Simplicial Belief

Christian Cachin David Lehnherr [1] Thomas Studer

Institute of Computer Science, University of Bern, Bern, Switzerland

Abstract

Recently, much work has been carried out to study simplicial interpretations of modal logic. While notions of (distributed) knowledge have been well investigated in this context, it has been open how to model belief in simplicial models. We introduce polychromatic simplicial complexes, which naturally impose a plausibility relation on states. From this, we can define various notions of belief.

Keywords: Simplicial complex, epistemic logic, plausibility model, belief modality.

1 Introduction

Simplicial interpretations for modal logic are currently avidly researched; see, e.g., [3,7,8,10,12] due to their close connection with distributed computing [9]. At its core lies the epistemic interpretation of simplicial complexes of various kinds. Let $\mathcal{V}$ be a set of vertices. Each vertex corresponds to a local state of an agent, and we say that this vertex is of that agent's color. In the simplest case, a simplicial complex $(S, \mathcal{V})$ is a pair where S is a set of subsets of $\mathcal{V}$ that is closed under set inclusion. Vertices that belong to the same set must be of different colors, and maximal elements of S represent global states. An agent a cannot distinguish two global states if its local state is included in both. Hence, simplicial complexes offer sufficient structure for an epistemic interpretation. While (distributed) knowledge has been studied extensively in this context, it has been open, see [4], how to model belief on simplicial structures such that

(i) belief depends only on the topological structure of the simplicial complex;

(ii) the principle of knowledge-yields-belief holds.

In this brief announcement, we present polychromatic simplicial complexes, i.e., complexes that are not necessarily properly colored. We define a plausibility relation between the states based on the multiplicity of a color within a state. If the color of an agent a has a lower or equal multiplicity in a state X than in a state Y, then a considers X to be at least as plausible as Y. This relation is a wellfounded preorder, and hence, we can use the machinery of plausibility models [1,2] to define various notions of belief such as plausible belief

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and safe belief. Moreover, our structures also satisfy the knowledge-yields-belief principle.

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2 Simplicial Knowledge

We quickly recall the standard definitions for distributed knowledge on simplicial complexes [7,8,12]. In the subsequent section, we will extend them to incorporate notions of belief.

Let Ag be the set of finitely many agents, and let Prop be a countable set of atomic propositions. We define the language of knowledge $\mathcal{L}_{\mathcal{K}}$ for $G \subseteq \mathsf{Ag}$ and $p \in \mathsf{Prop}$ inductively by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [\sim]_G\phi.$$

The remaining Boolean connectives are defined as usual. In particular, we set $\bot := p \wedge \neg p$ for some fixed $p \in \mathsf{Prop}$. We write $\mathsf{alive}(G)$ for $\neg[\sim]_G\bot$ and $\mathsf{dead}(G)$ for $[\sim]_G\bot$.

Definition 2.1 Let $\mathcal{V}$ be a set of vertices. $C = (S, \mathcal{V})$ with $S \subseteq \mathsf{Pow}(\mathcal{V}) \setminus \{\emptyset\}$ is called a *simplicial complex* if

$$\text{for each } X \in S \text{ and each } \emptyset \neq Y \subseteq X, \text{ we have } Y \in S.$$

We call the elements of S *faces*. A face that is maximal under inclusion is called a *facet*. We denote the set of facets of C by $\mathcal{F}(C)$. A *coloring* is a mapping $\chi : \mathcal{V} \to \mathsf{Ag}$. A coloring is *proper* if it assigns a different agent to each vertex within a face. We use $\chi(U)$ for the set $\{\chi(u) \mid u \in U\}$.

Definition 2.2 Let $C = (S, \mathcal{V})$ be a simplicial complex. A *simplicial model* $\mathcal{C} = (C, \chi, W, \ell)$ is a quadruple where

(i) C is a simplicial complex;

(ii) $\chi : \mathcal{V} \to \mathsf{Ag}$ is a proper coloring;

(iii) $\mathcal{F}(C) \subseteq W \subseteq S$ is a set of worlds;

(iv) $\ell : W \to \mathsf{Pow}(\mathsf{Prop})$ is a valuation.

Given a simplicial model, a group of agents $G \subseteq \mathsf{Ag}$ cannot distinguish two worlds $X, Y \in W$, denoted by $X \sim_G Y$, if and only if $G \subseteq \chi(X \cap Y)$. We call $\sim_G$ the *epistemic indistinguishability relation.* If G contains only a single agent a, we write $X \sim_a Y$ and $[\sim]_a$ instead of $X \sim_{\{a\}} Y$ and $[\sim]_{\{a\}}$, respectively.

Definition 2.3 For a simplicial model $\mathcal{C} = (C, W, \chi, \ell)$, a world $X \in W$, and a formula $\phi \in \mathcal{L}_{\mathcal{K}}$, we define the relation $\mathcal{C}, X \Vdash \phi$ inductively by

$$\begin{array}{lll} \mathcal{C}, X \Vdash p & \text{iff} & p \in \ell(X) \\ \mathcal{C}, X \Vdash \neg\phi & \text{iff} & \mathcal{C}, X \nVdash \phi \\ \mathcal{C}, X \Vdash \phi \wedge \psi & \text{iff} & \mathcal{C}, X \Vdash \phi \text{ and } \mathcal{M}, X \Vdash \psi \\ \mathcal{C}, X \Vdash [\sim]_G \phi & \text{iff} & X \sim_G Y \text{ implies } \mathcal{C}, Y \Vdash \phi \quad \text{for all } Y \in W. \end{array}$$

We say that *agent a is alive in a world X* if $a \in \chi(X)$. The set of worlds in which a group $G \subseteq \mathsf{Ag}$ is alive is defined as

$$\mathsf{Alive}_{\mathcal{C}}(G) = \{X \in W \mid G \subseteq \chi(X)\}.$$

Lemma 2.4 *Let $\mathcal{C} = (C, \chi, W, \ell)$ be a simplicial model. For each $G \subseteq \mathsf{Ag}$, the relation $\sim_G$ is an equivalence relation on $\mathsf{Alive}_{\mathcal{C}}(G)$ and empty otherwise.*

3 Simplicial belief

We now drop the requirement that the coloring of a simplicial model must be proper. The resulting models are called polychromatic. We will define a wellfounded preorder on the states of a polychromatic model, which will serve as a plausibility relation [1,2]. This makes it possible to interpret various notions of belief on simplicial models.

It is straightforward to verify that Lemma 2.4 does not hold for polychromatic models because $\sim_G$ need not be transitive. Indeed, consider the set of vertices $\{0, 1, 2, 3\}$ and the complex consisting of the facets

$$X := \{0, 1\}, \quad Y := \{1, 2\}, \quad \text{and} \quad Z = \{2, 3\}$$

with a coloring χ that assigns the same agent a to every vertex. We find that $X \sim_a Y$ and $Y \sim_a Z$, but not $X \sim_a Z$.

In order to re-establish transitivity of $\sim_G$, we must require that for any three worlds $X, Y, Z \in W$ and any group of agents $G \subseteq \mathsf{Ag}$:

$$G \subseteq \chi(X \cap Y) \text{ and } G \subseteq \chi(Y \cap Z) \text{ implies } G \subseteq \chi(X \cap Z). \qquad (\star)$$

Definition 3.1 A *polychromatic model* is a simplicial model where:

(i) the coloring is not required to be proper;

(ii) condition $(\star)$ holds.

Definition 3.2 Let (C, χ, W, ℓ) be a polychromatic model. We define the *multiplicity* of $a \in \mathsf{Ag}$ in a world X by

$$m_a(X) = |\{v \in X \mid \chi(v) = a\}|$$

where $|\cdot|$ denotes the cardinality of a set. Note that if agent a is alive in a world X, then $m_a(X) \geq 1$.

For $X, Y \in W$ and $a \in \mathsf{Ag}$, we write

$$X \leq_a Y \quad \text{iff} \quad m_a(X) \leq_a m_a(Y).$$

The multiplicity of a color within a face induces for each agent a a wellfounded relation $\leq_a$ on worlds. We call this the (a priori) plausibility relation. Notice that $\leq_a$ is a priori in the sense that it does not refer to the actual world, i.e., it does not account for possibility. We introduce a local plausibility relation

$$\trianglelefteq_a \ := \ \leq_a \cap \sim_a,$$

which captures the agent's plausibility relation at a given state. Further, we write $X \geq_a Y$ if and only if $m_X(a) \geq m_Y(a)$ and we use $\trianglerighteq_a$ and $\triangleleft_a$ in the obvious way. The following lemma shows that the indistinguishability relation can be given in terms of the local plausibility relation.

Lemma 3.3 $\sim_a = \trianglelefteq_a \cup \trianglerighteq_a$.

From the relation $\trianglerighteq_a$, we get a corresponding modal operator $[\trianglerighteq]_a$, which is referred to in the literature as safe belief [2]. Our language of knowledge and belief $\mathcal{L}_{\mathcal{KB}}$ extends $\mathcal{L}_{\mathcal{K}}$ by the modal operator $[\trianglerighteq]_a$ for each agent $a \in \mathsf{Ag}$. It is inductively defined by as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [\sim]_G\phi \mid [\trianglerighteq]_a\phi$$

where $p \in \mathsf{Prop}$. As usual, the dual of safe belief is defined as $\langle\trianglerighteq\rangle_a\varphi \equiv \neg[\trianglerighteq]_a\neg\varphi$.

Definition 3.4 For a polychromatic model $\mathcal{C} = (C, \chi, W, \ell)$, a world $X \in W$, and a formula $\phi \in \mathcal{L}_{\mathcal{KB}}$, we define the relation $\mathcal{C}, X \Vdash \phi$ inductively by

$$\begin{array}{lll}
\mathcal{C}, X \Vdash p & \text{iff} & p \in \ell(X) \\
\mathcal{C}, X \Vdash \neg\phi & \text{iff} & \mathcal{C}, X \nVdash \phi \\
\mathcal{C}, X \Vdash \phi \wedge \psi & \text{iff} & \mathcal{C}, X \Vdash \phi \text{ and } \mathcal{M}, X \Vdash \psi \\
\mathcal{C}, X \Vdash [\sim]_G\phi & \text{iff} & X \sim_G Y \quad \text{implies} \quad \mathcal{C}, Y \Vdash \phi \quad \text{for all } Y \in W \\
\mathcal{C}, X \Vdash [\trianglerighteq]_a\phi & \text{iff} & X \trianglerighteq_a Y \quad \text{implies} \quad \mathcal{C}, Y \Vdash \phi \quad \text{for all } Y \in W.
\end{array}$$

As usual with plausibility models, we can not only define safe belief but also other notions of belief.

Definition 3.5 Let $\mathcal{C} = (C, \chi, W, \ell)$ be a polychromatic model. For $X \in W$ we define

$$\mathsf{Min}_{\trianglelefteq_a}(X) = \{Y \in W \mid Y \sim_a X \text{ and } \nexists Z \in W. Z \triangleleft_a Y\}.$$

Since $\leq_a$ is wellfounded, we find that $\mathsf{Min}_{\trianglelefteq_a}(X) \neq \emptyset$ if agent a is alive in the world X.

We can now extend our language $\mathcal{L}_{\mathcal{KB}}$ with a new modality $\mathcal{B}_a$ for each agent a. We use the following truth definition.

Definition 3.6 For a polychromatic model $\mathcal{C} = (C, \chi, W, \ell)$, a world $X \in W$, and a formula $\phi \in \mathcal{L}_{\mathcal{KB}}$, we define

$$\mathcal{C}, X \Vdash \mathcal{B}_a\varphi \quad \text{iff} \quad Y \in \mathsf{Min}_{\trianglelefteq_a}(X) \text{ implies } \mathcal{C}, Y \Vdash \varphi \quad \text{for all } Y \in W.$$

The modality $\mathcal{B}_a$ models agent a's (most plausible) belief. It is well-known that $\mathcal{B}_a$ can be expressed in terms of the $[\trianglerighteq]_a$ modalitiy [2,11].

Lemma 3.7 *Let $\mathcal{C} = (C, \chi, W, \ell)$ be a polychromatic model, a an agent, and $X \in W$ such that a is alive in X. We find that*

$$\mathcal{C}, X \Vdash \mathcal{B}_a \phi \quad \textit{if and only if} \quad \mathcal{C}, X \Vdash \langle \trianglerighteq \rangle_a [\trianglerighteq]_a \phi.$$

Our model satisfies the knowledge-yields-belief principle. In particular, we have the following lemma.

Lemma 3.8 *Let $\mathcal{C} = (C, \chi, W, \ell)$ be a polychromatic model and $X \in W$. For any agent a and any formula φ, we have*

$$\mathcal{C}, X \Vdash [\sim]_a \varphi \to [\trianglerighteq]_a \varphi \quad \textit{and} \quad \mathcal{C}, X \Vdash [\trianglerighteq]_a \varphi \to \mathcal{B}_a \varphi.$$

4 Conclusion and future work

We presented the first interpretation of belief on a simplicial structure that depends only on the topological structure without requiring additional machinery like belief functions. Our approach consists of dropping the requirement that the coloring must be proper and using the multiplicity of color within a face as an inverted plausibility measure.

The study of polychromatic models is still in its infancy, and many basic properties need further investigation. For instance, simplicial models are proper, i.e. different worlds can be distinguished by at least one agent. Formally, Goubault et al. [6] express this as

$$\mathsf{alive}(G) \land \mathsf{dead}(G^c) \land \varphi \to [\sim]_G(\mathsf{dead}(G^c) \to \varphi)$$

being valid, where G^c stands for the complement of G. This no longer holds for polychromatic models.

Moreover, the analysis of polychromatic models is an important step towards simplicial models that are based on simplicial sets [5]. Informally, one could say that the actual vertex is repeated in such a model, and not just the color. In this case, the property $(\star)$ trivially holds and must not be imposed as a restriction on the model.

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