where $\mathsf{d_V}$ is a new constant which we add to $\mathbb{L}$.

Theorem 5.4. *The theory* $\mathsf{EET} + (\mathsf{d_V}) + (\mathrm{Inv}_{\mathsf{E}})$ *is inconsistent.*

Proof. We proof in $\mathsf{EET} + (\mathsf{d_V})$ that there is no operation f with $\mathrm{Inv_E}[f]$. Assume that f is an operation with $\mathrm{Inv}[f]$ and let t be the term

$$\lambda gx.f(\lambda y.1, g, x).$$

Then we have by the properties of f and the strictness axioms of the logic of partial terms that $\forall g, x(t(g,x) = 0 \lor t(g,x) = 1)$ and

$$t(g,x) = \begin{cases} 1 & \text{if } gx\downarrow, \\ 0 & \text{if } \neg gx\downarrow. \end{cases}$$

for every g and x. Now let s be the term

$$\lambda gx.(\ \mathsf{d_N}(\lambda gx.\mathsf{d_V}(0,1,gx,1), \lambda gx.0, t(g,x), 1)\ )(g,x).$$

Then we have by the axioms about the operations $\mathsf{d_N}$ and $\mathsf{d_V}$ that $\mathsf{EET} + (\mathsf{d_V})$ proves for all g, x:

$$s(g,x) = \begin{cases} (\lambda gx.\mathsf{d_V}(0,1,gx,1))(g,x) = 1 & \text{if } gx\downarrow \text{ and } \neg gx = 1, \\ (\lambda gx.\mathsf{d_V}(0,1,gx,1))(g,x) = 0 & \text{if } gx\downarrow \text{ and } gx = 1, \\ (\lambda gx.0)(g,x) = 0 & \text{if } \neg gx\downarrow. \end{cases}$$

So, if we let r be the term $\lambda x.s(x,x)$, $\mathsf{EET} + (\mathsf{d_V})$ proves $r\downarrow$ as well as $\forall x(rx = 0 \vee rx = 1)$. By the properties of s it follows therefore that

$$rr = 1 \text{ iff. } s(r,r) = 1 \text{ iff. } \neg rr = 1.$$

Hence $\mathsf{EET} + (\mathsf{d_V})$ disproves the existence of an operation f with $\mathrm{Inv_E}[f]$. $\square$

Conclusion

In the following we write $\mathcal{S} \equiv_{\text{con}} \mathcal{T}$ if the theories $\mathcal{S}$ and $\mathcal{T}$ are equiconsitent, and $\mathcal{S} \equiv_{\Delta} \mathcal{T}$ if they prove the same absolute sentences. Furthermore we write IOST^- for the theory OST^- but based on the intuitionistic logic of partial terms.

We have seen in this thesis that we can define a form of operational set theory OST^- without an axiom for a choice operation $\mathbb{C}$ but with the same strength as OST. By Theorem 1.33 and Corollary 4.18 we can note

$$\mathsf{OST} \ \equiv_{\Delta} \ \mathsf{OST}^- \ \equiv_{\Delta} \ \mathsf{KP}.$$

By Remark 5 on p. 76, KP also can be interpreted in the theory IOST^-, therefore we have

$$\mathsf{IOST}^- \ \equiv_{\text{con}} \ \mathsf{OST} \ \equiv_{\text{con}} \ \mathsf{OST}^- \ \equiv_{\text{con}} \ \mathsf{KP}.$$

It is known that KPI and $\mathsf{KP}+(\text{Beta})$ are proof-theoretically equivalent (c.f. Jäger [16, Theorem 8.5, p. 142]). Therefore we can combine this information with Corollary 4.18 and we get

$$\mathsf{OST}^- + (\mathbb{B}) \ \equiv_{\Delta} \ \mathsf{KP} + (\text{Beta}) \ \equiv_{\text{con}} \ \mathsf{KPI} \ \equiv_{\Delta} \ \mathsf{OST}^- + \mathcal{AD} + (\mathbb{A}).$$

It is well known that second order arithmetic with a comprehension schema for arbitrary formulas is stronger than KPI. Since arithmetic with full comprehension can certainly be embedded into ZFC^-, we know by Remark 1 and Corollary 4.10 that also OST + (Inac) is stronger than KPI and therefore stronger than conjectured in Feferman [14]. The theories discussed in this thesis of this strength are by Corollary 4.10 the theories

$$\mathsf{OST} + (\mathsf{Inac}) \ \equiv_{\Delta} \ \mathsf{KPS} \ \equiv_{\Delta} \ \mathsf{KPS} + (\mathbf{V}{=}\mathbf{L}) \ \equiv_{\Delta} \ \mathsf{KPSd}.$$

The last group of discussed theories of some specific strength is by the Remarks 2 on p. 36 and 5 on p. 76 as well as Corollary 4.18

$$\mathsf{IOST}^- + (\mathbb{P}) \ \equiv_{\text{con}} \ \mathsf{OST}^- + (\mathbb{P}). \ \equiv_{\Delta} \ \mathsf{KP} + (\mathcal{P}).$$

We did not clarify whether the axiom $(\mathbb{C})$ causes any difference in the proof-theoretic strength in the presence of the operational power set axiom $(\mathbb{P})$. I.e., it stays an open problem whether $\mathsf{OST} + (\mathbb{P})$ is proof-theoretically stronger than $\mathsf{OST}^- + (\mathbb{P})$ or not.

In chapter 5 we have detected two ontological differences between operational set theory and explicit mathematics:

(i) In all discussed operational set theories we can prove that there are operations which translates graphs of functions (i.e. set theoretic functions) to operations (c.f. Proposition 1.36). On the other hand, in explicit mathematics we can prove that such translation operations do not exist (c.f. Lemma 5.1).

(ii) There are operational set theories, for instance OST, which prove that $\lambda x.\mathbb{C}(\lambda y.\mathsf{el}(y,x))$ is a global choice operation. Others, as it seems, as for instance OST^- do not prove that. The provability of the existence of a global choice operation makes, at least for some systems of operational set theory, no difference in the proof-theoretic strength. On the other hand, the provability of the existence of a global choice operation in explicit mathematics (in the sense of p. 91) leads directly to inconsistent theories (c.f. Theorem 5.2).

Furthermore we have seen that it is not so easy to formulate an inverse image axiom in operational set theory which does not lead to inconsistent theories (c.f. Theorem 5.3).

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