# Forcing for Hat Inductive Definitions in Arithmetic
## — One of the Simplest Applications of Forcing —

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**Abstract**

By forcing, we give a direct interpretation of $\widehat{\mathbf{ID}}_\omega$ into Avigad's **FP**. To the best of the author's knowledge, this is one of the simplest applications of forcing to "real problems".

# 1 Introduction

It is reasonable to say that, to prove equiconsistency or relative consistency results on mathematical theories (i.e., those formal systems in which ordinary mathematical practice can be formalized) is one of the central aims both of proof theory and of set theory. Despite this common feature, the tools used in the two fields are quite different. In set theory, the forcing (or generic extension) method is, in effect, the only way to do this besides the inner model method. Indeed in some subcommunities the terms "consistent" and "forcible" are synonymous. In proof theory, on the other hand, the most commonly used methods are cut-elimination and, for intuitionistic theories, realizability interpretations. The forcing method is much less frequently used, though there is an excellent survey paper by Avigad [2] summarizing applications of forcing (including those for theories not necessarily on classical logic but on intuitionistic logic).

In this note, we give one application of classical forcing (i.e., forcing for theories on classical logic) in proof theory. Namely, we answer a question asked in the context of proof theory: we show that $\widehat{\mathbf{ID}}_\omega$ is interpretable in $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) + (\Delta^1_0\text{-}\mathbf{FP})$, by giving a concrete interpretation with the forcing method. Here $\widehat{\mathbf{ID}}_\omega$ is the first-order theory of $\omega$-times iterated fixed points, and $(\Delta^1_0\text{-}\mathbf{FP})$ asserts the existence of fixed points for any operators defined by positive arithmetical formulae with parameters. This is not a useless toy example: Jäger and Strahm mentioned in [5, p.498] that they "do not know whether a direct interpretation of $\widehat{\mathbf{ID}}_\omega$ in **FP** is possible", where **FP** in their notation is $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) + (\Delta^1_0\text{-}\mathbf{FP})$ in ours. The difficulty arises from the non-uniqueness of fixed points, which prevents us from defining the hierarchy of fixed points straightforwardly. This is a typical kind of difficulty with which the forcing method can deal.

This could be one of the simplest applications of classical forcing: in forcing for set theory or second-order number theory, we have to change the domain of discourse and hence need some machinery to name those elements (i.e., $V^{\mathbb{P}}$); and the definition of the forcing relation for atomic formulae becomes complex because of this naming and, in set theory, because of extensionality (though we can avoid the latter, as shown in [7], by combining one more additional interpretation from extensional set theories into intensional ones). Here, in our application, we do not change the domain and so do not need a complex machinery for names; and we do not need to care about extensionality. Moreover, to interpret $\widehat{\mathbf{ID}}_\omega$, we need to define only the interpretations of new predicates and so forcing need not to be iterated, unlike for parameter-allowed second-order systems (e.g., [2, §3]). Thus our application could be said to be a "textbook example" of forcing.

Our use of the forcing method can be generalized to obtain the reducibility of **DC**, a kind of first-order system for dependent choice, into $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind})$, whereas the forcing treatment of genuinely second-order $(\Sigma^1_1\text{-}\mathbf{DC})$ cannot be among the simplest, for the reason given above.

## 2 Preliminaries

**Definition 1.** $\mathcal{L}_1$ and $\mathcal{L}_2$ are the standard languages of first- and second-order number theory respectively (see [9]). An $\mathcal{L}_2$-formula is called *arithmetical* if it contains no second-order quantifiers. $\mathcal{L}[x_0,..,x_m,X_0,...,X_n]$ denotes the set of all the arithmetical $\mathcal{L}_2$-formulae whose free variables are among $x_0,..,x_m,X_0,...,X_n$. An *operator form* is an abstracted $\mathcal{L}[x,y,X,Y]$-formula $\lambda X,x.O(x,y,X,Y)$, and is called *positive* if $X$ occurs only positively in $O(x,y,X,Y)$.

We introduce for all the $\mathcal{L}[y,X,Y]$-formulae $A$ new predicate symbols $P^A$ of arity 2. The language $\mathcal{L}_{\mathrm{dc}} = \mathcal{L}_1 \cup \{P^A \mid A \text{ is a formulae in } \mathcal{L}[y,X,Y]\}$ is the result of augmenting $\mathcal{L}_1$ by all such new predicate symbols. $P^A_k$ and $P^A_{<k}$ denote the unary abstracts $\{x \mid P^A(k,x)\}$ and $\{y \mid (\exists i,x)(y = \langle i,x\rangle \wedge i<k \wedge P^A(i,x))\}$ respectively, where $\langle$-,-$\rangle$ is a pairing function.

We use the similar abbreviations $(X)_k$ and $(X)_{<k}$ for second-order variables $X$.

Let $\mathbf{ACA}_0$ denote the $\mathcal{L}_2$-system of arithmetical comprehension axiom with restricted induction (see e.g., [9]). Though we do not give the detailed definition, it includes the basic axioms of discretely ordered semi-rings for natural numbers, and induction and comprehension axioms for arithmetical formulae. ($\mathcal{L}_2$-**Ind**) denotes the induction scheme for all $\mathcal{L}_2$-formulae.

**Definition 2.** We define the following axiom scheme:

$$(\Delta^1_0\text{-}\mathbf{FP}) \qquad\qquad (\forall y,Y)(\exists F)(\forall x)(x\in F \leftrightarrow O(x,y,F,Y))$$

for any arithmetical formula $O(x,y,X,Y)$ in which $X$ occurs only positively.

In some literature (e.g., [1] and [5]), the systems $\mathbf{ACA}_0 + (\Delta^1_0\text{-}\mathbf{FP})$ and $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) + (\Delta^1_0\text{-}\mathbf{FP})$ are denoted by $\mathbf{FP}_0$ and $\mathbf{FP}$ respectively.

**Definition 3.** $\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}]$ denotes Peano arithmetic formulated in $\mathcal{L}_{\mathrm{dc}}$, namely the system consisting of all the axioms of Peano arithmetic and the induction scheme for all $\mathcal{L}_{\mathrm{dc}}$-formulae.

For an $\mathcal{L}[y,X,Y]$ formula $A$, the axiom $(\forall k)A(k,P^A_k,P^A_{<k})$ is denoted by ($A$-**DC**).

$\widehat{\mathbf{ID}}_\omega$ is the $\mathcal{L}_{\mathrm{dc}}$-theory consisting of $\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}]$ and ($F_O$-**DC**) for all positive operator forms $O$ where $F_O(y,X,Y) \equiv (\forall x)(x\in X \leftrightarrow O(x,y,X,Y))$. In this theory, we denote $P^{F_O}$ by $F^O$.

**DC** consists of $\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}]$ and ($A$-**DC**) for all $A$ with $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) \vdash \forall y,Y\exists X A(y,X,Y)$.

As shown in e.g., [8, §3.1], $\widehat{\mathbf{ID}}_\omega$ is equivalent to $\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}] + (F_O\text{-}\mathbf{DC})$ where $O$ is a universal $\Pi^0_2$-formula (more precisely, a $\Pi^0_2$ formula $O(x,k,X,Y)$ in which $X$ occurs only positively such that for any $\Pi^0_2$ formula $B(x,k,X,Y)$ in which $X$ occurs only positively we have an equivalence $B(x,k,X,Y) \leftrightarrow O(\langle e,x\rangle,k,X,Y)$ for some natural number $e$). For convenience in what follows we consider that $\widehat{\mathbf{ID}}_\omega$ is formulated in $\mathcal{L}_{\mathrm{fix}} = \mathcal{L}_1 \cup \{F^O\}$ for such $O$.

$F^O_k$ is intended to denote a fixed point of the monotone operator $\Gamma_k : X \mapsto \{x \mid O(x,k,X,F^O_{<k})\}$ defined by an $\mathcal{L}_1$-formula $O$ with previous fixed points $F^O_{<k}$. We call $F^O$ a *hierarchy of fixed points.*

To define $\mathbf{ACA}^-_\omega$, the arithmetical comprehension counterpart of $\widehat{\mathbf{ID}}_\omega$, we restrict $X$ not to occur in $O$, or equivalently, set $\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}]$ plus all ($C_B$-**DC**) for $B \in \mathcal{L}[x,y,Y]$ where $C_B(y,X,Y) \equiv (\forall x)(x\in X \leftrightarrow B(x,y,Y))$. By the uniqueness of $X$ in $C_B(y,X,Y)$, we can interpret $\mathbf{ACA}^-_\omega$ in $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind})$: By induction on $n$ we can show $(\exists H)(\forall k<n)C_B(k,(H)_k,(H)_{<k})$ in the latter system. If we interpret $P^{C_B}(n,x)$ by a formula $(\exists H)((\forall k<n+1)C_B(k,(H)_k,(H)_{<k}) \wedge x\in (H)_n)$, all the axioms of $\mathbf{ACA}^-_\omega$ are interpreted as provable formulae in $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind})$. However, since fixed points are not unique, we cannot interpret $\widehat{\mathbf{ID}}_\omega$ in $\mathbf{ACA}_0 + (\Delta^1_0\text{-}\mathbf{FP}) + (\mathcal{L}_2\text{-}\mathbf{Ind})$ in the same way. This is why we need the forcing method, and actually such a situation is the typical one in which the forcing method works well.

Our argument will be concentrated on the scheme ($A$-**DC**), which generalizes the schemata used in the definitions of $\widehat{\mathbf{ID}}_\omega$ and of $\mathbf{ACA}^-_\omega$. We call $P^A$ a *choice hierarchy* in accord with $A$.

## 3 Forcing Method

The main idea of forcing or the forcing relation $\Vdash_{\mathbb{P}}$ is as follows: Though we cannot define the whole $P^A$, we can define finite fragments of it, in the sense that $(n,X) \Vdash_{\mathbb{P}} \varphi(P^A)$ is intended to mean that any choice hierarchy $H$ must satisfy $\varphi(H)$ whenever $H$ extends $X \upharpoonright n$.

**Definition 4.** The forcing notion $\mathbb{P}_A$ (where $A \in \mathcal{L}[y, X, Y]$ is often omitted) consists of pairs $(n, X)$ of numbers $n$ and second-order objects $X$ with $(\forall k < n)A(k, (X)_k, (X)_{<k})$, ordered by

$$(n, X) \leq_{\mathbb{P}} (m, Y) \leftrightarrow (n \geq m \wedge (\forall k < m)((X)_k = (Y)_k)) \wedge (n, X) \in \mathbb{P}_A.$$

The forcing relation $(n, X) \Vdash_{\mathbb{P}} \varphi$ between forcing conditions and $\mathcal{L}_{\mathrm{dc}}$-formulae is defined by:

$$\begin{aligned}
(n, X) \Vdash_{\mathbb{P}} R(\vec{x}) &\leftrightarrow R(\vec{x}) \qquad \text{for any } \mathcal{L}_1\text{-atom } R(\vec{x});\\
(n, X) \Vdash_{\mathbb{P}} P^A(x, y) &\leftrightarrow (\forall (m, Y) \leq_{\mathbb{P}} (n, X))(\exists (l, Z) \leq_{\mathbb{P}} (m, Y))(l > x \wedge y \in (Z)_x);\\
(n, X) \Vdash_{\mathbb{P}} \neg\varphi &\leftrightarrow (\forall (m, Y) \leq_{\mathbb{P}} (n, X))\neg((m, Y) \Vdash_{\mathbb{P}} \varphi);\\
(n, X) \Vdash_{\mathbb{P}} \varphi \wedge \psi &\leftrightarrow ((n, X) \Vdash_{\mathbb{P}} \varphi) \wedge ((n, X) \Vdash_{\mathbb{P}} \psi);\\
(n, X) \Vdash_{\mathbb{P}} (\forall x)\varphi(x) &\leftrightarrow (\forall x)((n, X) \Vdash_{\mathbb{P}} \varphi(x)).
\end{aligned}$$

Note that these can be formalized in $\mathcal{L}_2$, and that with reasonable axioms (at least with $\mathbf{ACA}_0$) the basic properties, e.g., the reflexivity and transitivity of $\leq_{\mathbb{P}}$, are proved. [1]

Among set theorists (or recursion theorists), forcing is better known as a procedure to construct a new model, called a generic model, and the forcing relation as a tool to describe such a model. While this view helps us to understand the intention above, we can consider the forcing relation as a syntactic interpretation. This is essential for our goal, that is, to obtain a direct interpretation of $\widehat{\mathbf{ID}}_\omega$ into $\mathbf{ACA}_0 + (\Delta^1_0\text{-}\mathbf{FP}) + (\mathcal{L}_2\text{-}\mathbf{Ind})$. Moreover, even for almost all applications in set theory, viewing the forcing method as a syntactic interpretation is equivalent (at least in theory) to viewing it as a model construction (as shown in e.g., [7]).[2]

**Lemma 5.** For any $\mathcal{L}_{\mathrm{dc}}$-formulae $\varphi, \psi$ and any $(n, X), (m, Y) \in \mathbb{P}$, the following hold:

$$\begin{aligned}
(n, X) \Vdash_{\mathbb{P}} \varphi \vee \psi &\leftrightarrow (\forall (m, Y) \leq_{\mathbb{P}} (n, X))(\exists (l, Z) \leq_{\mathbb{P}} (m, Y))((l, Z) \Vdash_{\mathbb{P}} \varphi \vee (l, Z) \Vdash_{\mathbb{P}} \psi);\\
(n, X) \Vdash_{\mathbb{P}} \varphi \to \psi &\leftrightarrow (\forall (m, Y) \leq_{\mathbb{P}} (n, X))((m, Y) \Vdash_{\mathbb{P}} \varphi \to (m, Y) \Vdash_{\mathbb{P}} \psi);\\
(n, X) \Vdash_{\mathbb{P}} (\exists x)\varphi(x) &\leftrightarrow (\forall (m, Y) \leq_{\mathbb{P}} (n, X))(\exists (l, Z) \leq_{\mathbb{P}} (m, Y))(\exists x)((l, Z) \Vdash_{\mathbb{P}} \varphi(x)).
\end{aligned}$$

**a) monotonicity:** $(n, X) \leq_{\mathbb{P}} (m, Y)$ and $(m, Y) \Vdash_{\mathbb{P}} \varphi$ imply $(n, X) \Vdash_{\mathbb{P}} \varphi$;

**b) density:** $(n, X) \Vdash_{\mathbb{P}} \varphi$ iff $(\forall (m, Y) \leq_{\mathbb{P}} (n, X))(\exists (l, Z) \leq_{\mathbb{P}} (m, Y))((l, Z) \Vdash_{\mathbb{P}} \varphi)$; and

**c) closure under entailment:** if $\varphi_0$ is a consequence of $\{\varphi_1, \cdots, \varphi_k\}$ in the sense of classical logic and if $(n, X) \Vdash_{\mathbb{P}} \varphi_i$ for all $1 \leq i \leq k$, then $(n, X) \Vdash_{\mathbb{P}} \varphi_0$.

A brief explanation of this lemma is as follows: The forcing $\Vdash_{\mathbb{P}}$ is equivalently defined by $(n, X) \Vdash_{\mathbb{P}} \varphi \leftrightarrow (n, X) \Vdash^i_{\mathbb{P}} \varphi^N$, where $N$ is Gödel-Gentzen's negative interpretation and where $\Vdash^i_{\mathbb{P}}$ is the Kripke semantics (formalized in our base theory, e.g., $\mathbf{ACA}_0$) over the reversed[3] preorder of $\leq_{\mathbb{P}}$ with the constant domain $\{x \mid x = x\}$ defined by $(n, X) \Vdash^i_{\mathbb{P}} R(\vec{x}) \leftrightarrow R(\vec{x})$ for an $\mathcal{L}_1$-atom $R(\vec{x})$ and by $(n, X) \Vdash^i_{\mathbb{P}} P^A_x(y) \leftrightarrow n > x \wedge y \in (Z)_x$. Now a) monotonicity of $\Vdash_{\mathbb{P}}$ follows from that of $\Vdash^i_{\mathbb{P}}$, b) density is nothing more than the Kripke validity of the negative interpretation of double negation elimination, and c) is from the closure of $\Vdash^i_{\mathbb{P}}$ of under intuitionistic entailment and the preservation of entailment under the negative translation.

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[1] We do not need the third-order object $\mathbb{P}$ as an "official entity" in the formalization.

[2] Particularly, for the closure under entailment as in the lemma, we do not need truth lemma for generic model.

[3] For Kripke semantics, it seems standard to let $q \leq p$ mean that $p$ has more information than $q$. For forcing, however, the reversal notation seems more standard, i.e., $q \leq p$ is read as "$q$ extends $p$". It should be mentioned that there is also a strong tradition to denote "$q$ extends $p$" by $q \geq p$ (Israeli notation) which we do not follow.

# 4 Main Result

In the following lemmata, we are working in $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) + (\forall y, Y \exists X A(y, X, Y))$. With the intuition mentioned at the beginning of the last section, the following lemma is quite natural, since $C(\vec{x}, P^A_k, P^A_{<k})$ does not refer to $P^A_j$ for $j > k$. $P^A_k$ and $P^A_{<k}$ are determined by $(n, X)$.

**Lemma 6.** For any $\mathcal{L}[\vec{x}, U, V]$-formula $C(\vec{x}, U, V)$ and $(n, X) \in \mathbb{P}$ with $k < n$,

$$C(\vec{x}, (X)_k, (X)_{<k}) \text{ iff } (n, X) \Vdash_{\mathbb{P}} C(\vec{x}, P^A_k, P^A_{<k}).$$

In particular, for any $\mathcal{L}_1$-formula $\varphi(\vec{x})$, $(\forall (n, X) \in \mathbb{P})(\varphi(\vec{x}) \leftrightarrow (n, X) \Vdash_{\mathbb{P}} \varphi(\vec{x}))$.

*Proof.* This can easily be shown by induction on $C(\vec{x}, U, V)$. $\square$

**Lemma 7.** For any $(n, X)$ and any $\mathcal{L}_{\mathrm{dc}}$-formula $\varphi(x, y)$,

$$(n, X) \Vdash_{\mathbb{P}} (\forall x < y)\varphi(x, y) \text{ iff } (\forall x < y)((n, X) \Vdash_{\mathbb{P}} \varphi(x, y)).$$

*Proof.* $(n, X) \Vdash_{\mathbb{P}} (\forall x < y)\varphi(x, y)$ iff, for all $x$, $(n, X) \Vdash_{\mathbb{P}} (x < y \to \varphi(x, y))$ iff, for all $x$ and $(m, Y) \leq_{\mathbb{P}} (n, X)$, $(m, Y) \Vdash_{\mathbb{P}} x < y$ implies $(m, Y) \Vdash \varphi(x, y)$ iff, for all $x < y$ and $(m, Y) \leq_{\mathbb{P}} (n, X)$, $(m, Y) \Vdash \varphi(x, y)$ iff, for all $x < y$, $(n, X) \Vdash_{\mathbb{P}} \varphi(x, y)$ by monotonicity. $\square$

**Lemma 8.** For any $(n, X) \in \mathbb{P}$, $(n, X) \Vdash$ "$\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}] + (A\text{-}\mathbf{DC})$".

*Proof.* If $\varphi$ is an axiom of Peano arithmetic other than induction, $(n, X) \Vdash_{\mathbb{P}} \varphi$ is from Lemma 6.

For induction, assume $(n, X) \Vdash_{\mathbb{P}} (\forall x)((\forall y < x)\varphi(y) \to \varphi(x))$ for an $\mathcal{L}_{\mathrm{dc}}$-formula $\varphi$. We have to show $(n, X) \Vdash_{\mathbb{P}} (\forall x)\varphi(x)$. By assumption, $(m, Y) \Vdash_{\mathbb{P}} (\forall y < x)\varphi(x)$ implies $(m, Y) \Vdash_{\mathbb{P}} \varphi(x)$ for any $x$ and $(m, Y) \leq_{\mathbb{P}} (n, X)$. Fix $(m, Y) = (n, X)$. This means $(\forall y < x)\psi(y) \to \psi(x)$ by Lemma 7, where $\psi(x) \equiv (n, X) \Vdash_{\mathbb{P}} \varphi(x)$. Thus by $(\mathcal{L}_2\text{-}\mathbf{Ind})$, we have $(\forall x)\psi(x)$, i.e., $(n, X) \Vdash_{\mathbb{P}} (\forall x)\varphi(x)$.

It remains to see $(0, \emptyset) \Vdash_{\mathbb{P}} A(k, P^A_k, P^A_{<k})$ for all $k \in \omega$. With $(\mathcal{L}_2\text{-}\mathbf{Ind})$, particularly with $(\Sigma^1_1\text{-}\mathbf{Ind})$, $(\forall y, Y \exists X A(y, X, Y))$ implies that for any $(n, X) \in \mathbb{P}$ there is $(m, Y) \leq_{\mathbb{P}} (n, X)$ such that $m > k$, and so Lemma 6 yields $(m, Y) \Vdash_{\mathbb{P}} A(k, P^A_k, P^A_{<k})$. By density we have $(0, \emptyset) \Vdash_{\mathbb{P}} A(k, P^A_k, P^A_{<k})$. $\square$

**Theorem 9.** The translation from $\mathcal{L}_{\mathrm{dc}}$ to $\mathcal{L}_2$ defined by $\varphi \mapsto (0, \emptyset) \Vdash_{\mathbb{P}} \varphi$ interprets $\mathbf{PA}[\mathcal{L}_{\mathrm{dc}}] + (A\text{-}\mathbf{DC})$ into $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) + (\forall y, Y, \exists X A(y, X, Y))$ in such a way that, for any $\mathcal{L}[\vec{x}, U, V]$-formula $C(\vec{x}, U, V)$, $C(\vec{x}, P^A_k, P^A_{<k})$ is interpreted as, up to equivalence, the following:

$$(\forall X)((k + 1, X) \in \mathbb{P} \to C(\vec{x}, (X)_k, (X)_{<k})). \tag{$*$}$$

In particular, the interpretation preserves $\mathcal{L}_1$-formulae (up to equivalence).

*Proof.* It remains to show the equivalence between $(0, \emptyset) \Vdash_{\mathbb{P}} C(\vec{x}, P^A_k, P^A_{<k})$ and $(*)$.

First assume $(0, \emptyset) \Vdash_{\mathbb{P}} C(\vec{x}, P^A_k, P^A_{<k})$ and $(k + 1, X) \in \mathbb{P}$. Then $(k + 1, X) \leq_{\mathbb{P}} (0, \emptyset)$. By monotonicity we have $(k + 1, X) \Vdash_{\mathbb{P}} C(\vec{x}, P^A_k, P^A_{<k})$ and, by Lemma 6, $C(\vec{x}, (X)_k, (X)_{<k})$.

Conversely assume $(*)$. For any $(m, Y) \leq_{\mathbb{P}} (0, \emptyset)$, by $(\Sigma^1_1\text{-}\mathbf{Ind})$ there is $(n, X) \leq_{\mathbb{P}} (m, Y)$ with $n \geq k + 1$. Then $(k + 1, X) \in \mathbb{P}$ and so $C(\vec{x}, (X)_k, (X)_{<k})$. By Lemma 6 we have $(n, X) \Vdash_{\mathbb{P}} C(\vec{x}, P^A_k, P^A_{<k})$. By density, we have $(0, \emptyset) \Vdash_{\mathbb{P}} C(\vec{x}, P^A_k, P^A_{<k})$. $\square$

**Corollary 10.** The translation in the theorem, for $A(y, X, Y) \equiv (\forall x)(x \in X \leftrightarrow O(x, y, X, Y))$, interprets $\widehat{\mathbf{ID}}_\omega$ into $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind}) + (\Delta^1_0\text{-}\mathbf{FP})$ with all $\mathcal{L}_1$ formulae being preserved.

Note that a finite number of $(A\text{-}\mathbf{DC})$'s can be put into one: $\bigwedge_{i<n}(A_i\text{-}\mathbf{DC})$ is equivalent to $(B\text{-}\mathbf{DC})$ with $B(y, X, Y) \equiv \bigwedge_{i<n}(y = i \bmod n) \wedge A_i(\lfloor y/n \rfloor, X, \{\langle j, x\rangle \mid \langle n \cdot j + i, x\rangle \in Y\})$. Since any proof in the system $\mathbf{DC}$ uses only a finite number of $(A\text{-}\mathbf{DC})$'s, we have the following local interpretability result, where the converse is immediate from the definability of $\emptyset^{(\omega)}$ in $\mathbf{DC}$.

**Corollary 11.** The translation *locally* interprets $\mathbf{DC}$ into $\mathbf{ACA}_0 + (\mathcal{L}_2\text{-}\mathbf{Ind})$ with $\mathcal{L}_1$ preserved.

## 5 Discussions

One might wonder if the interpretation we have given is a "direct interpretation" in the sense of the paper [5] from which the question originates. The author has to admit that there are many notions of (syntactic) interpretation between formal systems, as explained below.

One of the narrowest notions allows only the restriction of the ranges of quantifiers by the formulae associated to the sort of the quantifiers and the replacement of atomic formulae by some associated formulae, and does not allow changes in the meaning of Boolean connectives. For example, the interpretation of arithmetic (say Peano arithmetic) in set theory (say Zermelo-Fraenkel set theory) is an interpretation in this sense, and so is the interpretation of $\mathbf{ACA}_0 + (\Delta^1_0\text{-}\mathbf{FP})+(\mathcal{L}_2\text{-}\mathbf{Ind})$ in $\widehat{\mathbf{ID}}_\omega$, given by replacing $n \in X_i$ ($i$-th second-order variable) by $F^O_{x_i}(\langle y_i, n\rangle)$ and the second-order $(\forall X_i)$ and $(\exists X_i)$ by $(\forall x_i, y_i)$ and $(\exists x_i, y_i)$ respectively (the reversal of our main result). The interpretation given by forcing does not satisfy this condition.

One of the widest notions, on the other hand, is called *proof-theoretic reduction*, which effectively transforms proofs in the interpreted system into proofs in the interpreting system in such a way that relatively simple conclusions (at least including $\Pi^0_1$ formulae) are preserved. The typical example is cut-elimination method, employed by Jäger and Strahm [5]. Thus there is no reason to consider that what Jäger and Strahm mean by "direct interpretation" is the narrowest sense. However, since this wide notion of interpretation includes what they would call "*indirect* interpretation", for example the reduction of $\widehat{\mathbf{ID}}_\omega$ to $\mathbf{ACA}_0 + (\Delta^1_0\text{-}\mathbf{FP}) + (\mathcal{L}_2\text{-}\mathbf{Ind})$ via ordinal analysis, which they themselves mention. What does "direct interpretation" mean after all?

As long as the theories concerned are reflexive, the two notions are equivalent as has been known (see e.g., [4, III.2.39 Theorem]). Nevertheless the interpretation (in the narrowest sense) given by this general theorem is usually not intuitively clear and preserves only the $\Pi^0_1$ sentences (while ours preserves all $\mathcal{L}_1$ sentences). What Jäger and Strahm [5] concerned seems to be the intuitive clearness of a concretely defined interpretation or the preservation of relevant fragments.

A significant difference between forcing (and realizability) and proof-theoretic reduction in general is that, whereas the transformation in the latter depends on the global structure of proofs (like local interpretations), in the former it depends only on formulae. Moreover, the transformations of compound formulae are determined uniformly from those of subfomulae (particularly, in the case of quantifiers, as opposed to the famous result by Pour-El and Kripke [6] [4]).

It could be interesting to formulate a general notion of such "direct interpretations", and to develop a general theory in a similar sense to those for the narrowest and the widest notions.

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[4]Feferman classified three notions of "interpretation" in [3, Appendix]: (a) relative interpretation, basically the same as the narrowest above, (b) translation, which preserves negation, and (c) proof theoretic reduction. (b) contains Pour-El and Kripke's, but not forcing nor realizability, because the negation is not preserved.