Explicit Evidence Systems with Common Knowledge

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Abstract. Justification logics are epistemic logics that explicitly include justifications for the agents' knowledge. We develop a multi-agent justification logic with evidence terms for individual agents as well as for common knowledge. We define a Kripke-style semantics that is similar to Fitting's semantics for the Logic of Proofs LP. We show the soundness, completeness, and finite model property of our multi-agent justification logic with respect to this Kripke-style semantics. We demonstrate that our logic is a conservative extension of Yavorskaya's minimal bimodal explicit evidence logic, which is a two-agent version of LP. We discuss the relationship of our logic to the multi-agent modal logic S4 with common knowledge. Finally, we give a brief analysis of the coordinated attack problem in the newly developed language of our logic.

1 Introduction

Justification logics [Art08] are epistemic logics that explicitly include justifications for the agents' knowledge. The first logic of this kind, the *Logic of Proofs* LP, was developed by Artemov [Art95, Art01] to provide the modal logic S4 with provability semantics. The language of justification logics has also been used to create a new approach to the logical omniscience problem [AK09] and to study self-referential proofs [Kuz10].

Instead of statements *A is known*, denoted $\Box A$, justification logics reason about justifications for knowledge by using the construct $[t]A$ to formalize statements *t is a justification for A*, where *evidence term t* can be viewed as an informal justification or a formal mathematical proof depending on the application. Evidence terms are built by means of operations that correspond to the axioms of S4, as is illustrated in Fig. 1.

Artemov [Art01] has shown that the Logic of Proofs LP is an *explicit counterpart* of the modal logic S4 in the following formal sense: each theorem of LP becomes a theorem of S4 if all terms are replaced with the modality $\Box$; and, vice versa, each theorem of S4 can be transformed into a theorem of LP if occurrences of modality are replaced with suitable evidence terms. The latter process

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S4 axioms	**LP axioms**	
$\Box(A \to B) \to (\Box A \to \Box B)$	$[t](A \to B) \to ([s]A \to [t \cdot s]B)$	(application)
$\Box A \to A$	$[t]A \to A$	(reflexivity)
$\Box A \to \Box\Box A$	$[t]A \to [!t][t]A$	(inspection)
	$[t]A \vee [s]A \to [t + s]A$	(sum)

Fig. 1. Axioms of S4 and LP

is called *realization*, and the statement of correspondence is called a *realization theorem*. Note that the operation + introduced by the sum axiom in Fig. 1 does not have a modal analog, but it is an essential part of the proof of the realization theorem in [Art01]. Explicit counterparts for many normal modal logics between K and S5 have been developed (see a recent survey in [Art08] and a uniform proof of realization theorems for all single-agent justification logics forthcoming in [BGK10]).

The notion of *common knowledge* is essential in the area of multi-agent systems, where coordination among agents is a central issue. The standard textbooks [FHMV95, MvdH95] provide excellent introductions to epistemic logics in general and common knowledge in particular. Informally, common knowledge of A is defined as the infinitary conjunction *everybody knows A and everybody knows that everybody knows A and so on.* This is equivalent to saying that common knowledge of A is the greatest fixed point of

$$\lambda X.(\text{everybody knows } A \text{ and everybody knows } X) \ . \tag{1}$$

Artemov [Art06] has created an explicit counterpart of McCarthy's *any fool knows* common knowledge modality [MSHI78], where common knowledge of A is defined as an arbitrary fixed point of (1). The relationship between the traditional common knowledge from [FHMV95, MvdH95] and McCarthy's version is studied in [Ant07].

In this paper, we present a multi-agent justification logic with evidence terms for individual agents as well as for common knowledge, with the intention to provide an explicit counterpart of the h-agent modal logic of traditional common knowledge $\mathsf{S4}_h^{\mathsf{C}}$.

Multi-agent justification logics with evidence terms for each agent have been considered in [Yav08, Ren09a, Art10], although common knowledge is not present in any of them. Artemov's interest [Art10] lies mostly in exploring a case of two agents with unequal epistemic powers, e.g., Artemov's Observer has sufficient evidence to reproduce his Object Agent's thinking, but not vice versa. Yavorskaya [Yav08] studies various operations of evidence transfer between agents. Among their systems, Yavorskaya's minimal[1] bimodal explicit evidence logic, which is an explicit counterpart of $\mathsf{S4}_2$, is the closest to our system. We will show that in the case of two agents our system is its conservative extension. Finally, Renne's system [Ren09a] combines features of modal and dynamic epistemic logics, and hence cannot be directly compared to our system.

[1] Minimality here is understood in the sense of the minimal transfer of evidence.

An epistemic semantics for LP, *F-models*, was created by Fitting in [Fit05] by augmenting Kripke models with an *evidence function* that specifies which formulae are evidenced by a term at a given world. It is easily extended to the whole family of single-agent justification logics (for details, see [Art08]). In [Art06] Artemov extends F-models to justification terms for McCarthy's common knowledge modality in the presence of several ordinary modalities, creating the most general type of epistemic models, sometimes called *AF-models*, where common evidence terms are given their own accessibility relation not directly dependent on the accessibility relations for individual modalities. Yavorskaya in [Yav08] proves a stronger completeness theorem with respect to singleton F-models, independently introduced by Mkrtychev [Mkr97] and now known as *M-models*, where the role of the accessibility relation is completely taken over by the evidence function.

The paper is organized as follows. In Sect. 2, we introduce the language and give the axiomatization of a family of multi-agent justification logics with common knowledge. In Sect. 3, we prove their basic properties including the internalization property, which is characteristic of all justification logics. In Sect. 4, we give a Fitting-style semantics similar to AF-models and prove soundness and completeness with respect to this semantics as well as with respect to singleton models, thereby demonstrating the finite model property. In Sect. 5, we show that for the two-agent case, our logic is a conservative extension of Yavorskaya's minimal bimodal explicit evidence logic. In Sect. 6, we show how our logic is related to the modal logic of traditional common knowledge and discuss the problem of realization. Finally, in Sect. 7, we provide an analysis of the coordinated attack problem in our logic.

2 Syntax

To create an explicit counterpart of the modal logic of common knowledge $\mathsf{S4}_h^{\mathsf{C}}$, we use its axiomatization via the induction axiom from [MvdH95] rather than via the induction rule to facilitate the proof of the internalization property for the resulting justification logic. We supply each agent with its own copy of terms from the Logic of Proofs, while terms for common and mutual knowledge employ additional operations. As motivated in [BKS09], a proof of $\mathsf{C}A$ can be thought of as an infinite list of proofs of the conjuncts $\mathsf{E}^m A$ in the representation of common knowledge through an infinite conjunction. To generate a finite representation of this infinite list, we use an explicit counterpart of the induction axiom

$$A \wedge [t]_{\mathsf{C}}(A \to [s]_{\mathsf{E}}A) \to [\mathsf{ind}(t,s)]_{\mathsf{C}}A$$

with a binary operation $\mathsf{ind}(\cdot,\cdot)$. To access the elements of the list, explicit counterparts of the co-closure axiom provide evidence terms that can be seen as splitting the infinite list into its head and tail,

$$[t]_{\mathsf{C}}A \to [\mathsf{ccl}_1(t)]_{\mathsf{E}}A \ , \quad [t]_{\mathsf{C}}A \to [\mathsf{ccl}_2(t)]_{\mathsf{E}}\,[t]_{\mathsf{C}}A \ ,$$

by means of two unary co-closure operations $\mathsf{ccl}_1(\cdot)$ and $\mathsf{ccl}_2(\cdot)$. Evidence terms for mutual knowledge are represented as tuples of the individual agents' evidence terms with the standard operation of tupling and with h unary projections. While only two of the three operations on LP terms are adopted for common knowledge evidence and none for mutual knowledge evidence, it will be shown in Sect. 3 that most remaining operations are definable with the notable exception of inspection for mutual knowledge.

We consider a system of h agents. Throughout the paper, i always denotes an element of $\{1,\dots,h\}$, $*$ always denotes an element of $\{1,\dots,h,\mathsf{C}\}$, and $\circledast$ always denotes an element of $\{1,\dots,h,\mathsf{E},\mathsf{C}\}$.

Let $\mathrm{Cons}_\circledast := \{c_1^\circledast, c_2^\circledast, \dots\}$ and $\mathrm{Var}_\circledast := \{x_1^\circledast, x_2^\circledast, \dots\}$ be countable sets of *proof constants* and *proof variables* respectively for each $\circledast$. The sets $\mathrm{Tm}_1, \dots, \mathrm{Tm}_h$, Tm_E, and Tm_C of *evidence terms for individual agents* and for *mutual* and *common knowledge* respectively are inductively defined as follows:

1. $\mathrm{Cons}_\circledast \subseteq \mathrm{Tm}_\circledast$;
2. $\mathrm{Var}_\circledast \subseteq \mathrm{Tm}_\circledast$;
3. $!_i t \in \mathrm{Tm}_i$ for any $t \in \mathrm{Tm}_i$;
4. $t +_* s \in \mathrm{Tm}_*$ and $t \cdot_* s \in \mathrm{Tm}_*$ for any $t, s \in \mathrm{Tm}_*$;
5. $\langle t_1, \dots, t_h\rangle \in \mathrm{Tm}_\mathsf{E}$ for any $t_1 \in \mathrm{Tm}_1, \dots, t_h \in \mathrm{Tm}_h$;
6. $\pi_i t \in \mathrm{Tm}_i$ for any $t \in \mathrm{Tm}_\mathsf{E}$;
7. $\mathsf{ccl}_1(t) \in \mathrm{Tm}_\mathsf{E}$ and $\mathsf{ccl}_2(t) \in \mathrm{Tm}_\mathsf{E}$ for any $t \in \mathrm{Tm}_\mathsf{C}$;
8. $\mathsf{ind}(t,s) \in \mathrm{Tm}_\mathsf{C}$ for any $t \in \mathrm{Tm}_\mathsf{C}$ and any $s \in \mathrm{Tm}_\mathsf{E}$.

$\mathrm{Tm} := \mathrm{Tm}_1 \cup \dots \cup \mathrm{Tm}_h \cup \mathrm{Tm}_\mathsf{E} \cup \mathrm{Tm}_\mathsf{C}$ denotes the set of all evidence terms. The indices of the operations $!$, $+$, and $\cdot$ will usually be omitted if they can be inferred from the context.

Let $\mathrm{Prop} := \{P_1, P_2, \dots\}$ be a countable set of *propositional variables. Formulae* are denoted by A, B, C, etc. and defined by the following grammar

$$A ::= P_j \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid [t]_\circledast A \ ,$$

where $t \in \mathrm{Tm}_\circledast$. The set of all formulae is denoted by $\mathrm{Fm}_{\mathsf{LP}_h^\mathsf{C}}$. We adopt the following convention: whenever a formula $[t]_\circledast A$ is used, it is assumed to be well-formed, i.e., it is implicitly assumed that term $t \in \mathrm{Tm}_\circledast$. This enables us to omit the explicit typification of terms.

Axioms of LP_h^C:

1. all propositional tautologies
2. $[t]_*(A \to B) \to ([s]_* A \to [t \cdot s]_* B)$ (application)
3. $[t]_* A \to [t+s]_* A, \quad [s]_* A \to [t+s]_* A$ (sum)
4. $[t]_i A \to A$ (reflexivity)
5. $[t]_i A \to [!t]_i\, [t]_i A$ (inspection)
6. $[t_1]_1 A \wedge \dots \wedge [t_h]_h A \to [\langle t_1, \dots, t_h\rangle]_\mathsf{E} A$ (tupling)
7. $[t]_\mathsf{E} A \to [\pi_i t]_i A$ (projection)
8. $[t]_\mathsf{C} A \to [\mathsf{ccl}_1(t)]_\mathsf{E} A, \quad [t]_\mathsf{C} A \to [\mathsf{ccl}_2(t)]_\mathsf{E}\, [t]_\mathsf{C} A$ (co-closure)
9. $A \wedge [t]_\mathsf{C}(A \to [s]_\mathsf{E} A) \to [\mathsf{ind}(t,s)]_\mathsf{C} A$ (induction)

A *constant specification* $\mathcal{CS}$ is any subset

$$\mathcal{CS} \subseteq \bigcup_{\circledast \in \{1,\dots,h,E,C\}} \left\{ [c]_\circledast A \ : \ c \in \mathrm{Cons}_\circledast \text{ and } A \text{ is an axiom of } \mathsf{LP}_h^{\mathsf{C}} \right\} \ .$$

A constant specification $\mathcal{CS}$ is called C-*axiomatically appropriate* if for each axiom A, there is a proof constant $c \in \mathrm{Cons}_{\mathsf{C}}$ such that $[c]_{\mathsf{C}} A \in \mathcal{CS}$. A constant specification $\mathcal{CS}$ is called *pure*, if $\mathcal{CS} \subseteq \{[c]_\circledast A \ : \ c \in \mathrm{Cons}_\circledast \text{ and } A \text{ is an axiom}\}$ for some fixed $\circledast$, i.e., if for all $[c]_\circledast A \in \mathcal{CS}$, the constants c are of the same type.

Let $\mathcal{CS}$ be a constant specification. The deductive system $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS})$ is the Hilbert system given by the axioms of $\mathsf{LP}_h^{\mathsf{C}}$ above and rules modus ponens and axiom necessitation:

$$\frac{A \quad A \to B}{B} \ , \qquad\qquad \frac{}{[c]_\circledast A} \ , \text{ where } [c]_\circledast A \in \mathcal{CS}.$$

By $\mathsf{LP}_h^{\mathsf{C}}$ we denote the system $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS})$ with

$$\mathcal{CS} = \left\{ [c]_{\mathsf{C}} A \ : \ c \in \mathrm{Cons}_{\mathsf{C}} \text{ and } A \text{ is an axiom of } \mathsf{LP}_h^{\mathsf{C}} \right\} \ . \tag{2}$$

For an arbitrary $\mathcal{CS}$, we write $\Delta \vdash_{\mathcal{CS}} A$ to state that A is derivable from Δ in $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS})$ and omit the mention of $\mathcal{CS}$ when working with the constant specification from (2) by writing $\Delta \vdash A$. We use Δ, A to mean $\Delta \cup \{A\}$.

3 Basic Properties

In this section, we show that our logic possesses the standard properties expected of any justification logic. In addition, we show that the operations on terms introduced in the previous section are sufficient to express the operations of sum and application for mutual knowledge evidence and the operation of inspection for common knowledge evidence. This is the reason why $+_{\mathsf{E}}$, $\cdot_{\mathsf{E}}$, and $!_{\mathsf{C}}$ are not primitive connectives in the language. It should be noted that no inspection operation for mutual evidence terms can be defined, which follows from Lemma 27 in Sect. 6 and the fact that $\mathsf{E}A \to \mathsf{EE}A$ is not a valid modal formula.

We begin with the following observation:

Lemma 1. *For any constant specification $\mathcal{CS}$ and any formulae A and B:*
1. $\vdash_{\mathcal{CS}} [t]_{\mathsf{E}} A \to A$ *for all* $t \in \mathrm{Tm}_{\mathsf{E}}$*;* ($\mathsf{E}$-*reflexivity*)
2. *for any* $t, s \in \mathrm{Tm}_{\mathsf{E}}$*, there is a term* $t \cdot_{\mathsf{E}} s \in \mathrm{Tm}_{\mathsf{E}}$ *such that* $\vdash_{\mathcal{CS}} [t]_{\mathsf{E}}(A \to B) \to ([s]_{\mathsf{E}} A \to [t \cdot_{\mathsf{E}} s]_{\mathsf{E}} B)$*;* ($\mathsf{E}$-*application*)
3. *for any* $t, s \in \mathrm{Tm}_{\mathsf{E}}$*, there is a term* $t +_{\mathsf{E}} s \in \mathrm{Tm}_{\mathsf{E}}$ *such that* $\vdash_{\mathcal{CS}} [t]_{\mathsf{E}} A \to [t +_{\mathsf{E}} s]_{\mathsf{E}} A$ *and* $\vdash_{\mathcal{CS}} [s]_{\mathsf{E}} A \to [t +_{\mathsf{E}} s]_{\mathsf{E}} A$*;* ($\mathsf{E}$-*sum*)
4. *for any* $t \in \mathrm{Tm}_{\mathsf{C}}$ *and any* $i \in \{1, \dots, h\}$*, there is a term* $\downarrow_i t \in \mathrm{Tm}_i$ *such that* $\vdash_{\mathcal{CS}} [t]_{\mathsf{C}} A \to [\downarrow_i t]_i A$*;* ($i$-*conversion*)
5. $\vdash_{\mathcal{CS}} [t]_{\mathsf{C}} A \to A$ *for all* $t \in \mathrm{Tm}_{\mathsf{C}}$*.* ($\mathsf{C}$-*reflexivity*)

Proof. 1. Immediate by the projection and reflexivity axioms.

2. Set $t \cdot_{\mathsf{E}} s := \langle \pi_1 t \cdot_1 \pi_1 s, \ldots, \pi_h t \cdot_h \pi_h s \rangle$.
3. Set $t +_{\mathsf{E}} s := \langle \pi_1 t +_1 \pi_1 s, \ldots, \pi_h t +_h \pi_h s \rangle$.
4. Set $\downarrow_i t := \pi_i \mathsf{ccl}_1(t)$.
5. Immediate by 4. and the reflexivity axiom. □

Unlike Lemma 1, the next lemma requires that a constant specification $\mathcal{CS}$ be C-axiomatically appropriate.

Lemma 2. *Let $\mathcal{CS}$ be C-axiomatically appropriate and A be a formula.*
1. *For any $t \in \mathrm{Tm}_{\mathsf{C}}$, there is a term $!_{\mathsf{C}} t \in \mathrm{Tm}_{\mathsf{C}}$ such that* $\vdash_{\mathcal{CS}} [t]_{\mathsf{C}} A \to [!_{\mathsf{C}} t]_{\mathsf{C}} \, [t]_{\mathsf{C}} A$. ($\mathsf{C}$-*inspection*)
2. *For any $t \in \mathrm{Tm}_{\mathsf{C}}$, there is a term $\Lleftarrow t \in \mathrm{Tm}_{\mathsf{C}}$ such that* $\vdash_{\mathcal{CS}} [t]_{\mathsf{C}} A \to [\Lleftarrow t]_{\mathsf{C}} \, [\mathsf{ccl}_1(t)]_{\mathsf{E}} A$. ($\mathsf{C}$-*shift*)

Proof. 1. Set $!_{\mathsf{C}} t := \mathsf{ind}(c, \mathsf{ccl}_2(t))$, where $[c]_{\mathsf{C}}([t]_{\mathsf{C}} A \to [\mathsf{ccl}_2(t)]_{\mathsf{E}} \, [t]_{\mathsf{C}} A) \in \mathcal{CS}$.
2. Set $\Lleftarrow t := c \cdot_{\mathsf{C}} (!_{\mathsf{C}} t)$, where $[c]_{\mathsf{C}}([t]_{\mathsf{C}} A \to [\mathsf{ccl}_1(t)]_{\mathsf{E}} A) \in \mathcal{CS}$. □

The following two theorems are standard in justification logics. Their proofs can be taken almost word for word from [Art01] and are, therefore, omitted here.

Lemma 3 (Deduction Theorem). *Let $\mathcal{CS}$ be a constant specification and $\Delta \cup \{A, B\} \subseteq \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}$. Then $\Delta, A \vdash_{\mathcal{CS}} B$ if and only if $\Delta \vdash_{\mathcal{CS}} A \to B$.*

Lemma 4 (Substitution). *For any constant specification $\mathcal{CS}$, any propositional variable P, any $\Delta \cup \{A, B\} \subseteq \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}$, any $x \in \mathrm{Var}_{\circledast}$, and any $t \in \mathrm{Tm}_{\circledast}$,*

$$\text{if } \Delta \vdash_{\mathcal{CS}} A, \text{ then } \Delta(x/t, P/B) \vdash_{\mathcal{CS}(x/t,P/B)} A(x/t, P/B) \ ,$$

where $A(x/t, P/B)$ denotes the formula obtained by simultaneously replacing all occurrences of x in A with t and all occurrences of P in A with B, accordingly for $\Delta(x/t, P/B)$ and $\mathcal{CS}(x/t, P/B)$.

The following lemma states that our logic can internalize its own proofs, which is an important property of justification logics.

Lemma 5 (C-lifting). *Let $\mathcal{CS}$ be a pure C-axiomatically appropriate constant specification. If*

$$[s_1]_{\mathsf{C}} B_1, \ldots, [s_n]_{\mathsf{C}} B_n, C_1, \ldots, C_m \vdash_{\mathcal{CS}} A \ ,$$

then for each $\circledast$, there is a term $t_{\circledast}(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathrm{Tm}_{\circledast}$ such that

$$[s_1]_{\mathsf{C}} B_1, \ldots, [s_n]_{\mathsf{C}} B_n, [y_1]_{\circledast} C_1, \ldots, [y_m]_{\circledast} C_m \vdash_{\mathcal{CS}} [t_{\circledast}(s_1, \ldots, s_n, y_1, \ldots, y_m)]_{\circledast} A$$

for fresh variables $y_1, \ldots, y_m \in \mathrm{Tm}_{\circledast}$.

Proof. We proceed by induction on the derivation of A.

If A is an axiom, there is a constant $c \in \mathrm{Tm}_{\mathsf{C}}$ such that $[c]_{\mathsf{C}} A \in \mathcal{CS}$ because $\mathcal{CS}$ is C-axiomatically appropriate. Then take

$$t_{\mathsf{C}} := c, \qquad t_i := \downarrow_i c, \qquad t_{\mathsf{E}} := \mathsf{ccl}_1(c)$$

and use axiom necessitation, axiom necessitation and i-conversion, or axiom necessitation and the co-closure axiom respectively.

For $A = [s_j]_{\mathsf{C}} B_j$, $1 \le j \le n$, take

$$t_{\mathsf{C}} := !_{\mathsf{C}} s_j, \qquad t_i := \downarrow_i !_{\mathsf{C}} s_j, \qquad t_{\mathsf{E}} := \mathsf{ccl}_2(s_j)$$

and use C-inspection, C-inspection and i-conversion, or the co-closure axiom respectively.

For $A = C_j$, $1 \le j \le m$, take $t_{\circledast} := y_j \in \mathrm{Var}_{\circledast}$ for a fresh variable y_j.

For A derived by modus ponens from $D \to A$ and D, by induction hypothesis there are terms $r_{\circledast}, s_{\circledast} \in \mathrm{Tm}_{\circledast}$ such that $[r_{\circledast}]_{\circledast}(D \to A)$ and $[s_{\circledast}]_{\circledast} D$ are provable. Take $t_{\circledast} := r_{\circledast} \cdot_{\circledast} s_{\circledast}$ and use $\circledast$-application, which is an axiom for $\circledast = i$ and for $\circledast = \mathsf{C}$ or follows from Lemma 1 for $\circledast = \mathsf{E}$.

For $A = [c]_{\mathsf{C}} E \in \mathcal{CS}$ derived by axiom necessitation, take

$$t_{\mathsf{C}} := !_{\mathsf{C}} c, \qquad t_i := \downarrow_i !_{\mathsf{C}} c, \qquad t_{\mathsf{E}} := \mathsf{ccl}_2(c)$$

and use C-inspection, C-inspection and i-conversion, or the co-closure axiom respectively. □

Corollary 6 (Constructive necessitation). *Let $\mathcal{CS}$ be a pure C-axiomatically appropriate constant specification. For any formula A, if $\vdash_{\mathcal{CS}} A$, then for each $\circledast$, there is a ground term $t \in \mathrm{Tm}_{\circledast}$ such that $\vdash_{\mathcal{CS}} [t]_{\circledast} A$.*

The following two lemmas show that our system $\mathsf{LP}_h^{\mathsf{C}}$ can internalize versions of the induction rule used in various axiomatizations of $\mathsf{S4}_h^{\mathsf{C}}$ (see [BKS09] for a discussion of several axiomatizations of this kind).

Lemma 7 (Internalized induction rule 1). *Let $\mathcal{CS}$ be a pure C-axiomatically appropriate constant specification. For any formula A, if $\vdash_{\mathcal{CS}} A \to [s]_{\mathsf{E}} A$, there is a term $t \in \mathrm{Tm}_{\mathsf{C}}$ such that $\vdash_{\mathcal{CS}} A \to [\mathsf{ind}(t,s)]_{\mathsf{C}} A$.*

Proof. By constructive necessitation, there exists a term $t \in \mathrm{Tm}_{\mathsf{C}}$ such that $\vdash_{\mathcal{CS}} [t]_{\mathsf{C}}(A \to [s]_{\mathsf{E}} A)$. It remains to use the induction axiom and propositional reasoning. □

Lemma 8 (Internalized induction rule 2). *Let $\mathcal{CS}$ be a pure C-axiomatically appropriate constant specification. For any formulae A and B, if we have $\vdash_{\mathcal{CS}} B \to [s]_{\mathsf{E}}(A \wedge B)$, then there exist a term $t \in \mathrm{Tm}_{\mathsf{C}}$ and a constant $c \in \mathrm{Tm}_{\mathsf{C}}$ such that $\vdash_{\mathcal{CS}} B \to [c \cdot \mathsf{ind}(t,s)]_{\mathsf{C}} A$, where $[c]_{\mathsf{C}}(A \wedge B \to A) \in \mathcal{CS}$.*

Proof. Assume

$$\vdash_{\mathcal{CS}} B \to [s]_{\mathsf{E}}(A \wedge B) \ . \tag{3}$$

From this we immediately get $\vdash_{\mathcal{CS}} A \wedge B \to [s]_{\mathsf{E}}(A \wedge B)$. Thus, by Lemma 7, there is a $t \in \mathrm{Tm}_{\mathsf{C}}$ with

$$\vdash_{\mathcal{CS}} A \wedge B \to [\mathsf{ind}(t,s)]_{\mathsf{C}}(A \wedge B) \ . \tag{4}$$

Since $\mathcal{CS}$ is C-axiomatically appropriate, there is a constant $c \in \mathrm{Tm}_{\mathsf{C}}$ such that

$$\vdash_{\mathcal{CS}} [c]_{\mathsf{C}}(A \wedge B \to A) \ . \tag{5}$$

Making use of C-application, we find by (4) and (5) that

$$\vdash_{\mathcal{CS}} A \wedge B \to [c \cdot \mathsf{ind}(t,s)]_{\mathsf{C}}(A) \ . \tag{6}$$

From (3) we get by E-reflexivity that $\vdash_{\mathcal{CS}} B \to A \wedge B$. This, together with (6), finally yields $\vdash_{\mathcal{CS}} B \to [c \cdot \mathsf{ind}(t,s)]_{\mathsf{C}}(A)$. □

4 Soundness and Completeness

Definition 9. *An* AF-model meeting a constant specification $\mathcal{CS}$ *is a structure* $\mathcal{M} = (W, R, \mathcal{E}, \nu)$, *where* (W, R, ν) *is a Kripke model for* $\mathsf{S4}_h$ *with a* set of possible worlds $W \neq \varnothing$, *with a function* $R \colon \{1, \ldots, h\} \to \mathcal{P}(W \times W)$ *that assigns a reflexive and transitive* accessibility relation *on* W *to each agent* $i \in \{1, \ldots, h\}$, *and with a* truth valuation $\nu \colon \mathrm{Prop} \to \mathcal{P}(W)$. *We always write* R_i *instead of* $R(i)$ *and define the accessibility relations for mutual and common knowledge in the standard way:* $R_{\mathsf{E}} := R_1 \cup \cdots \cup R_h$ *and* $R_{\mathsf{C}} := \bigcup_{n=1}^{\infty} (R_{\mathsf{E}})^n$.

An evidence function $\mathcal{E} \colon W \times \mathrm{Tm} \to \mathcal{P}\left(\mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}\right)$ *determines the formulae evidenced by a term at a world. We define* $\mathcal{E}_{\circledast} := \mathcal{E} \restriction (W \times \mathrm{Tm}_{\circledast})$. *Note that whenever* $A \in \mathcal{E}_{\circledast}(w, t)$, *it follows that* $t \in \mathrm{Tm}_{\circledast}$. *The evidence function* $\mathcal{E}$ *must satisfy the following closure conditions: for any worlds* $w, v \in W$,

1. $\mathcal{E}_*(w,t) \subseteq \mathcal{E}_*(v,t)$ *whenever* $(w,v) \in R_*$; (*monotonicity*)
2. *if* $[c]_{\circledast} A \in \mathcal{CS}$, *then* $A \in \mathcal{E}_{\circledast}(w,c)$; (*constant specification*)
3. *if* $(A \to B) \in \mathcal{E}_*(w,t)$ *and* $A \in \mathcal{E}_*(w,s)$, *then* $B \in \mathcal{E}_*(w, t \cdot s)$; (*application*)
4. $\mathcal{E}_*(w,s) \cup \mathcal{E}_*(w,t) \subseteq \mathcal{E}_*(w, s+t)$; (*sum*)
5. *if* $A \in \mathcal{E}_i(w,t)$, *then* $[t]_i A \in \mathcal{E}_i(w, !t)$; (*inspection*)
6. *if* $A \in \mathcal{E}_i(w, t_i)$ *for all* $1 \leq i \leq h$, *then* $A \in \mathcal{E}_{\mathsf{E}}(w, \langle t_1, \ldots, t_h \rangle)$; (*tupling*)
7. *if* $A \in \mathcal{E}_{\mathsf{E}}(w,t)$, *then* $A \in \mathcal{E}_i(w, \pi_i t)$; (*projection*)
8. *if* $A \in \mathcal{E}_{\mathsf{C}}(w,t)$, *then* $A \in \mathcal{E}_{\mathsf{E}}(w, \mathsf{ccl}_1(t))$ *and* $[t]_{\mathsf{C}} A \in \mathcal{E}_{\mathsf{E}}(w, \mathsf{ccl}_2(t))$; (*co-closure*)
9. *if* $A \in \mathcal{E}_{\mathsf{E}}(w,s)$ *and* $(A \to [s]_{\mathsf{E}} A) \in \mathcal{E}_{\mathsf{C}}(w,t)$,
   then $A \in \mathcal{E}_{\mathsf{C}}(w, \mathsf{ind}(t,s))$. (*induction*)

When the model is clear from the context, we will directly refer to $R_1, \ldots, R_h$, R_{E}, R_{C}, $\mathcal{E}_1, \ldots, \mathcal{E}_h$, $\mathcal{E}_{\mathsf{E}}$, $\mathcal{E}_{\mathsf{C}}$, W, and ν.

Definition 10. *A ternary relation* $\mathcal{M}, w \Vdash A$ *for* formula A being satisfied at a world $w \in W$ in an AF-model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ *is defined by induction on the structure of the formula* A:

1. $\mathcal{M}, w \Vdash P$ *if and only if* $w \in \nu(P)$;
2. $\Vdash$ *behaves classically with respect to the propositional connectives;*
3. $\mathcal{M}, w \Vdash [t]_{\circledast} A$ *if and only if 1)* $A \in \mathcal{E}_{\circledast}(w,t)$ *and 2)* $\mathcal{M}, v \Vdash A$ *for all* $v \in W$ *with* $(w,v) \in R_{\circledast}$.

We write $\mathcal{M} \Vdash A$ if $\mathcal{M}, w \Vdash A$ for all $w \in W$. We write $\Vdash_{\mathcal{CS}} A$ and say that formula A is valid with respect to $\mathcal{CS}$ *if $\mathcal{M} \Vdash A$ for all AF-models $\mathcal{M}$ meeting $\mathcal{CS}$.*

Lemma 11 (Soundness). *Provable formulae are valid: $\vdash_{\mathcal{CS}} A$ implies $\Vdash_{\mathcal{CS}} A$.*

Proof. Let $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ be an AF-model meeting $\mathcal{CS}$ and let $w \in W$. We show soundness by induction on the derivation of A. The cases for propositional tautologies, for the application, sum, reflexivity, and inspection axioms, and for modus ponens rule are the same as for the single-agent case in [Fit05] and are, therefore, omitted. We show the remaining five cases:

(tupling) Assume $\mathcal{M}, w \Vdash [t_i]_i A$ for all $1 \le i \le h$. Then for all $1 \le i \le h$, we have 1) $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_i$ and 2) $A \in \mathcal{E}_i(w, t_i)$. So, by the tupling closure condition, $A \in \mathcal{E}_{\mathsf{E}}(w, \langle t_1, \ldots, t_h \rangle)$ from 2). Since by definition $R_{\mathsf{E}} = \bigcup_{i=1}^{h} R_i$, it follows from 1) that $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_{\mathsf{E}}$. Hence, $\mathcal{M}, w \Vdash [\langle t_1, \ldots, t_h \rangle]_{\mathsf{E}} A$.

(projection) Assume $\mathcal{M}, w \Vdash [t]_{\mathsf{E}} A$. Then 1) $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_{\mathsf{E}}$ and 2) $A \in \mathcal{E}_{\mathsf{E}}(w, t)$. By the projection closure condition, it follows from 2) that $A \in \mathcal{E}_i(w, \pi_i t)$. In addition, since $R_{\mathsf{E}} = \bigcup_{i=1}^{h} R_i$, we get $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_i$ by 1). Thus, $\mathcal{M}, w \Vdash [\pi_i t]_i A$.

(co-closure) Assume $\mathcal{M}, w \Vdash [t]_{\mathsf{C}} A$. Then 1) $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_{\mathsf{C}}$ and 2) $A \in \mathcal{E}_{\mathsf{C}}(w, t)$. It follows from 1) that for all $v' \in W$ with $(w, v') \in R_{\mathsf{E}}$, we have $\mathcal{M}, v' \Vdash A$ since $R_{\mathsf{E}} \subseteq R_{\mathsf{C}}$; also, due to the monotonicity closure condition, $\mathcal{M}, v' \Vdash [t]_{\mathsf{C}} A$ since $R_{\mathsf{E}} \circ R_{\mathsf{C}} \subseteq R_{\mathsf{C}}$. From 2), by the co-closure closure condition, $A \in \mathcal{E}_{\mathsf{E}}(w, \mathsf{ccl}_1(t))$ and $[t]_{\mathsf{C}} A \in \mathcal{E}_{\mathsf{E}}(w, \mathsf{ccl}_2(t))$. Hence, $\mathcal{M}, w \Vdash [\mathsf{ccl}_1(t)]_{\mathsf{E}} A$ and $\mathcal{M}, w \Vdash [\mathsf{ccl}_2(t)]_{\mathsf{E}}\, [t]_{\mathsf{C}} A$.

(induction) Assume $\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash [t]_{\mathsf{C}}(A \to [s]_{\mathsf{E}} A)$. From the second assumption and the reflexivity of R_{C}, we get $\mathcal{M}, w \Vdash A \to [s]_{\mathsf{E}} A$; thus, $\mathcal{M}, w \Vdash [s]_{\mathsf{E}} A$ by the first assumption. So $A \in \mathcal{E}_{\mathsf{E}}(w, s)$ and, by the second assumption, $A \to [s]_{\mathsf{E}} A \in \mathcal{E}_{\mathsf{C}}(w, t)$. By the induction closure condition, we have $A \in \mathcal{E}_{\mathsf{C}}(w, \mathsf{ind}(t, s))$. To show $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in R_{\mathsf{C}}$, we prove that $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $(w, v) \in (R_{\mathsf{E}})^n$ by induction on the positive integer n.

The **base case** $n = 1$ immediately follows from $\mathcal{M}, w \Vdash [s]_{\mathsf{E}} A$.

Induction step. Let $(w, v') \in (R_{\mathsf{E}})^n$ and $(v', v) \in R_{\mathsf{E}}$ for some $v, v' \in W$. By induction hypothesis, $\mathcal{M}, v' \Vdash A$. Since $\mathcal{M}, w \Vdash [t]_{\mathsf{C}}(A \to [s]_{\mathsf{E}} A)$, we get $\mathcal{M}, v' \Vdash A \to [s]_{\mathsf{E}} A$. Thus, $\mathcal{M}, v' \Vdash [s]_{\mathsf{E}} A$, which yields $\mathcal{M}, v \Vdash A$.

Finally, we conclude that $\mathcal{M}, w \Vdash [\mathsf{ind}(t, s)]_{\mathsf{C}} A$.

(axiom necessitation) Let A be an axiom and c be a proof constant such that $[c]_{\circledast} A \in \mathcal{CS}$. Since A is an axiom, $\mathcal{M}, w \Vdash A$ for all $w \in W$, as shown above. Since $\mathcal{M}$ is an AF-model meeting $\mathcal{CS}$, we also have $A \in \mathcal{E}_{\circledast}(w, c)$ for all $w \in W$ by the constant specification closure condition. Thus, $\mathcal{M}, w \Vdash [c]_{\circledast} A$ for all $w \in W$. □

Definition 12. *Let $\mathcal{CS}$ be a constant specification. A set Φ of formulae is called* $\mathcal{CS}$-consistent *if $\Phi \nvdash_{\mathcal{CS}} \phi$ for some formula ϕ. A set Φ is called* maximal $\mathcal{CS}$-consistent *if it is $\mathcal{CS}$-consistent and has no $\mathcal{CS}$-consistent proper extensions.*

Whenever safe, we do not mention the constant specification and only talk about consistent and maximal consistent sets. It can be easily shown that maximal consistent sets contain all axioms of $\mathsf{LP}_h^{\mathsf{C}}$ and are closed under modus ponens.

Definition 13. *For a set Φ of formulae, we define*

$$\Phi/\circledast := \{A \ : \ \textit{there is a } t \in \mathrm{Tm}_{\circledast} \textit{ such that } [t]_{\circledast} A \in \Phi\} \ .$$

Definition 14. *Let $\mathcal{CS}$ be a constant specification. The* canonical AF-model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ meeting $\mathcal{CS}$ *is defined as follows:*

1. $W := \{w \subseteq \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}} \ : \ w$ *is a maximal $\mathcal{CS}$-consistent set*$\}$;

2. $R_i := \{(w, v) \in W \times W \ : \ w/i \subseteq v\}$;

3. $\mathcal{E}_{\circledast}(w, t) := \{A \in \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}} \ : \ [t]_{\circledast} A \in w\}$;

4. $\nu(P_n) := \{w \in W \ : \ P_n \in w\}$.

Lemma 15. *Let $\mathcal{CS}$ be a constant specification. The canonical AF-model meeting $\mathcal{CS}$ is an AF-model meeting $\mathcal{CS}$.*

Proof. The proof of reflexivity and transitivity of each R_i, as well as the argument for the constant specification, application, sum, and inspection closure conditions, is the same as in the single-agent case (see [Fit05]). We show the remaining five closure conditions:

(tupling) Assume $A \in \mathcal{E}_i(w, t_i)$ for all $1 \le i \le h$. By definition of $\mathcal{E}_i$, we have $[t_i]_i A \in w$ for all $1 \le i \le h$. Therefore, by the tupling axiom and maximal consistency, $[\langle t_1, \dots, t_h \rangle]_{\mathsf{E}} A \in w$. Thus, $A \in \mathcal{E}_{\mathsf{E}}(w, \langle t_1, \dots, t_h \rangle)$.

(projection) Assume $A \in \mathcal{E}_{\mathsf{E}}(w, t)$. Thus, we have $[t]_{\mathsf{E}} A \in w$. Then, by the projection axiom and maximal consistency, $[\pi_i t]_i A \in w$, and thus $A \in \mathcal{E}_i(w, \pi_i t)$.

(co-closure) Assume $A \in \mathcal{E}_{\mathsf{C}}(w, t)$. Thus, $[t]_{\mathsf{C}} A \in w$, and, by the co-closure axioms and maximal consistency, $[\mathsf{ccl}_1(t)]_{\mathsf{E}} A \in w$ and $[\mathsf{ccl}_2(t)]_{\mathsf{E}} [t]_{\mathsf{C}} A \in w$. Hence, $A \in \mathcal{E}_{\mathsf{E}}(w, \mathsf{ccl}_1(t))$ and $[t]_{\mathsf{C}} A \in \mathcal{E}_{\mathsf{E}}(w, \mathsf{ccl}_2(t))$.

(induction) Assume $A \in \mathcal{E}_{\mathsf{E}}(w, s)$ and $(A \to [s]_{\mathsf{E}} A) \in \mathcal{E}_{\mathsf{C}}(w, t)$. Then we have $[s]_{\mathsf{E}} A \in w$ and $[t]_{\mathsf{C}}(A \to [s]_{\mathsf{E}} A) \in w$. From $\vdash_{\mathcal{CS}} [s]_{\mathsf{E}} A \to A$ (Lemma 1.1) and the induction axiom, it follows by maximal consistency that $A \in w$ and $[\mathsf{ind}(t, s)]_{\mathsf{C}} A \in w$. Therefore, $A \in \mathcal{E}_{\mathsf{C}}(w, \mathsf{ind}(t, s))$.

(monotonicity) We show only the case of $* = \mathsf{C}$ since the other cases are the same as in [Fit05]. It is sufficient to prove by induction on the positive integer n that

$$\text{if } [t]_{\mathsf{C}} A \in w \text{ and } (w, v) \in (R_{\mathsf{E}})^n, \text{ then } [t]_C A \in v \ . \tag{7}$$

Base case $n = 1$. Assume $(w, v) \in R_{\mathsf{E}}$, i.e., $w/i \subseteq v$ for some i. As $[t]_{\mathsf{C}} A \in w$, $[\pi_i \mathsf{ccl}_2(t)]_i [t]_{\mathsf{C}} A \in w$ by maximal consistency, and hence $[t]_{\mathsf{C}} A \in w/i \subseteq v$. The argument for the **induction step** is similar.

Now assume $(w, v) \in R_{\mathsf{C}} = \bigcup_{n=1}^{\infty} (R_{\mathsf{E}})^n$ and $A \in \mathcal{E}_{\mathsf{C}}(w, t)$, i.e., $[t]_{\mathsf{C}} A \in w$. As shown above, $[t]_{\mathsf{C}} A \in v$. Thus, $A \in \mathcal{E}_{\mathsf{C}}(v, t)$. ☐

Remark 16. Let R'_{C} denote the binary relation on W given by

$$(w, v) \in R'_{\mathsf{C}} \quad \text{if and only if} \quad w/\mathsf{C} \subseteq v \ .$$

An argument similar to the one just used for monotonicity shows that $R_{\mathsf{C}} \subseteq R'_{\mathsf{C}}$. However, the converse does not hold for any pure C-axiomatically appropriate constant specification $\mathcal{CS}$, which we demonstrate by adapting an example from [MvdH95]. Let

$$\Phi := \{[s_n]_{\mathsf{E}} \dots [s_1]_{\mathsf{E}} P \ : \ n \geq 1,\, s_1, \dots, s_n \in \mathrm{Tm}_{\mathsf{E}}\} \cup \{\neg\, [t]_{\mathsf{C}} P \ : \ t \in \mathrm{Tm}_{\mathsf{C}}\} \ .$$

This set is $\mathcal{CS}$-consistent for any $P \in \mathrm{Prop}$.

To see this, let $\Phi' \subseteq \Phi$ be finite and let m denote the maximal number of terms such that $[s_m]_{\mathsf{E}} \dots [s_1]_{\mathsf{E}} P \in \Phi'$. Define the model $\mathcal{N} := (\mathbb{N}, R^{\mathcal{N}}, \mathcal{E}^{\mathcal{N}}, \nu^{\mathcal{N}})$ by

- $R^{\mathcal{N}}_i := \{(n, n+1) \in \mathbb{N}^2 \ : \ n \mod h = i\} \cup \{(n, n) \in \mathbb{N}^2 \ : \ n \in \mathbb{N}\}$;
- $\mathcal{E}^{\mathcal{N}}(n, s) := \mathrm{Fm}_{\mathsf{LP}^{\mathsf{C}}_h}$ for all $n \in \mathbb{N}$ and terms $s \in \mathrm{Tm}$;
- $\nu^{\mathcal{N}}(P) := \{1, 2, \dots, m+1\} \subseteq \mathbb{N}$.

Clearly, $\mathcal{N}$ meets any constant specification; in particular, it meets $\mathcal{CS}$. It can also be easily verified that $\mathcal{N}, 1 \Vdash \Phi'$; therefore, Φ' is $\mathcal{CS}$-consistent.

Since Φ is $\mathcal{CS}$-consistent, there exists a maximal $\mathcal{CS}$-consistent set $w \supseteq \Phi$. Let us show that the set $\Psi := \{\neg P\} \cup (w/\mathsf{C})$ is also $\mathcal{CS}$-consistent. Indeed, if it were not the case, there would exist formulae $B_1, \dots, B_n \in w/\mathsf{C}$ such that

$$\vdash_{\mathcal{CS}} B_1 \to (B_2 \to \dots \to (B_n \to P) \dots) \ .$$

Then, by Corollary 6, there would exist a term $s \in \mathrm{Tm}_{\mathsf{C}}$ such that

$$\vdash_{\mathcal{CS}} [s]_{\mathsf{C}}(B_1 \to (B_2 \to \dots \to (B_n \to P) \dots)) \ .$$

But this would imply $[(\dots(s \cdot t_1) \cdots t_{n-1}) \cdot t_n]_{\mathsf{C}} P \in w$ for $[t_j]_{\mathsf{C}} B_j \in w$, $1 \leq j \leq n$, a contradiction with the consistency of w.

Let v be a maximal $\mathcal{CS}$-consistent set that contains Ψ, i.e., $v \supseteq \Psi$. Clearly, $w/\mathsf{C} \subseteq v$, i.e., $(w, v) \in R'_{\mathsf{C}}$, but $(w, v) \notin R_{\mathsf{C}}$ because this would imply $P \in v$, which cannot happen. It follows that $R_{\mathsf{C}} \subsetneq R'_{\mathsf{C}}$.

Similarly, we can define R'_{E} by $(w, v) \in R'_{\mathsf{E}}$ if and only if $w/\mathsf{E} \subseteq v$. However, $R'_{\mathsf{E}} = R_{\mathsf{E}}$ for any C-axiomatically appropriate constant specification $\mathcal{CS}$. Indeed, is easy to show that $R_{\mathsf{E}} \subseteq R'_{\mathsf{E}}$. For the converse, assume $(w, v) \notin R_{\mathsf{E}}$, then $(w, v) \notin R_i$ for all $1 \leq i \leq h$. So there are formulae $A_1, \dots, A_h$ such that $[t_i]_i A_i \in w$ for some $t_i \in \mathrm{Tm}_i$, but $A_i \notin v$. Now let $[c_i]_{\mathsf{C}}(A_i \to A_1 \vee \dots \vee A_h) \in \mathcal{CS}$ for constants $c_1, \dots, c_h$. Then $[\downarrow_i c_i \cdot t_i]_i (A_1 \vee \dots \vee A_h) \in w$ for all $1 \leq i \leq h$, so $[\langle \downarrow_1 c_1 \cdot t_1, \dots, \downarrow_h c_h \cdot t_h \rangle]_{\mathsf{E}} (A_1 \vee \dots \vee A_h) \in w$. However, $A_i \notin v$ for any $1 \leq i \leq h$; therefore, by the maximal consistency of v, $A_1 \vee \dots \vee A_h \notin v$ either. Hence, $w/\mathsf{E} \not\subseteq v$, so $(w, v) \notin R'_{\mathsf{E}}$.

Lemma 17 (Truth Lemma). *Let $\mathcal{CS}$ be a constant specification and $\mathcal{M}$ be the canonical AF-model meeting $\mathcal{CS}$. For all formulae A and all worlds $w \in W$,*

$$A \in w \text{ if and only if } \mathcal{M}, w \Vdash A \ .$$

Proof. The proof is by induction on the structure of A. The cases for propositional variables and propositional connectives are immediate by the definition of $\Vdash$ and by the maximal consistency of w. We check the remaining cases:
Case A is $[t]_i B$. Assume $A \in w$. Then $B \in w/i$ and $B \in \mathcal{E}_i(w,t)$. Consider any v such that $(w,v) \in R_i$. Since $w/i \subseteq v$, it follows that $B \in v$, and thus, by induction hypothesis, $\mathcal{M}, v \Vdash B$. And $\mathcal{M}, w \Vdash A$ immediately follows from this.

For the converse, assume $\mathcal{M}, w \Vdash [t]_i B$. By definition of $\Vdash$ we get $B \in \mathcal{E}_i(w,t)$, from which $[t]_i B \in w$ immediately follows by definition of $\mathcal{E}_i$.
Case A is $[t]_{\mathsf{E}} B$. Assume $A \in w$ and consider any v such that $(w,v) \in R_{\mathsf{E}}$. Then $(w,v) \in R_i$ for some $1 \leq i \leq h$, i.e., $w/i \subseteq v$. By definition of $\mathcal{E}_{\mathsf{E}}$, we get $B \in \mathcal{E}_{\mathsf{E}}(w,t)$. By maximal consistency of w, it follows that $[\pi_i t]_i B \in w$, and thus $B \in w/i \subseteq v$. Since, by induction hypothesis, $\mathcal{M}, v \Vdash B$, we conclude that $\mathcal{M}, w \Vdash A$. The argument for the converse repeats the one from the previous case.
Case A is $[t]_{\mathsf{C}} B$. Assume $A \in w$ and consider any v such that $(w,v) \in R_{\mathsf{C}}$, i.e., $(w,v) \in (R_{\mathsf{E}})^n$ for some $n \geq 1$. As in the previous cases, $B \in \mathcal{E}_{\mathsf{C}}(w,t)$ by definition of $\mathcal{E}_{\mathsf{C}}$. By (7) we find $A \in v$, and thus, by C-reflexivity and maximal consistency, also $B \in v$. Hence, by the induction hypothesis $\mathcal{M}, v \Vdash B$. Now $\mathcal{M}, w \Vdash A$ immediately follows. The argument for the converse repeats the one from the previous cases. □

Note that the converse directions in the proof above are far from trivial in the modal case, see e.g. [MvdH95]. The last case, in particular, usually requires more sophisticated methods that guarantee the finiteness of the model.

Theorem 18 (Completeness). $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS})$ *is sound and complete with respect to the class of AF-models meeting* $\mathcal{CS}$*, i.e., for all formulae* $A \in \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}$*,*

$$\vdash_{\mathcal{CS}} A \text{ if and only if } \Vdash_{\mathcal{CS}} A \ .$$

Proof. Soundness has already been shown in Lemma 11. For completeness, let $\mathcal{M}$ be the canonical AF-model meeting $\mathcal{CS}$ and assume $\nvdash_{\mathcal{CS}} A$. Then $\{\neg A\}$ is $\mathcal{CS}$-consistent and hence is contained in some maximal $\mathcal{CS}$-consistent set $w \in W$. So, by Lemma 17, $\mathcal{M}, w \Vdash \neg A$, and hence, by Lemma 15, $\nVdash_{\mathcal{CS}} A$. □

M-models were introduced as semantics for LP by Mkrtychev [Mkr97]. They form a subclass of F-models (see [Fit05]).

Definition 19. *An* M-model *is a singleton AF-model.*

Theorem 20 (Completeness with respect to M-models). $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS})$ *is also sound and complete with respect to the class of M-models meeting* $\mathcal{CS}$*.*

Proof. Soundness follows immediately from Lemma 11. Now assume that $\nvdash_{\mathcal{CS}} A$, then $\{\neg A\}$ is $\mathcal{CS}$-consistent, and hence $\mathcal{M}, w \Vdash \neg A$ for some world $w_0 \in W$ in the canonical AF-model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ meeting $\mathcal{CS}$.

Let $\mathcal{M}' = (W', R', \mathcal{E}', \nu')$ be the restriction of $\mathcal{M}$ to $\{w_0\}$, i.e., $W' := \{w_0\}$, $R'_{\circledast} := \{(w_0, w_0)\}$ for any $\circledast$, $\mathcal{E}' := \mathcal{E} \upharpoonright (W' \times \mathrm{Tm})$, and $\nu'(P_n) := \nu(P_n) \cap W'$.

Since $\mathcal{M}'$ is clearly an M-model meeting $\mathcal{CS}$, it remains to demonstrate that $\mathcal{M}', w_0 \Vdash B$ if and only if $\mathcal{M}, w_0 \Vdash B$ for all formulae B. We proceed by induction on the structure of B. The cases where either B is a propositional variable or its primary connective is propositional are trivial. Therefore, we only show the case of $B = [t]_{\circledast} C$. First, observe that

$$\mathcal{M}, w_0 \Vdash [t]_{\circledast} C \text{ if and only if } C \in \mathcal{E}'_{\circledast}(w_0, t) \ . \tag{8}$$

Indeed, by Lemma 17, $\mathcal{M}, w_0 \Vdash [t]_{\circledast} C$ if and only if $[t]_{\circledast} C \in w_0$, which, by definition of the canonical AF-model, is equivalent to $C \in \mathcal{E}_{\circledast}(w_0, t) = \mathcal{E}'_{\circledast}(w_0, t)$.

If $\mathcal{M}, w_0 \Vdash [t]_{\circledast} C$, then $\mathcal{M}, w_0 \Vdash C$ since $R_{\circledast}$ is reflexive. By induction hypothesis, $\mathcal{M}', w_0 \Vdash C$. By (8) we have $C \in \mathcal{E}'_{\circledast}(w_0, t)$, and thus $\mathcal{M}', w_0 \Vdash [t]_{\circledast} C$.

If $\mathcal{M}, w_0 \nVdash [t]_{\circledast} C$, then by (8) we have $C \notin \mathcal{E}'_{\circledast}(w, t)$, so $\mathcal{M}', w_0 \nVdash [t]_{\circledast} C$. $\square$

Corollary 21 (Finite model property). $\mathsf{LP}^{\mathsf{C}}_h(\mathcal{CS})$ *enjoys the finite model property with respect to AF-models.*

5 Conservativity

Yavorskaya in [Yav08] introduced a two-agent version of LP, which we extend to an arbitrary h in the natural way:

Definition 22. *The language of* LP_h *is obtained from that of* $\mathsf{LP}^{\mathsf{C}}_h$ *by restricting the set of operations to* $\cdot_i$, $+_i$, *and* $!_i$ *and by dropping all terms from* Tm_{E} *and* Tm_{C}. *The axioms are restricted to application, sum, reflexivity, and inspection for each* i. *The definition of constant specification is changed accordingly.*

We show that $\mathsf{LP}^{\mathsf{C}}_h$ is conservative over LP_h by adapting a technique from [Fit08].

Definition 23. *The mapping* $\times : \mathrm{Fm}_{\mathsf{LP}^{\mathsf{C}}_h} \to \mathrm{Fm}_{\mathsf{LP}_h}$ *is defined as follows:*

1. $P^\times := P$ *for propositional variables* $P \in \mathrm{Prop}$;
2. $\times$ *commutes with propositional connectives;*
3. $([t]_{\circledast} A)^\times := \begin{cases} A^\times & \text{if } t \text{ contains a subterm } s \in \mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}, \\ [t]_{\circledast} A^\times & \text{otherwise.} \end{cases}$

Theorem 24. *Let* $\mathcal{CS}$ *be a constant specification for* $\mathsf{LP}^{\mathsf{C}}_h$. *For an arbitrary formula* $A \in \mathrm{Fm}_{\mathsf{LP}_h}$, *if* $\mathsf{LP}^{\mathsf{C}}_h(\mathcal{CS}) \vdash A$, *then* $\mathsf{LP}_h(\mathcal{CS}^\times) \vdash A$.

Proof. Since $A^\times = A$ for any $A \in \mathrm{Fm}_{\mathsf{LP}_h}$, it suffices to demonstrate that for any formula $D \in \mathrm{Fm}_{\mathsf{LP}^{\mathsf{C}}_h}$, if $\mathsf{LP}^{\mathsf{C}}_h(\mathcal{CS}) \vdash D$, then $\mathsf{LP}_h(\mathcal{CS}^\times) \vdash D^\times$, which can be done by induction on the derivation of D.

Case when D is a propositional tautology, then so is $D^\times$.

Case when $D = [t]_i B \to B$ is an instance of the reflexivity axiom. Then $D^\times$ is either $[t]_i B^\times \to B^\times$ or $B^\times \to B^\times$, i.e., an instance of the reflexivity axiom of LP_h or a propositional tautology respectively.

Case when $D = [t]_*(B \to C) \to ([s]_* B \to [t \cdot s]_* C)$ is an instance of the application axiom. We distinguish the following possibilities:

1. Both t and s contain a subterm from $\mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}$. Then $D^\times$ has the form $(B^\times \to C^\times) \to (B^\times \to C^\times)$, which is a propositional tautology and, thus, an axiom of LP_h.
2. Neither t nor s contains a subterm from $\mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}$. Then $D^\times$ is an instance of the application axiom of LP_h.
3. Term t contains a subterm from $\mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}$ while s does not. Then $D^\times$ is $(B^\times \to C^\times) \to ([s]_i B^\times \to C^\times)$, which can be derived in $\mathsf{LP}_h(\mathcal{CS}^\times)$ from the reflexivity axiom $[s]_i B^\times \to B^\times$ by propositional reasoning. In this case, translation $\times$ does not map an axiom of $\mathsf{LP}^{\mathsf{C}}_h$ to an axiom of LP_h.
4. Term s contains a subterm from $\mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}$ while t does not. Then $D^\times$ is $[t]_i(B^\times \to C^\times) \to (B^\times \to C^\times)$, an instance of the reflexivity axiom of LP_h.

Case when $D = [t]_* B \to [t+s]_* B$ is an instance of the sum axiom. Then $D^\times$ becomes $B^\times \to B^\times$, $[t]_i B^\times \to B^\times$, or $[t]_i B^\times \to [t+s]_i B^\times$, i.e., a propositional tautology, an instance of the reflexivity axiom of LP_h, or an instance of the sum axiom of LP_h respectively. The sum axiom $[s]_* B \to [t+s]_* B$ is treated in the same manner.

Case when $D = [t]_i B \to [!t]_i\, [t]_i B$ is an instance of the inspection axiom. Then $D^\times$ is either the propositional tautology $B^\times \to B^\times$ or $[t]_i B^\times \to [!t]_i\, [t]_i B^\times$, an instance of the inspection axiom of LP_h.

Case when $D = [t_1]_1 B \wedge \dots \wedge [t_h]_h B \to [\langle t_1, \dots, t_h \rangle]_{\mathsf{E}} B$ is an instance of the tupling axiom. We distinguish the following possibilities:

1. At least one of the t_i's contains a subterm from $\mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}$. Then $D^\times$ has the form $C_1 \wedge \dots \wedge C_h \to B^\times$ with at least one $C_i = B^\times$ and is, therefore, a propositional tautology.
2. None of the t_i's contains a subterm from $\mathrm{Tm}_{\mathsf{E}} \cup \mathrm{Tm}_{\mathsf{C}}$. Then $D^\times$ has the form $[t_1]_1 B^\times \wedge \dots \wedge [t_h]_h B^\times \to B^\times$, which can be derived in $\mathsf{LP}_h(\mathcal{CS}^\times)$ from the reflexivity axiom. This is another case when translation $\times$ does not map an axiom of $\mathsf{LP}^{\mathsf{C}}_h$ to an axiom of LP_h.

Case when D is an instance of the projection axiom $[t]_{\mathsf{E}} B \to [\pi_i t]_i B$ or of the co-closure axiom, i.e., $[t]_{\mathsf{C}} B \to [\mathsf{ccl}_1(t)]_{\mathsf{E}} B$ or $[t]_{\mathsf{C}} B \to [\mathsf{ccl}_2(t)]_{\mathsf{E}}\, [t]_{\mathsf{C}} B$. Then $D^\times$ is the propositional tautology $B^\times \to B^\times$.

Case when $D = B \wedge [t]_{\mathsf{C}}(B \to [s]_{\mathsf{E}} B) \to [\mathsf{ind}(t,s)]_{\mathsf{C}} B$ is an instance of the induction axiom. Then $D^\times$ is $B^\times \wedge (B^\times \to B^\times) \to B^\times$, a propositional tautology.

Case when D is derived by modus ponens is trivial.

Case when D is $[c]_{\circledast} B \in \mathcal{CS}$. Then $D^\times$ is either $B^\times$ or $[c]_i B^\times$. In the former case, B is an axiom of $\mathsf{LP}^{\mathsf{C}}_h$, and hence $B^\times$ is derivable in $\mathsf{LP}_h(\mathcal{CS}^\times)$, as shown above; in the latter case, $[c]_i B^\times \in \mathcal{CS}^\times$. □

Remark 25. Note that $\mathcal{CS}^\times$ need not, in general, be a constant specification for LP_h because, as noted above, for an axiom D of $\mathsf{LP}^{\mathsf{C}}_h$, its image $D^\times$ is not always an axiom of LP_h. To ensure that $\mathcal{CS}^\times$ is a proper constant specification, $(A \to B) \to ([s]_i A \to B)$ and $[t_1]_1 A \wedge \dots \wedge [t_h]_h A \to A$ have to be made axioms of LP_h. Another option is to use Fitting's concept of *embedding* one justification logic into another, which involves replacing constants in D with more complicated terms in $D^\times$ (see [Fit08] for details).

6 Forgetful Projection and a Word on Realization

Most justification logics are introduced as explicit counterparts to particular modal logics in the strict sense described in Sect. 1. Although the realization theorem for $\mathsf{LP}_h^{\mathsf{C}}$ remains an open problem, in this section we prove that each theorem of our logic $\mathsf{LP}_h^{\mathsf{C}}$ states a valid modal fact if all terms are replaced with the corresponding modalities, which is one direction of the realization theorem. We also discuss approaches to the harder opposite direction.

We start with recalling the modal language of common knowledge. Modal formulae are defined by the following grammar

$$A ::= P_j \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \Box_i A \mid \mathsf{E}A \mid \mathsf{C}A \ ,$$

where $P_j \in \mathrm{Prop}$. The set of all modal formulae is denoted by $\mathrm{Fm}_{\mathsf{S4}_h^{\mathsf{C}}}$.

The Hilbert system $\mathsf{S4}_h^{\mathsf{C}}$ [MvdH95] is given by the modal axioms of $\mathsf{S4}$ for individual agents, by the necessitation rule for $\Box_1, \dots, \Box_h$, and C, by modus ponens, and by the axioms

$$\mathsf{C}(A \to B) \to (\mathsf{C}A \to \mathsf{C}B), \qquad \mathsf{C}A \to A, \qquad \mathsf{E}A \leftrightarrow \Box_1 A \wedge \dots \wedge \Box_h A,$$
$$A \wedge \mathsf{C}(A \to \mathsf{E}A) \to \mathsf{C}A, \qquad \mathsf{C}A \to \mathsf{E}(A \wedge \mathsf{C}A).$$

Definition 26 (Forgetful projection). *The mapping* $\circ\colon \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}} \to \mathrm{Fm}_{\mathsf{S4}_h^{\mathsf{C}}}$ *is defined as follows:*

1. $P^\circ := P$ *for propositional variables* $P \in \mathrm{Prop}$*;*
2. $\circ$ *commutes with propositional connectives;*
3. $([t]_i A)^\circ := \Box_i A^\circ$*;*
4. $([t]_{\mathsf{E}} A)^\circ := \mathsf{E}A^\circ$*;*
5. $([t]_{\mathsf{C}} A)^\circ := \mathsf{C}A^\circ$*.*

Lemma 27. *Let* $\mathcal{CS}$ *be any constant specification. For any formula* $A \in \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}$*, if* $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS}) \vdash A$*, then* $\mathsf{S4}_h^{\mathsf{C}} \vdash A^\circ$*.*

Proof. The proof is by easy induction on the derivation of A. ☐

Definition 28 (Realization). *A realization is a mapping* $r\colon \mathrm{Fm}_{\mathsf{S4}_h^{\mathsf{C}}} \to \mathrm{Fm}_{\mathsf{LP}_h^{\mathsf{C}}}$ *such that* $(r(A))^\circ = A$*. We usually write* A^r *instead of* $r(A)$*.*

We can think of a realization as a function that replaces occurrences of modal operators (including E and C) with evidence terms of the corresponding type. The problem of realization for a given pure C-axiomatically appropriate constant specification $\mathcal{CS}$ can be stated as follows:

Is there a realization r such that $\mathsf{LP}_h^{\mathsf{C}}(\mathcal{CS}) \vdash A^r$ for any theorem A of $\mathsf{S4}_h^{\mathsf{C}}$?

A positive answer to this question would constitute the harder direction of the realization theorem, which is often demonstrated using induction on a cut-free sequent proof of the modal formula.

Cut-free systems for $\mathsf{S4}_h^{\mathsf{C}}$ are presented in [AJ05] and [BS09]. They are based on an infinitary ω-rule of the form

$$\frac{\mathsf{E}^m A, \Gamma \quad \text{for all } m \geq 1}{\mathsf{C}A, \Gamma} \qquad (\omega).$$

However, realization of such a rule meets with serious difficulties in reaching uniformity among the realizations of the approximants $\mathsf{E}^m A$.

A finitary cut-free system is obtained in [JKS07] by finitizing this ω-rule via the finite model property. Unfortunately, the "somewhat unusual" structural properties of the resulting system (see discussion in [JKS07]) make it hard to use it for realization.

The non-constructive, semantic realization method from [Fit05] cannot be applied directly because of the non-standard behavior of the canonical model (see Remark 16).

Perhaps the infinitary system presented in [BKS09], which is finitely branching but admits infinite branches, can help in proving the realization theorem for $\mathsf{LP}_h^{\mathsf{C}}$. For now this remains work in progress.

7 Coordinated attack

To illustrate our logic, we will now analyze the coordinated attack problem along the lines of [FHMV95], where additional references can be found. Let us briefly recall this classical problem. Suppose two divisions of an army, located in different places, are about to attack an enemy. They have some means of communication, but these may be unreliable, and the only way to secure a victory is to attack simultaneously. How should generals G and H who command the two divisions coordinate their attacks? Of course, general G could send a message m_1^G with the time of attack to general H. Let us use the proposition del to denote the fact that the message with the time of attack has been delivered. If the generals trust the authenticity of the message, say because of a signature, the message itself can be taken as evidence that it has been delivered. So general H, upon receiving the message, knows the time of attack, i.e., $\left[m_1^G\right]_H del$. However, since communication is unreliable, G considers it possible that his message has not been delivered. But if general H sends an acknowledgment m_2^H, he in turn cannot be sure whether the acknowledgment has reached G, which prompts yet another acknowledgment m_3^G by general G, and so on.

In fact, common knowledge of del is a necessary condition for the attack. Indeed, it is reasonable to assume it to be common knowledge between the generals that they should only attack simultaneously or not attack at all, i.e., that they attack only if both know that they attack: $[t]_{\mathsf{C}}(att \to [s]_{\mathsf{E}} att)$ for some terms s and t. Thus, by the induction axiom, we get $att \to [\mathsf{ind}(t,s)]_{\mathsf{C}} att$. Another reasonable assumption is that it is common knowledge that neither general attacks unless the message with the time of attack has been delivered: $[r]_{\mathsf{C}}(att \to del)$ for some term r. Using the application axiom, we obtain $att \to [r \cdot \mathsf{ind}(t,s)]_{\mathsf{C}} del$.

We now show that common knowledge of *del* cannot be achieved and that, therefore, no attack will take place, no matter how many messages and acknowledgments m_1^G, m_2^H, m_3^G, ... are sent by the generals even if all the messages are successfully delivered.

In the classical modeling without evidence, the reason is that the sender of the last message always considers the possibility that his last message, say m_{2k}^H, has not been delivered. To give a flavor of the argument carried out in detail in [FHMV95], we provide a countermodel where m_2^H is the last message, it has been delivered, but H is unsure of that, i.e., $\left[m_1^G\right]_H del$, $\left[m_2^H\right]_G \left[m_1^G\right]_H del$, but $\neg [s]_H \left[m_2^H\right]_G \left[m_1^G\right]_H del$ for all terms s. Indeed, consider the model $\mathcal{M}$ with $W := \{0, 1, 2, 3\}$, $\nu(del) := \{0, 1, 2\}$, R_G being the reflexive closure of $\{(1, 2)\}$, R_H being the reflexive closure of $\{(0, 1), (2, 3)\}$, and any evidence function $\mathcal{E}$ such that $del \in \mathcal{E}_H(0, m_1^G)$ and $\left[m_1^G\right]_H del \in \mathcal{E}_G(0, m_2^H)$. Then, whatever $\mathcal{E}_\mathsf{C}$ is, we have $\mathcal{M}, 0 \nVdash [s]_H \left[m_2^H\right]_G \left[m_1^G\right]_H del$ and $\mathcal{M}, 0 \nVdash [t]_\mathsf{C} del$ for any s and t because $\mathcal{M}, 3 \nVdash del$.

In our models with explicit evidence, there is an alternative possibility for the lack of knowledge: the absence of evidence. For example, G may receive the acknowledgment m_2^H but not consider it to be evidence for $\left[m_1^G\right]_H del$ because the signature of H is missing.

We now demonstrate that common knowledge of the time of attack cannot emerge, basing the argument solely on the lack of common knowledge evidence. A corresponding M-model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ is obtained as follows: $W := \{w\}$, $R_i := \{(w, w)\}$, $\nu(del) := \{w\}$, and $\mathcal{E}$ is the minimal evidence function such that $del \in \mathcal{E}_H(w, m_1^G)$ and $\left[m_1^G\right]_H del \in \mathcal{E}_G(w, m_2^H)$. In this model $M, w \nVdash [t]_\mathsf{C} del$ for any evidence term t because $del \notin \mathcal{E}_\mathsf{C}(w, t)$ for any t. To show the latter statement, note that for any term t, by Lemma 27,

$$\nvdash \left[m_1^G\right]_H del \wedge \left[m_2^H\right]_G \left[m_1^G\right]_H del \to [t]_\mathsf{C} del \tag{9}$$

because

$$\mathsf{S4}_h^\mathsf{C} \nvdash \Box_H del \wedge \Box_G \Box_H del \to \mathsf{C} del \ ,$$

which is easy to demonstrate. Thus, the negation of the formula from (9) is satisfiable, and for each t there is a world w_t in the canonical AF-model with evidence function $\mathcal{E}^{\text{can}}$ such that $del \in \mathcal{E}_H^{\text{can}}(w_t, m_1^G)$ and $\left[m_1^G\right]_H del \in \mathcal{E}_G^{\text{can}}(w_t, m_2^H)$, but by the Truth Lemma 17, $del \notin \mathcal{E}_\mathsf{C}^{\text{can}}(w_t, t)$. Since $\mathcal{E}^{\text{can}} \upharpoonright (\{w_t\} \times \text{Tm})$ satisfies all the closure conditions, minimality of $\mathcal{E}$ implies that $\mathcal{E}_\mathsf{C}(w, s) \subseteq \mathcal{E}_\mathsf{C}^{\text{can}}(w_t, s)$ for any term s. In particular, $del \notin \mathcal{E}_\mathsf{C}(w, t)$ for any term t.

8 Conclusions

We have presented an explicit evidence system LP_h^C with common knowledge, which is a conservative extension of the multi-agent explicit evidence logic LP_h. The major open problem at the moment remains proving the realization theorem, one direction of which we have demonstrated.

Our analysis of the coordinated attack problem in the language of $\mathsf{LP}_h^{\mathsf{C}}$ shows that access to explicit evidence creates more alternatives than the classical modal approach. In particular, the lack of knowledge can occur either because messages are not delivered or because evidence of authenticity is missing.

We have mostly concentrated on the study of C-axiomatically appropriate constant specifications. For modeling distributed systems with different reasoning capabilities of agents, it is also interesting to consider i-axiomatic appropriate, E-axiomatic appropriate, and mixed constant specifications, where only certain aspects of reasoning are common knowledge.

We established soundness and completeness with respect to AF-models and singleton M-models. Can other semantics for justification logics such as (arithmetical) provability semantics [Art95, Art01] and game semantics [Ren09b] be adapted to $\mathsf{LP}_h^{\mathsf{C}}$?

There are further interesting questions: Is $\mathsf{LP}_h^{\mathsf{C}}$ decidable and, if yes, what is its complexity compared to that of $\mathsf{S4}_h^{\mathsf{C}}$? How robust is our treatment of common knowledge if the individual modalities are taken to be of type K, $\mathsf{K5}$, etc.?

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