Finitely stratified inductive definitions

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The theory ID_1

The classical theory ID_1 is formulated in an extension of the language of Peano arithmetic by predicate symbols $P^{\mathfrak{A}}$ for each positive arithmetical operator form $\mathcal{A}(X,x)$. Its characteristic axioms are:

$$\forall x(\mathfrak{A}(P^{\mathfrak{A}},x) \leftrightarrow P^{\mathfrak{A}}(x)) \qquad \text{(Fixed Point)}$$

$$\forall x(\mathfrak{A}(B,x) \to B(x)) \to \forall x(P^{\mathfrak{A}}(x) \to B(x)) \qquad \text{(Fixed Point Induction)}$$

Here B is any formula in the language of ID_1.

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It is well-known that $|\mathsf{ID}_1^*| = |\widehat{\mathsf{ID}}_1| = \varphi_{\varepsilon_0}(0)$.

Stratifying fixed point induction

The general idea is to consider a sequence of approximations

$$P_1^{\mathfrak{A}}, P_2^{\mathfrak{A}}, \ldots, P_n^{\mathfrak{A}},$$

where each $P_i^{\mathfrak{A}}$ with $i \leq n$ is a fixed point of $\mathfrak{A}$, and fixed point induction on $P_i^{\mathfrak{A}}$ is only allowed for formulas B containing fixed point constants $P_j^{\mathfrak{A}}$ with $j < i$.

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$$P_1^{\mathfrak{A}} \supseteq P_2^{\mathfrak{A}} \supseteq \cdots \supseteq P_n^{\mathfrak{A}}.$$

The resulting theory is called SID_n and we let $\mathsf{SID}_{<\omega}$ be the union of the systems SID_n for $n < \omega$.

Theorem

$|\mathsf{SID}_{<\omega}| = \varphi_{\varepsilon_0}(0)$.

Relation to Leivant's work

The definition of $\mathsf{SID}_{<\omega}$ bears some similarities with D. Leivant's ramified theories for finitary inductive definitions. In particular, Leivant uses a family of predicates $N_0, N_1, \dots$ satisfying the usual closure conditions for the natural numbers and the schema of complete induction in the form

$$A(0) \wedge \forall x(A(x) \to A(x')) \to (\forall x \in N_i)A(x)$$

where A only refers to predicates N_j with $j < i$.

The language $\mathcal{L}_n$ of SID_n

Definition ($\mathcal{L}_n$)

For each positive operator $\mathfrak{A}$ and $1 \leq n < \omega$ let $P_n^{\mathfrak{A}}$ denote a new and distinguished unary relation symbol. Furthermore, define for each $n < \omega$:

$$\mathcal{L}_0 := \mathcal{L}_{\mathrm{PA}} \qquad \mathcal{L}_{n+1} := \mathcal{L}_n \cup \{\, P_{n+1}^{\mathfrak{A}} : \mathfrak{A} \text{ positive operator form} \,\}$$

Further, let $\mathcal{L}_{<\omega} := \bigcup_{n<\omega} \mathcal{L}_n$.

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$$\forall x(\mathfrak{A}(B,x) \to B(x)) \to \forall x(x \in P^{\mathfrak{A}}_m \to B(x))$$

3. **Fixed-point axioms for $1 \leq m \leq n$:**

$$\forall x(\mathfrak{A}(P^{\mathfrak{A}}_m, x) \leftrightarrow x \in P^{\mathfrak{A}}_m)$$

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The theory $\mathrm{SID}_{<\omega}$ with language $\mathcal{L}_{<\omega}$ is the collection $\bigcup_{n<\omega} \mathrm{SID}_n$.

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- For each $n < \omega$, set up an infinitary proof-system SID_n^∞. For $n > 0$, we obtain a useful result on *partial cut elimination (p.c.e.)*, while for the case $n = 0$, we can even achieve *full predicative cut-elimination (f.c.e.)*.

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- *Asymmetric interpretation (a.i.)* is used to establish the connection between the systems $\mathsf{SID}_{n+1}^\infty$ and SID_n^∞ for any $n < \omega$, given that we deal with derivations where we partially removed cuts first. In particular, the symbols $P_{n+1}^{\mathfrak{A}}$ are interpreted by $Q_{\mathfrak{A}}^{<\xi}$ for suitable ξ.

Strategy for upper bound (ctd.)

- The theme is to start with a formal derivation in SID_{n+1} of an *arithmetical* formula A, embed it into $\mathsf{SID}^{\infty}_{n+1}$ such that the proof complexity stays below ε_0, combine a p.c.e. followed by an a.i. iteratively, and end up with a derivation in SID^{∞}_0 with proof complexity still below ε_0. Then f.c.e. yields the desired sharp bound $\varphi_{\varepsilon_0}(0)$ for $|\mathsf{SID}_{<\omega}|$ via a standard boundedness argument:

$$\mathsf{SID}_{n+1} \overset{\text{embed}}{\rightsquigarrow} \mathsf{SID}^{\infty}_{n+1} \overset{\text{p.c.e.}}{\rightsquigarrow} \mathsf{SID}^{\infty}_{n+1} \overset{\text{a.i.}}{\rightsquigarrow} \mathsf{SID}^{\infty}_{n} \rightsquigarrow \cdots \rightsquigarrow \mathsf{SID}^{\infty}_{0} \overset{\text{f.c.e.}}{\rightsquigarrow} \mathsf{SID}^{\infty}_{0}$$

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- In the following we also assume that $\ell \in \omega$ is a (global) bound to the length of cut formulas occurring in a given formal derivation in SID_{n+1}.

The language $\mathcal{L}_n^\infty$ of SID_n^∞

Definition ($\mathcal{L}_n^\infty$)

Let $Q_{\mathfrak{A}}^{<\xi}$ be a fresh unary relation symbol for each $\mathfrak{A}$ and ξ. For each $n < \omega$, let

$$\mathcal{L}_n^\infty := \mathcal{L}_n \cup \{ Q_{\mathfrak{A}}^{<\xi} : \xi < \Gamma_0 \ \& \ \mathfrak{A} \text{ is a positive operator form} \}$$

The infinitary system SID_n^∞

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- **Number-theoretic and logical axioms:**

$$\begin{array}{ll} \Gamma, A & \text{if } A \text{ is a true } \mathcal{L}_{\mathsf{PA}} \text{ literal without set-parameters} \\ \Gamma, A(s), \neg A(t) & \text{if } s^{\mathbb{N}} = t^{\mathbb{N}} \text{ and } A(z) \in \mathcal{L}_n \text{ is atomic} \end{array}$$

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- **Stratified induction axioms for each $1 \leq m \leq n$ and $B(z) \in \mathcal{L}_{m-1}$:**

$$\Gamma, \exists x(\mathfrak{A}(B, x) \wedge \neg B(x)), t \notin P_m^{\mathfrak{A}}, B(t)$$

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- **Fixed-point rules for $1 \leq m \leq n$:**

$$\frac{\Gamma, \mathfrak{A}(P_m^{\mathfrak{A}}, t)}{\Gamma, t \in P_m^{\mathfrak{A}}} \qquad \frac{\Gamma, \neg\mathfrak{A}(P_m^{\mathfrak{A}}, t)}{\Gamma, t \notin P_m^{\mathfrak{A}}}$$

The infinitary system SID_n^∞ (ctd.)

- **Predicative rules:**

$$\frac{\Gamma, A}{\Gamma, A \vee B} \qquad \frac{\Gamma, B}{\Gamma, A \vee B} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$\frac{\Gamma, A_x(s)}{\Gamma, \exists x A} \qquad \frac{\ldots\ \Gamma, A_x(t)\ \ldots\ (t \text{ closed term})}{\Gamma, \forall x A}$$

$$\frac{\Gamma, \mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)}{\Gamma, t \in Q_{\mathfrak{A}}^{<\tau}} \text{ for } \xi < \tau \qquad \frac{\ldots\ \Gamma, \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)\ \ldots\ (\xi < \tau)}{\Gamma, t \notin Q_{\mathfrak{A}}^{<\tau}}$$

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- **Cut rule:**

$$\frac{\Gamma, C \qquad \Gamma, \neg C}{\Gamma}$$

The n-rank of an $\mathcal{L}_n^\infty$ formula

Definition (rk_n)

Let $\mathrm{rk}_0(A) := 0$ for each $A \in \mathcal{L}_0^\infty$. For $1 \le n < \omega$, we say that $A \in \mathcal{L}_n^\infty$ is *n-atomic* if $A \in \mathcal{L}_{n-1}^\infty$ or if it is a literal of the form $t \in P_n^{\mathfrak{A}}$ or $t \notin P_n^{\mathfrak{A}}$. The *n-rank* $\mathrm{rk}_n(A) < \omega$ is defined for $1 \le n < \omega$ and formulas $A \in \mathcal{L}_n^\infty$ by

$$\mathrm{rk}_n(A) := \begin{cases} 0 & \text{if } A \text{ is } n\text{-atomic, or otherwise} \\ \max(\mathrm{rk}_n(B), \mathrm{rk}_n(C)) + 1 & \text{if } A = B \wedge C \text{ or } A = B \vee C \\ \mathrm{rk}_n(B) + 1 & \text{if } A = \forall x B \text{ or } A = \exists x B \end{cases}$$

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The ordinal rank of an $\mathcal{L}_n^\infty$ formula

Definition (rk)

The *ordinal-rank* $\mathrm{rk}(A) < \Gamma_0$ is defined for formulas $A \in \mathcal{L}_{<\omega}^\infty$ by

$$\mathrm{rk}(A) := \begin{cases} 0 & \text{if } A \text{ is a literal and } A \in \mathcal{L}_{<\omega} \\ \omega \cdot \xi & \text{if } A = t \in Q_{\mathfrak{A}}^{<\xi} \text{ or } A = t \notin Q_{\mathfrak{A}}^{<\xi} \\ \max(\mathrm{rk}(B), \mathrm{rk}(C)) + 1 & \text{if } A = B \wedge C \text{ or } A = B \vee C \\ \mathrm{rk}(B) + 1 & \text{if } A = \forall x B \text{ or } A = \exists x B \end{cases}$$

$\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha} \Gamma$

The derivability notion $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha} \Gamma$ for $n, r < \omega$ is defined inductively on α to mean that there is a SID_n^∞ proof of Γ of depth less than or equal to α so that all its cut formulas have ordinal rank less than ρ and n rank less than r.

Cut-elimination

Theorem (Cut-elimination)

1. *Partial cut-elimination:* $\mathsf{SID}_n^\infty \vdash^{\alpha}_{\rho,1+r} \Gamma$ *implies* $\mathsf{SID}_n^\infty \vdash^{\omega_r(\alpha)}_{\rho,1} \Gamma$ *for each* $1 \le n < \omega$*, where* $\omega_0(\alpha) := \alpha$ *and* $\omega_{k+1}(\alpha) := \omega_k(\omega^\alpha)$.
2. *Full predicative cut-elimination:* $\mathsf{SID}_0^\infty \vdash^{\alpha}_{\gamma+\omega^\delta,1} \Gamma$ *implies* $\mathsf{SID}_0^\infty \vdash^{\varphi_\delta(\alpha)}_{\gamma,1} \Gamma$.

Asymmetric interpretation

Definition

For $\mathcal{L}^{\infty}_{n+1}$ formulas A, $\mathcal{L}^{\infty}_{n+1}$ sequents Γ, and ordinals ξ we write

A^{ξ} — for the $\mathcal{L}^{\infty}_{n}$ formula that is obtained from A by substituting any $P^{\mathfrak{A}}_{n+1}$ that occurs in A with the corresponding symbol $Q^{<\xi}_{\mathfrak{A}}$

$[\Gamma]^{\xi}$ — for the $\mathcal{L}^{\infty}_{n}$ sequent obtained from Γ by substituting every occurring formula A with A^{ξ}

Asymmetric interpretation theorem

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For $A \in \mathcal{L}_{n+1}^{\infty}$, we write $A \in \mathrm{Pos}_{n+1}$ to denote that $P_{n+1}^{\mathfrak{A}}$ occurs at most positively in A for every $\mathfrak{A}$, and we write $A \in \mathrm{Neg}_{n+1}$ to denote $\neg A \in \mathrm{Pos}_{n+1}$.

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Theorem (Asymmetric interpretation)

Assume that we have

$$\mathsf{SID}_{n+1}^{\infty} \vdash_{\rho,1}^{\alpha} \Delta^{-}, \Delta^{+}$$

for some $\Delta^{-} \subseteq \mathrm{Neg}_{n+1}$ *and* $\Delta^{+} \subseteq \mathrm{Pos}_{n+1}$. *Let* ν *and* π *be given such that* $\pi = \nu + 2^{\alpha}$ *and* $\rho \leq \omega \cdot \pi$ *hold, then we have*

$$\mathsf{SID}_{n}^{\infty} \vdash_{\omega \cdot \pi, \ell}^{\omega \cdot \pi + \alpha} [\Delta^{-}]^{\nu}, [\Delta^{+}]^{\pi}$$

Summary and outlook to G. Jäger's talk

ordinal	stratification	iteration
$\varphi_{\varepsilon_0}(0)$	$\mathsf{SID}_{<\omega}$	$\widehat{\mathsf{ID}}_1$
$\varphi_{\varepsilon_{\varepsilon_0}}(0)$	$\mathsf{SID}_{<\omega+\omega}$	—
$\varphi_{\varphi_\omega(0)}(0)$	$\mathsf{SID}_{<\omega^\omega}$	—
$\varphi_{\varphi_{\varepsilon_0}(0)}(0)$	$\mathsf{SID}_{<\varepsilon_0}$	$\widehat{\mathsf{ID}}_2$