A Decidable Multi-agent Logic with Iterations of Upper and Lower Probability Operators

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Abstract. We present a propositional logic for reasoning about higher-order upper and lower probabilities. The main technical result is the proof of decidability of the introduced logical system. We also show that the axiomatization for the corresponding logic without iterations of operators, which we developed in our previous work, is also complete for the new class of models presented in this paper.

Keywords: Probabilistic Logic, Upper and Lower Probabilities, Decidability, Completeness theorem.

1 Introduction

In the last few decades, uncertain reasoning has become an active topic of investigation for researchers in the fields of computer science, artificial intelligence and cognitive science. One particular line of research concerns the formalization in terms of logic. The frameworks designed for reasoning about uncertainty often use probability-based interpretation of knowledge or belief. In the first of those papers [28] motivated by development of an expert system in medicine, Nilsson tried to give a logic with probabilistic operators as a well-founded framework for uncertain reasoning. The question of providing an axiomatization and decision procedure for Nilsson's logic attracted the attention of other researchers in the field, and triggered investigation about formal systems for probabilistic reasoning [6–9, 11, 16, 25, 29–32].

However, in many applications, sharp numerical probabilities appear too simple for modeling uncertainty. In order to model some situations of interest, various imprecise probability models are developed [4, 5, 23, 26, 35–37, 39]. Some of those approaches use sets of probability measures instead of one fixed measure, and the uncertainty is represented by two boundaries, called *lower probability* and *upper probability* [14, 22]. Halpern and Pucella [13] give the following example: a bag contains 100 marbles, 30 of them are red and the remaining 70 are either blue or yellow, but we do not know their exact proportion. Obviously, we can assign exact probability 0.3 to the event that a randomly picked ball from the bag is red. On the other hand, for each possible probability p for picking a blue ball, we know that the remaining probability for yellow one is 0.7-p. This way we obtain a set of possible probability measures P. Based on P we can define the following two functions: the upper probability and the lower probability measure, which assign to an event X the supremum (resp. the infimum) of the probabilities assigned to X by the measures in P. Formally, if the uncertainty about probabilities is modeled by a set P of probability measures defined on given algebra H, then the lower probability measure $P_\star$ and the upper probability measure $P^\star$ are defined by $P_\star(X) = \inf\{\mu(X) \mid \mu \in P\}$ and $P^\star(X) = \sup\{\mu(X) \mid \mu \in P\}$, for every $X \in H$. Those two functions are related by the formula $P_\star(X) = 1 - P^\star(X^c)$.

Those probability notions were previously formalized in the logic developed in [13], where lower and upper probability operators are applied to propositional formulas, and in [33], where first-order logic is considered (a formula is a Boolean combination of formulas in which lower and upper probability operators are applied to first-order sentences).

In this paper, we use the papers [13, 33] as a starting point and generalize them in a way that we reason not only about lower and upper probabilities an agent assigns to a certain event, but also about her uncertain belief about other agent's imprecise probabilities. Thus, we introduce separate lower and upper probability operators for different agents, and we allow nesting of the operators, similarly as it has been done in [7], in the case of simple probabilities[1]. Our preliminary research on the topic is published in [34], where we axiomatized a first-order logic with nesting of lower and upper probability operators. However, since that logic extends standard first-order logic, it is obviously undecidable. To overcome that problem, in this paper we present a propositional variant of this logic, which we denote by ILUPP[2]; we prove that the logic is decidable and we propose a sound and strongly complete axiomatization for the logic.

Our language contains the upper and lower probability operators $U^a_{\geq r}$ and $L^a_{\geq r}$, for every agent a and every rational number r from the unit interval (we also introduce the operators with other types of inequalities, like $U^a_{=r}$). Consider the following example, essentially taken from [34]. *Suppose that an agent a is planning to visit a city based on the weather reports from several sources, and she decides to take an action if the probability of rain is at most $\frac{1}{10}$, according to all reports she considers. Since she wishes to go together with b, she should be sure with probability at least $\frac{9}{10}$ that b (who might consult different weather reports) has the same conclusion about the possibility of rain.* In our language,

[1] For a discussion on higher-order probabilities we refer the reader to [10].

[2] The notation is motivated by the logic LUPP from [34], where LUP stands for "lower and upper probability", while the second P indicates that the logic is propositional. We add I to denote iteration of upper and lower operators.

this situation can be formalized as

$$U^a_{\leq \frac{1}{10}} Rain \wedge L^a_{\geq \frac{9}{10}} (U^b_{\leq \frac{1}{10}} Rain),$$

where *Rain* is a primitive proposition of the corresponding language. The appropriate modal semantics consists of a specific class of Kripke models, in which every world is equipped with sets of probability measures (one set for each agent).

Our main technical result is that the satisfiability problem for ILUPP logic is decidable. In the proof, we combine the method of filtration [15] and a reduction to linear programming. In the first part of the proof, we show that a formula α is satisfiable in a world w of an ILUPP model if and only if it is satisfiable in a finite model, i.e., a model with a finite number of worlds, bounded by a number which is a function of the length of α, and such that the sets of probabilitiy measures are finite in every world of the model. Note that, while in a standard modal framework this is enough to prove decidability, since for every natural number k there are only finitely many modal models with k worlds, this is not the case for our logic. Indeed, since our models involve sets of probability measures, for every finite set of k worlds, there are uncountably many probability measures defined on them, and uncountably many models with k worlds. However, in the second part of the proof we use a reduction to linear programming to solve the probabilistic satisfiability in a finite number of steps.

We also propose a sound and strongly complete axiomatization of the logic. Interestingly, we use the same axiomatization that we used in [33] for the logic LUPP, and we show that it is also complete for the richer logic ILUPP. Of course, the instances of the axiom schemata are different, because the sets of formulas of ILUPP is larger, due to nesting of lower and upper probability operators, and due to the presence of more agents. Also, the definition of the syntactical consequence (proof) $\vdash$ is different, due to the different interpretation of classical formulas. Since the class of formulas and the class of models are different, the proof techniques are modified. In order to achieve completeness, we use a Henkin-like construction, following some of our earlier developed methods [17, 19, 20, 29, 32, 33].

The interesting situation that one axiomatic system is sound and complete for more than one class of models is not an exception. For example, modal system K is also sound and complete with respect to the class of all irreflexive models [15].

The paper is organized as follows: in Section 2 we introduce the set of formulas of the logic ILUPP and we define the corresponding semantics. Then, in Section 3 we prove that the satisfiability problem for the logic ILUPP is decidable. In Section 4 we provide an axiomatic system for the logic, and we prove that the axiomatization is strongly complete. Finally, Section 6 contains some concluding remarks.

2 The logic ILUPP

In this section we introduce the syntax and the semantics of the logic ILUPP.

2.1 Syntax

Let $\Sigma = \{a, b, \dots\}$ be a finite, non-empty set of agents. Let $S = \mathbb{Q} \cap [0, 1]$ and let $\mathcal{L} = \{p, q, r, \dots\}$ be a denumerable set of propositional letters. The language of the logic ILUPP consists of:

- the elements of set $\mathcal{L}$,
- classical propositional connectives $\neg$ and $\wedge$,
- the list of upper probability operators $U^a_{\geq s}$, for every $a \in \Sigma$ and every $s \in S$,
- the list of lower probability operators $L^a_{\geq s}$, for every $a \in \Sigma$ and every $s \in S$.

Definition 1 (Formula) *The set For_{ILUPP} of formulas is the smallest set containing all elements of $\mathcal{L}$ and that is closed under following formation rules: if α, β are formulas, then $L^a_{\geq s}\alpha$, $U^a_{\geq s}\alpha$, $\neg\alpha$ and $\alpha \wedge \beta$ are formulas as well. The formulas from For_{ILUPP} will be denoted by $\alpha, \beta, \dots$*

Intuitively, $U^a_{\geq s}\alpha$ means that according to an agent a, upper probability that a formula α is true is greater or equal to s and analoguosly $L^a_{\geq s}\alpha$ means that according to an agent a lower probability that a formula α is true is greater or equal to s.

Note that we use conjunction and negation as primitive connectives, while $\vee$, $\rightarrow$ and $\leftrightarrow$ are introduced in the usual way. We also use abbreviations to introduce other types of inequalities:

- $U^a_{<s}\alpha$ is $\neg U^a_{\geq s}\alpha$, $U^a_{\leq s}\alpha$ is $L^a_{\geq 1-s}\neg\alpha$, $U^a_{=s}\alpha$ is $U^a_{\leq s}\alpha \wedge U^a_{\geq s}\alpha$, $U^a_{>s}\alpha$ is $\neg U^a_{\leq s}\alpha$,
- $L^a_{<s}\alpha$ is $\neg L^a_{\geq s}\alpha$, $L^a_{\leq s}\alpha$ is $U^a_{\geq 1-s}\neg\alpha$, $L^a_{=s}\alpha$ is $L^a_{\leq s}\alpha \wedge L^a_{\geq s}\alpha$, $L^a_{>s}\alpha$ is $\neg L^a_{\leq s}\alpha$.

For example, the expression

$$p \wedge U^a_{=0.9} L^b_{=0.3}(p \vee q)$$

is a formula of our language.

2.2 Semantics

The semantics for the logic ILUPP is based on the possible-world approach. Every world is equipped with an evaluation function on propositional letters, and one generalized probability space for each agent.

Definition 2 (ILUPP-structure) *An ILUPP-structure is a tuple $\langle W, LUP, v\rangle$, where:*

- *W is a nonempty set of worlds,*
- *LUP assigns, to every $w \in W$ and every $a \in \Sigma$, a space, such that $LUP(a, w) = \langle W(a, w), H(a, w), P(a, w)\rangle$, where:*
  - *$\emptyset \neq W(a, w) \subseteq W$,*
  - *$H(a, w)$ is an algebra of subsets of $W(a, w)$, i.e. a set of subsets of $W(a, w)$ such that:*
    - *$W(a, w) \in H(a, w)$,*
    - *if $A, B \in H(a, w)$, then $W(a, w) \setminus A \in H(a, w)$ and $A \cup B \in H(a, w)$,*

- *$P(a,w)$ is a set of finitely additive probability measures defined on $H(a,w)$, i.e. for every $\mu(a,w) \in P(a,w)$, $\mu(a,w) : H(a,w) \longrightarrow [0,1]$ and the following conditions hold:*
  - $\mu(a,w)(W(a,w)) = 1$,
  - $\mu(a,w)(A \cup B) = \mu(a,w)(A) + \mu(a,w)(B)$, *whenever* $A \cap B = \emptyset$.
- *$v : W \times \mathcal{L} \longrightarrow \{true, false\}$ provides for each world $w \in W$ a two-valued evaluation of the primitive propositions.*

Now we define satisfiability of the formulas from For_{ILUPP} in the worlds of ILUPP-structures. As we mentioned in the introduction, for any set P of probability measures defined on given algebra H, the lower probability measure $P_\star$ and the upper probability measure $P^\star$ are defined by

- $P_\star(X) = \inf\{\mu(X) \mid \mu \in P\}$ and
- $P^\star(X) = \sup\{\mu(X) \mid \mu \in P\}$,

for every $X \in H$. It is easy to check that

$$P_\star(X) = 1 - P^\star(X^c), \tag{1}$$

for every $X \in H$. In the context of the definition of an ILUPP-structure, we will denote $P_\star(a,w)([\alpha]^a_{M,w}) = \inf\{\mu([\alpha]^a_{M,w}) \mid \mu \in P(a,w)\}$ and $P^\star(a,w)([\alpha]^a_{M,w}) = \sup\{\mu([\alpha]^a_{M,w}) \mid \mu \in P(a,w)\}$, where $[\alpha]^a_{M,w} = \{u \in W(a,w) \mid M,u \models \alpha\}$.

Definition 3 (Satisfiability relation) *For every ILUPP structure $M = \langle W, LUP, v\rangle$ and every $w \in W$, the satisfiability relation $\models$ fulfills the following conditions:*

- *if $p \in \mathcal{L}$, $M,w \models p$ iff $v(w)(p) = true$,*
- *$M,w \models \neg\alpha$ iff it is not the case that $M,w \models \alpha$,*
- *$M,w \models \alpha \wedge \beta$ iff $M,w \models \alpha$ and $M,w \models \beta$,*
- *$M,w \models U^a_{\geq s}\alpha$ iff $P^\star(a,w)([\alpha]^a_{M,w}) \geq s$,*
- *$M,w \models L^a_{\geq s}\alpha$ iff $P_\star(a,w)([\alpha]^a_{M,w}) \geq s$.*

We will omit M when it's clear from context. The possible problem with the previous definition is that it might happen that for some M, w , a and α the set $[\alpha]^a_{M,w}$ doesn't belong to $W(a,w)$. For that reason, we will consider only so called measurable structures.

Definition 4 (Measurable structure) *The structure M is measurable if for every $a \in \Sigma$ and every $w \in W$, $H(a,w) = \{[\alpha]^a_w \mid \alpha \in For_{\mathsf{ILUPP}}\}$. The class of all measurable structures of the logic ILUPP will be denoted by ILUPP_{Meas}.*

Definition 5 (Satisfiability of a formula) *A formula $\alpha \in For_{\mathsf{ILUPP}}$ is satisfiable if there is a world w in an ILUPP_{Meas}-model M such that $w \models \alpha$; α is valid if it is satisfied in every world in every ILUPP_{Meas}-model M. A set of formulas T is satisfiable if there is a world w in an ILUPP_{Meas}-model M such that $w \models \alpha$ for every $\alpha \in T$.*

3 Decidability

In this section, we prove our main technical result. Recall the satisfiability problem: given an ILUPP-formula α, we want to determine if there exists a world w in an ILUPP_{Meas}-model M such that $w \models \alpha$. Decidability for ILUPP will be proved in two steps:

- first, we show that an ILUPP-formula is satisfiable iff it is satisfiable in a measurable structures with a finite number of worlds,
- second, we show that we can consider only finite measurable structures, i.e., measurable structure with finite number of worlds and with finite sets of probability measures in every world and for every agent, and
- third, we reduce the satisfiability problem in those finite models to a decidable linear programming problem.

In the first part of the proof, we will use the method of filtration [15]. Like the previous papers on the logical formalization of upper and lower probabilities [13, 33], we also use the characterization theorem by Anger and Lembcke [2]. It uses the notion of (n, k)-cover.

Definition 6 ((n,k)-cover) *A set A is said to be covered n times by a multiset $\{\{A_1, \dots, A_m\}\}$ of sets if every element of A appears in at least n sets from $A_1, \dots, A_m$, i.e., for all $x \in A$, there exists $i_1, \dots, i_n$ in $\{1, \dots, m\}$ such that for all $j \leq n$, $x \in A_{i_j}$. An (n, k)-cover of (A, W) is a multiset $\{\{A_1, \dots, A_m\}\}$ that covers W k times and covers A $n + k$ times.*

Now we can state the characterization theorem.

Theorem 1 (Anger and Lembcke [2]) *Let W be a set, H an algebra of subsets of W, and f a function $f : H \longrightarrow [0, 1]$. There exists a set P of probability measures such that $f = P^\star$ iff f satisfies the following three properties:*

(1) $f(\emptyset) = 0$,
(2) $f(W) = 1$,
(3) for all natural numbers m, n, k and elements $A_1, \dots, A_m$ in H, if the multiset $\{\{A_1, \dots, A_m\}\}$ is an (n, k)-cover of (A, W), then $k + nf(A) \leq \sum_{i=1}^{m} f(A_i)$.

Let $SF(\alpha)$ denote the set of all subformulas of a formula α, i.e.

$$SF(\alpha) = \{\beta \mid \beta \text{ is a subformula of } \alpha\}.$$

Theorem 2 *If a formula α is satisfiable, then it is satisfiable in an ILUPP_{Meas}-model with at most $2^{|SF(\alpha)|}$ worlds.*

Proof. Suppose that a formula α holds in some world of the model $M = \langle W, LUP, v \rangle$ and let $k = |SF(\alpha)|$. By $\approx$, we will denote an equivalence relation over W^2, such that

$$w \approx u \text{ if and only if for every } \beta \in SF(\alpha), w \models \beta \text{ iff } u \models \beta.$$

Since there are finitely many subformulas of α, we know that the quotient set

$$W_{/\approx} = \{C_{w_i} \mid w_i \in W\}$$

is finite, where

$$C_{w_i} = \{u \in W \mid u \approx w_i\}$$

is the class of equivalence of w_i. More precsely,

$$|W_{/\approx}| \leq 2^k.$$

Next, from each class of equivalence C_{w_i}, we choose an element w_i.
Consider a tuple $\overline{M} = \langle \overline{W}, \overline{LUP}, \overline{v} \rangle$, where:

– $\overline{W} = \{w_1, w_2, \dots\}$,
– For every a and for every w_i $\overline{LUP}(a, w_i) = \langle \overline{W}(a, w_i), \overline{H}(a, w_i), \overline{P}(a, w_i) \rangle$ is defined as follows:
  - $\overline{W}(a, w_i) = \{w_j \in \overline{W} \mid (\exists u \in C_{w_j}) u \in W(a, w_i)\}$
  - $\overline{H}(a, w_i) = 2^{\overline{W}(a,w_i)}$
  - $\overline{P}(a, w_i)$ is any set of finitely additive measures, such that for every $D \in \overline{H}(a, w_i)$, $\overline{P}^\star(a, w_i)(D) = P^\star(a, w_i)(\bigcup_{w_j \in D}(C_{w_j} \cap W(a, w_i)))$
– $\overline{v}(w_i)(p) = v(w_i)(p)$, for every primitive proposition $p \in \mathcal{L}$.

First, we have to prove that $\overline{P}^\star(a, w_i)$ satisfies the conditions $(1) - (3)$ from Theorem 1, which will guarantee the existence of sets $\overline{P}(a, w_i)$, for every agent a and each $w_i \in \overline{W}$.

(1) $\overline{P}^\star(a, w_i)(\emptyset) = P^\star(a, w_i)(\bigcup_{w_j \in \emptyset}(C_{w_j} \cap W(a, w_i))) = P^\star(a, w_i)(\emptyset) = 0$;
(2) $\overline{P}^\star(a, w_i)(\overline{W}(a, w_i)) = P^\star(a, w_i)(\bigcup_{w_j \in \overline{W}(a,w_i)}(C_{w_j} \cap W(a, w_i))) =$
$= P^\star(a, w_i)(W(a, w_i)) = 1$;
(3) Let $\{\{D_1, \dots, D_m\}\}$ be an (n, k)-cover of $(D, \overline{W}(a, w_i))$. That means:
  i) every element from D appears in at least $n + k$ sets from $D_1, \dots, D_m$;
  ii) every element from $\overline{W}(a, w_i)$ appears in at least k sets from $D_1, \dots, D_m$.
  Therefore,
  iii) every element from $(\bigcup_{u \in D}(C_u \cap W(a, w_i))$ appears in at least $n + k$ sets from $\bigcup_{u \in D_1}(C_u \cap W(a, w_i)), \dots, \bigcup_{u \in D_m}(C_u \cap W(a, w_i))$;
  iv) every element from $W(a, w_i)$ appears in at least k sets from $\bigcup_{u \in D_1}(C_u \cap W(a, w_i)), \dots, \bigcup_{u \in D_m}(C_u \cap W(a, w_i))$.
  Hence, by definition, we obtain that a multiset

$$\{\{ \bigcup_{u \in D_1} (C_u \cap W(a, w_i)), \dots, \bigcup_{u \in D_m} (C_u \cap W(a, w_i))\}\}$$

  is an (n, k)-cover of

$$(\bigcup_{u \in D} (C_u \cap W(a, w_i)), W(a, w_i)).$$

Hence, using the fact that $P^\star(a, w_i)$ is an upper probability, from Theorem 1, we have that

$$k + nP^\star(a, w_i)(\bigcup_{u\in D}(C_u \cap W(a, w_i))) \leq \sum_{j=1}^{m} P^\star(a, w_i)(\bigcup_{u\in D_j}(C_u \cap W(a, w_i))),$$

and therefore

$$k + n\overline{P}^\star(a, w_i)(D) \leq \sum_{j=1}^{m} \overline{P}^\star(a, w_i)(D_j).$$

Using induction on the complexity of a formula from the set $SF(\alpha)$, we can prove that for every $w \in \overline{W}$ and every $\beta \in SF(\alpha)$,

$$M, w \models \beta \quad \text{if and only if} \quad \overline{M}, w \models \beta.$$

If a formula is a propositional letter or obtained using Boolean connectives, the claim is trivial. So, let us consider the case when $\beta = U^a_{\geq s}\gamma$:

$$\begin{aligned} M, w \models U^a_{\geq s}\gamma & \quad \text{iff} \\ P^\star(a, w)(\{u \in W(a, w) \mid M, u \models \gamma\}) \geq s & \quad \text{iff} \\ P^\star(a, w)(\bigcup_{M,u\models\gamma} C_u \cap W(a, w)) \geq s & \quad \text{iff (ind. hyp)} \\ \overline{P}^\star(a, w)(\{u \in \overline{W}^\star(a, w) \mid \overline{M}, u \models \gamma\}) \geq s & \quad \text{iff} \\ \overline{M}, w \models U^a_{\geq s}\gamma. & \end{aligned}$$

Using the equation (1) and the fact that $\overline{P}^\star(a, w)$ is an upper probability, the case when $\beta = L^a_{\geq s}\gamma$ can be proved analogously. □

In the second part of the proof, we use the following result of Halpern and Pucella [13].

Theorem 3 ([13]) *Let P be a set of probability measures defined on an algebra H over a finite set W. Then there exists a set P' of probability measures such that, for each $X \in H$, $P^*(X) = (P')^*(X)$. Moreover, there is a probability measure $\mu_X \in P'$ such that*

$$\mu_X(X) = P^*(X).$$

As a direct consequence of Theorem 2 and Theorem 3, we obtain the following result.

Lemma 1 *If a formula α is satisfiable, then it is satisfiable in an ILUPP_{Meas}-model with at most $2^{|SF(\alpha)|}$ worlds and for every agent $a \in \Sigma$ and every $w \in W$, $H(a, w) = 2^{W(a,w)}$ and*

$$|P(a, w)| = |H(a, w)|.$$

Furthermore, for each $X \in H(a, w)$, there exists a $\mu_X \in P(a, w)$ such that

$$\mu_X(a, w)(X) = P^*(a, w)(X).$$

With this lemma we are ready to prove the decidability result for the ILUPP logic.

Theorem 4 *Satisfiability problem for* ILUPP_{Meas} *is decidable.*

Proof. Let $M = \langle W, LUP, v\rangle$ be an ILUPP_{Meas}-model and α an arbitrary formula. Also, let

$$SF(\alpha) = \{\beta_1, \dots, \beta_k\}.$$

In every $w \in W$, exactly one of the formulas of the following form:

$$\pm\beta_1 \wedge \dots \wedge \pm\beta_k$$

holds, where $\pm\beta_i$ denotes β_i or $\neg\beta_i$. We will call that formula a characteristic formula for a world w (characteristic formula for a world w_i will be denoted by α_i).

By Lemma 1, we know that there exists an ILUPP_{Meas}-model $\overline{M}$ with

1) at most 2^k worlds and
2) at most 2^{2^k} probabilistic measures (for any agent and any world),

such that α holds in some world of the model $\overline{M}$ iff α holds in some world of a model M.

For every $l \leq 2^k$, we will consider models with

- l worlds, $w_1, \dots, w_l$, and
- for every agent a and every world w, sets of probability measures $P(a, w)$, such that $|P(a, w)| = 2^{|W(a,w)|}$, for every $W(a, w) \subseteq \{w_1, \dots w_l\}$.

In each of these worlds, exactly one characteristic formula holds. So, for each l, we will consider all possible sets of l characteristic formulas such that:

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(a) Let α_i be a characteristic formula. In α_i we replace every occurrence of a formula starting with a probabilistic operator with an atomic proposition (all the occurrences of the same formula are assigned the same atomic proposition). Then we obtain a propositional formula, α_i'. Using any algorithm for propositional satisfiability we check whether α_i' is satisfiable. If α_i' passes the test, then α_i is further considered for probabilistic tests (as in the paper). If α_i' does not pass the test, then α_i is no longer considered;
(b) At least one formula contains α.

For each choice, and each world w_i, we will consider following set of linear equalities and inequalities (by $\beta \in (\alpha_j)^+$ we will denote that β is a conjuct in α_j and by $\beta \in (\alpha_j)^-$ we will denote that $\neg\beta$ is a conjuct in α_j):

1) $\mu(a, w_i)(\{w_j\}) \geq 0$, for each $\mu(a, w_i) \in P(a, w_i)$ and $j = 1, \dots, l$;
2) $\sum_{w_j \in W(a,w_i)} \mu(a, w_i)(\{w_j\}) = 1$, for every $\mu(a, w_i) \in P(a, w_i)$;

3) $\sum_{w_j \in X} \mu_X(a, w_i)(\{w_j\}) \geq \sum_{w_j \in X} \mu_Y(a, w_i)(\{w_j\})$, for every $X, Y \subseteq W(a, w_i)$;
4) $\sum_{w_j : \beta \in (\alpha_j)^+} \mu_X(a, w_i)(\{w_j\}) \geq s$, if $U^a_{\geq s}\beta \in \alpha_i, \quad X = \{w_j \mid \beta \in (\alpha_j)^+\}$;
5) $\sum_{w_j : \beta \in (\alpha_j)^+} \mu_X(a, w_i)(\{w_j\}) < s$, if $\neg U^a_{\geq s}\beta \in \alpha_i, \quad X = \{w_j \mid \beta \in (\alpha_j)^+\}$;
6) $\sum_{w_j : \beta \in (\alpha_j)^-} \mu_X(a, w_i)(\{w_j\}) \leq 1 - s$, if $L^a_{\geq s}\beta \in \alpha_i, \quad X = \{w_j \mid \beta \in (\alpha_j)^-\}$;
7) $\sum_{w_j : \beta \in (\alpha_j)^-} \mu_X(a, w_i)(\{w_j\}) > 1 - s$, if $\neg L^a_{\geq s}\beta \in \alpha_i$, $X = \{w_j \mid \beta \in (\alpha_j)^-\}$.

- First inequality states that all the measures must be nonnegative.
- Second equality assures that the probability of the set of all possible worlds has to be equal to 1.
- Third inequality corresponds to the fact that $\mu_X(a, w)(X) = P^*(a, w)(X)$ and therefore

$$\mu_X(a, w)(X) \geq \mu(a, w)(X), \text{ for all } \mu(a, w) \in P(a, w).$$

- For the fourth and fifth inequality, note that if $X = \{w_j \mid \beta \in (\alpha_j)^+\}$

$$\sum_{w_j : \beta \in (\alpha_j)^+} \mu_X(a, w_i)(\{w_j\}) = P^*(a, w_i)([\beta]^a_{w_i}),$$

  so these inequalities reflect the appropriate constraints.
- In order to understand sixth and seventh inequality, first recall the equality connecting upper and lower probabilty:

$$P^*([\neg\beta]^a_{w_i}) = 1 - P_*([\beta]^a_{w_i}).$$

  Next, note that if $X = \{w_j \mid \beta \in (\alpha_j)^-\}$

$$\sum_{w_j : \beta \in (\alpha_j)^-} \mu_X(a, w_i)(\{w_j\}) = P^*(a, w_i)([\neg\beta]^a_{w_i}).$$

  Consequently, if

$$P_*([\beta]^a_{w_i}) \geq s,$$

  then

$$P^*([\neg\beta]^a_{w_i}) \leq 1 - s,$$

  and similarly for the case when $P_*([\beta]^a_{w_i}) < s$.

The equations and inequalities 1–7 form a finite system of linear equalities and inequalities and it is well known that solving this system is decidable. If for some fixed l and fixed choice of characteristic formulas, and each choice of subsets $W(a, w)$ of considered sets of worlds (for every agent a and every considered world w), corresponding system is solvable, then in each world, probabilistic space can be defined. Moreover, in every world w of the model, the characteristic formula of the world holds in w. Since α belongs to at least one of the corresponding characteristic formulas, we have that α is satisfiable.

If the test fails, and there is another possibility of choosing l and/or the set of l worlds and/or subsets $W(a, w)$ of chosen sets of worlds, we continue with the procedure. Otherwise, if for any l, any choice of characteristic formulas and any choice of subsets $W(a, w)$, appropriate system is not solvable, using Lemma 1, we conclude that α is not ILUPP_{Meas}-satisfiable.

Note that in the previously described method we consider only finitely many systems of linear equation and inequalities. Therefore, the satisfiability problem is decidable. □

4 A complete axiomatization

Having settled the decidability issue the for the logic ILUPP, we turn to the problem of developing an axiomatic system for the logic ILUPP. That system will be denoted by Ax_{ILUPP}.

4.1 The axiomatization Ax_{ILUPP}

We start with the observation that, like any other real-valued probabilistic logic, ILUPP is not compact. Indeed, consider the set of formulas $T = \{\neg U_{=0}\alpha\} \cup \{U_{<\frac{1}{n}}\alpha \mid n \text{ is a positive integer }\}$. Obviously, every finite subset of T is ILUPP_{Meas}-satisfiable, but the set T is not. Consequently, any finitary axiomatic system would be incomplete [38]. In order to achieve completeness, we use two infinitary rules of inference, with countably many premises and one conclusion.

In order to axiomatize upper and lower probabilities, we need to completely characterize them with a small number of properties. There are many complete characterizations in the literature, and the earliest appears to be by Lorentz [24]. We will use Theorem 1 from the previous section.

For the logic ILUPP, we use a minor modification of the axiomatic system for the logic LUPP in [33].

Axiom schemes

(1) all instances of the classical propositional tautologies
(2) $U^a_{\leq 1}\alpha \wedge L^a_{\leq 1}\alpha$
(3) $U^a_{\leq r}\alpha \to U^a_{<s}\alpha$, $s > r$
(4) $U^a_{<s}\alpha \to U^a_{\leq s}\alpha$
(5) $(U^a_{\leq r_1}\alpha_1 \wedge \cdots \wedge U^a_{\leq r_m}\alpha_m) \to U^a_{\leq r}\alpha$, if $\alpha \to \bigvee_{J\subseteq\{1,\ldots,m\},|J|=k+n} \bigwedge_{j\in J}\alpha_j$ and $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k} \bigwedge_{j\in J}\alpha_j$ are tautologies, where $r = \frac{\sum_{i=1}^m r_i - k}{n}$, $n \neq 0$
(6) $\neg(U^a_{\leq r_1}\alpha_1 \wedge \cdots \wedge U^a_{\leq r_m}\alpha_m)$, if $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k} \bigwedge_{j\in J}\alpha_j$ is a tautology and $\sum_{i=1}^m r_i < k$
(7) $L^a_{=1}(\alpha \to \beta) \to (U^a_{\geq s}\alpha \to U^a_{\geq s}\beta)$

Inference Rules

(1) From α and $\alpha \to \beta$ infer β

(2) From α infer $L^a_{\geq 1}\alpha$
(3) From the set of premises

$$\{\alpha \rightarrow U^a_{\geq s-\frac{1}{k}}\beta \mid k \geq \frac{1}{s}\}$$

infer $\alpha \rightarrow U^a_{\geq s}\beta$
(4) From the set of premises

$$\{\alpha \rightarrow L^a_{\geq s-\frac{1}{k}}\beta \mid k \geq \frac{1}{s}\}$$

infer $\alpha \rightarrow L^a_{\geq s}\beta$.

The axioms 5 and 6 together capture the third condition from the Theorem 1 (see [33]. The rules 3 and 4 are infinitary rules of inference and intuitively state that if an upper/lower probability is arbitrary close to a rational number s then it is at least s.

Now we define some proof theoretical notions.

- $\vdash \alpha$ (α is a theorem) iff there is an at most denumerable sequence of formulas $\alpha_1, \alpha_2, \dots, \alpha$, such that every α_i is an axiom or it is derived from the preceding formulas by an inference rule;
- $T \vdash \alpha$ (α is derivable from T) if there is an at most denumerable sequence of formulas $\alpha_1, \alpha_2, \dots, \alpha$, such that every α_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Inference Rule 2 can be applied only to the theorems;
- T is *consistent* if there is at least one formula $\alpha \in For_{\mathsf{ILUPP}}$ that is not deducible from T, otherwise T is inconsistent;
- T is *maximal consistent* set if it is consistent and for every $\alpha \in For_{\mathsf{ILUPP}}$, either $\alpha \in T$ or $\neg\alpha \in T$;
- T is *deductively closed* if for every $\alpha \in For_{\mathsf{ILUPP}}$, if $T \vdash \alpha$, then $\alpha \in T$.

Note that T is inconsistent iff $T \vdash \bot$. Also, it is easy to check that every maximal consistent set is deductively closed.

It is easy to check that the axiomatic system Ax_{ILUPP} is sound with respect to the class of ILUPP_{Meas}-models.

4.2 Completeness

We prove that the axiomatization Ax_{ILUPP} is complete, using a Henkin-like construction. Due to the presence of infinitary rules, the standard completion technique (Lindenbaum's theorem) has to be modified in the following way: if the current theory is inconsistent with the current formula and that formula can be derived by one of infinitary inference rules, than one of the premises must be blocked.

The proof of completeness is a direct combination of the proof techniques presented in our papers [33] and [34]. Thus, here we only present a sketch of the proof, and for details and the completion of the proof we refer the reader to [33, 34].

Theorem 5 (Strong completeness) *If α is a formula, and T is a set of formulas of the logic* ILUPP*, then $T \vdash \alpha$ iff $T \models \alpha$.*

Sketch of the proof. First we point out that the theorem follows from soundness of the axiomatic system Ax_{ILUPP}, and the following usual formulation of strong completeness:

Every consistent set of formulas T is satisfiable.

Let us prove this statement. First, we will extend T to a maximal consistent set T^*. We assume an enumeration $\alpha_0, \alpha_1, \dots$ of all formulas. Then we define the chain of sets T_i, $i = 0, 1, 2, \dots$ and the set T^* in the following way:

(1) $T_0 = T$,
(2) for every $i \geq 0$,
 (a) if $T_i \cup \{\alpha_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\alpha_i\}$, otherwise
 (b) if α_i is of the form $\beta \to U^a_{\geq s}\alpha$, then $T_{i+1} = T_i \cup \{\neg\alpha_i, \beta \to \neg U^a_{\geq s-\frac{1}{n}}\alpha\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
 (c) if α_i is of the form $\beta \to L^a_{\geq s}\alpha$, then $T_{i+1} = T_i \cup \{\neg\alpha_i, \beta \to \neg L^a_{\geq s-\frac{1}{n}}\alpha\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
 (d) $T_{i+1} = T_i \cup \{\neg\alpha_i\}$.
(3) $T^\star = \bigcup_{i=0}^{\infty} T_i$.

The proof that T^* is a maximal consistent set is based on the following observations:

- Natural numbers (n), from the steps $2(b)$ and $2(c)$ of the construction exist; this follows from Deduction Theorem, which holds in ILUPP logic (the deduction theorem can be proved using the implicative form of the two infinitary inference rules, and the fact that the application of Rule 2 is restricted to theorems only).
- Each T_i is consistent, by construction.
- T^* does not contain all the formulas, by construction, using the fact that all T_i's are consistent.
- For every formula α, either $\alpha \in T^*$ or $\neg\alpha \in T^*$, by construction (steps (1) and (2)).
- For every formula α, if $T^\star \vdash \alpha$, then $\alpha \in T^\star$ (the proof of this fact is by the induction on the length of the inference).
- By the last two facts, $T^\star$ is a deductively closed set, and $T^\star$ does not contain all the formulas, so it is consistent. Therefore, $T^\star$ is a maximal consistent set.

Now we define the canonical model $M_{Can} = \langle W, LUP, v\rangle$ such that:

- $W = \{w \mid w \text{ is a maximal consistent set of formulas}\}$,
- for every world w and every propositional letter p, $v(w)(p) = true$ iff $p \in w$,
- for every $a \in \Sigma$ and $w \in W$, $LUP(a, w) = \langle W(a, w), H(a, w), P(a, w)\rangle$ is defined in the following way:

- $W(a,w) = W$,
- $H(a,w) = \{\{u \mid u \in W(a,w), \alpha \in u\} \mid \alpha \in For_{\mathsf{ILUPP}}\}$,
- $P(a,w)$ is any set of probability measures such that $P^{\star}(a,w)(\{u \mid u \in W(a,w), \alpha \in u\}) = \sup\{s \mid U_{\geq s}\alpha \in w\}$.

We have the following properties of M_{Can}:

- For every formula α and every $w \in W$, $\alpha \in w$ iff $M_{Can}, w \models \alpha$ (the proof is on the complexity of the formula α).
- For every $a \in \Sigma$, every $w \in W$ and every formula α, $\{u \mid u \in W(a,w), \alpha \in u\} = [\alpha]^{a}_{w}$. (this follows from the previous item).
- M_{Can} is a well defined measurable structure (the proof that $P^{\star}(a,w)$ is an upper probability measure follows from Theorem 1 and the axioms 5 and 6).

Recall that we extended T to the maximal consistent set T^{*}. We showed that for every formula α, and every $w \in W$, $M_{Can}, w \models \alpha$ iff $\alpha \in w$. Since $T^{*} \in W$, we obtain $M_{Can}, T^{*} \models T$. □

5 Conclusion

In this paper we present the proof-theoretical analysis of a logic which allows making statements about upper and lower probabilities. The introduced formalism can be used for reasoning not only about lower and upper probabilities an agent assigns to a certain event, but also about her uncertain belief about other agent's imprecise probabilities. The language of ILUPP is a modal language which extends propositional logic with the unary operators $U^{a}_{\geq r}$ and $L^{a}_{\geq r}$, where a is an agent and r ranges over the unit interval of rational numbers. The corresponding semantics ILUPP_{Meas} consists of the measurable Kripke models with sets of finitely additive probability measures attached to each possible world world.

We prove that the satisfiability problem for ILUPP logic is decidable. In the proof, we use the method of filtration [15] to show that if a formula is satisfiable in a world w of an ILUPP structure, then it is satisfiable in a finite structure. We also use a reduction to linear programming to deal with infinitely many probability measures definable on finite algebras, and to solve the satisfiability problem in a finite number of steps.

We also prove that the proposed axiomatic system Ax_{ILUPP} is strongly complete with respect to the class of ILUPP_{Meas}-models. Since the logic is not compact, the axiomatization contains infinitary rules of inference. In [33] it is shown that the same axiomatic system (the only difference is that in [33] only one agent is considered) is sound and complete for a class of LUPP_{Meas}-models. This situation is not an exception. For example, modal system K is sound and complete with respect to the class of all modal models, but also with respect to the class of all irreflexive models [15].

We propose two topics for future work. First, we will try to prove decidability for the logic ILUPP by employing a tableau procedure. Such a method is developed in [21] for a probabilistic logic with iterations of standard probability operators. We believe that a similar tableaux method can be applied for

ILUPP. Finally, the upper and lower probabilities are just one approach in development of imprecise probability models. In future work, we also wish to logically formalize dierent imprecise probabilities.

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