

# Bayesian Confirmation and Justifications

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**Abstract.** We introduce a family of probabilistic justification logics that feature Bayesian confirmations. Our logics include new justification terms representing evidence that make a proposition firm in the sense of making it more probable. We present syntax and semantics of our logic and establish soundness and strong completeness. Moreover, we show how to formalize in our logic the screening-off condition for transitivity of Bayesian confirmations.

**Keywords:** Epistemic logic · justification logic · Bayesian confirmation

## 1 Introduction

Justification logic is a type of logic that explicitly includes justifications why something is known or believed [6, 18]. The first justification logic, the Logic of Proofs [2], has been developed to provide a classical provability semantics for intuitionistic logic. In that approach, justifications represent proofs in a formal system like Peano arithmetic [17]. Later justification logic was introduced into formal epistemology where justifications can represent not only proofs but evidence in general [4]. For instance, an agent's knowledge may be justified by direct observation or by communication with another agent. In this context, notions like common knowledge [3, 8] and public announcements [7, 9] have been studied in detail.

Milnikel [19] was the first to investigate uncertain justifications. This led to several further frameworks that model uncertain reasoning in justification logic: fuzzy justification logics [12, 21], possibilistic justification logics [11, 28], probabilistic justification logics [13, 14, 20], and logics for combining evidence and uncertainty [1, 25].

Having logics that contain justifications for belief as well as operators for conditional probabilities, it is natural to extend them to a framework in which justifications can represent Bayesian confirmations [29]. The main principle of Bayesian confirmation theory says that (for simplicity we do not consider a background theory here) evidence  $E$  confirms hypothesis  $H$  if the prior probability of  $H$  conditional on  $E$  is greater than the prior unconditional probability

of  $H$ , that is if  $P(H|E) > P(H)$ . Carnap [10] calls this condition *confirmation as increase in firmness*.

We aim at a probabilistic justification logic that implements the above idea, that is in which something like

$$P(H|E) > P(H) \text{ entails } j(E) : H \quad (1)$$

holds, where  $j(E)$  is a term that represents the evidence  $E$ . Hence in this logic we read the formula  $e : F$  as *evidence  $e$  confirms  $F$* .

In order to model this relationship between conditional probability and evidence, we need a way to consider formulas as evidence terms. In (1) this is the role of the operator  $j$ . It takes a formula  $E$  and produces an evidence term  $j(E)$  representing the evidence  $E$ .

A similar kind of justification operator has been considered in the treatment of public announcements [16] where the operator  $\text{up}$  transforms formulas to evidence terms. We will use a similar strategy for the  $j$ -operator of (1). Further we will employ operators for conditional probabilities  $\text{CP}_{\geq s}$  as in [20, 22] and operators for the degree of confirmation  $\text{D}_{\geq s}$  as in [26]. A formula  $\text{CP}_{\geq s}(A, B)$  means that *the conditional probability of  $A$  given  $B$  is at least  $s$*  and a formula  $\text{D}_{\geq s}(A, B)$  means that *the difference between the conditional probability of  $A$  given  $B$  and the probability of  $A$  is at least  $s$* .

The paper is organized as follows. In the next section we introduce syntax and semantics of Bayesian justification logic, i.e. we present the deductive system  $\text{BJ}_{\text{CS}}$  and we introduce the class of measurable Bayesian models. Then in Section 3 we establish soundness and completeness of  $\text{BJ}_{\text{CS}}$  with respect to those models. Section 4 discusses transitivity of Bayesian confirmations in the framework of justification logic. Finally, Section 5 concludes the paper.

## 2 Bayesian Justification Logic BJ

### 2.1 Syntax

We start with countably many constants  $c_i$ , countably many variables  $x_i$ , and countably many atomic propositions  $p_i$ . Further, we define  $S := \mathbb{Q} \cap [0, 1]$  and  $S^* := \mathbb{Q} \cap [-1, 1]$ , where  $\mathbb{Q}$  is the set of all rational numbers. The (evidence) terms and formulas of the language of BJ are defined by simultaneous induction as follows:

- Evidence terms.
  - Every constant  $c_i$  and every variable  $x_i$  is an atomic term. If  $A$  is a formula, then  $j_A$  is an atomic term. Every atomic term is a term.
  - If  $t$  and  $s$  are terms, then  $t \cdot s$  is a term and  $!t$  is a term.
- Formulas.
  - Every atomic proposition  $p_i$  is a formula.
  - $\perp$  is a formula.
  - If  $A$  and  $B$  are formulas,  $t$  is a term,  $s \in S$ , and  $r \in S^*$ , then  $A \rightarrow B$ ,  $t : A$ ,  $\text{CP}_{\geq s}(A, B)$ , and  $\text{D}_{\geq r}(A, B)$  are formulas.

The set of all constants is denoted by  $\text{Con}$ , the set of all terms is  $\text{Tm}$ , and we use  $t, s, u, v, \dots$  to denote terms. The set of atomic propositions and the set of justification formulas are denoted by  $\text{Prop}$  and  $\text{Fml}$ , respectively. We use  $A, B, \dots$  to denote formulas. The classical Boolean connectives  $\neg, \vee, \wedge, \leftrightarrow$  are defined as usual and we set

$$\text{CP}_{\leq s}(B, C) := \text{CP}_{\geq 1-s}(\neg B, C) \quad \text{and} \quad \text{D}_{\leq r}(B, C) := \text{D}_{\geq -r}(\neg B, C)$$

for  $s \in S$  and  $r \in S^*$ . Moreover, we use the standard abbreviations, see [22], for the following formulas:

$$\text{CP}_{< s}(A, B) \quad \text{CP}_{> s}(A, B) \quad \text{CP}_{= s}(A, B) \quad \text{P}_{\rho s}(A) \quad \text{for } \rho \in \{\geq, \leq, >, <, =\}$$

and similarly for  $\text{D}_{< s}(A, B)$ ,  $\text{D}_{> s}(A, B)$  and  $\text{D}_{= s}(A, B)$ .

The axiom schemes of BJ are the following where  $\bigcirc \in \{\text{CP}, \text{D}\}$ :

1. all classical tautologies
2.  $t : (A \rightarrow B) \rightarrow (s : A \rightarrow t \cdot s : B)$
3.  $\text{CP}_{\geq 0}(A, B)$
4.  $\bigcirc_{\leq s}(A, B) \rightarrow \bigcirc_{< t}(A, B)$ , for  $t > s$
5.  $\bigcirc_{< s}(A, B) \rightarrow \bigcirc_{\leq s}(A, B)$
6.  $\text{P}_{\geq 1}(A \leftrightarrow B) \rightarrow (\text{P}_{= s}A \rightarrow \text{P}_{= s}B)$
7.  $\text{P}_{= s}A \wedge \text{P}_{= t}B \wedge \text{P}_{\geq 1}\neg(A \wedge B) \rightarrow \text{P}_{=\min\{1, s+t\}}(A \vee B)$
8.  $\text{P}_{= 0}B \rightarrow \text{CP}_{= 1}(A, B)$
9.  $\text{P}_{\geq s}(A \wedge B) \wedge \text{P}_{< t}B \rightarrow \text{CP}_{\geq \frac{s}{t}}(A, B)$ , for  $t \neq 0$
10.  $\text{P}_{< s}(A \wedge B) \wedge \text{P}_{> t}B \rightarrow \text{CP}_{\leq \frac{s}{t}}(A, B)$ , for  $t \neq 0$
11.  $\text{CP}_{> s}(A, B) \wedge \text{P}_{< t}A \rightarrow \text{D}_{> s-t}(A, B)$
12.  $\text{CP}_{\leq s}(A, B) \wedge \text{P}_{> t}A \rightarrow \text{D}_{\leq s-t}(A, B)$
13.  $j_B : A \leftrightarrow \text{D}_{> 0}(A, B)$

Axioms 1 to 10 come from justification logic with conditional probabilities, see [20]. The main difference is that we replaced the axiom

$$\text{P}_{= s}(A \wedge B) \wedge \text{P}_{= t}B \rightarrow \text{CP}_{= \frac{s}{t}}(A, B) \quad \text{for } t \neq 0$$

from [20] with our axioms 9 and 10, which yields a slightly stronger system. This additional power is needed to prove Lemma 4. Axioms 11 and 12 formalize the relationship between conditional probabilities and degrees of confirmation as in [26]. Axiom 13 finally states that terms  $j_B$  represent Bayesian confirmations.

A *constant specification* is any set  $\text{CS}$  that satisfies

$$\begin{aligned} \text{CS} \subseteq \{&(c, A) \mid c \text{ is a constant and} \\ &A \text{ is an instance of some axiom of BJ}\}. \end{aligned}$$

Let  $\text{CS}$  be any constant specification. The deductive system  $\text{BJ}_{\text{CS}}$  is the Hilbert system obtained by adding to the axioms of BJ the rules (MP), (CE), (ST.1), (ST.2) and (AN!) as given in Figure 1.

Note that (ST.1) and (ST.2) are infinitary rules, which we need to obtain strong completeness. Observe also the difference in the definitions of rules (MP), (ST.1), (ST.2), and (CE) in Figure 1. Rule (CE) can only be applied to theorems of BJ (i.e. formulas that are deducible from the empty set), whereas (MP), (ST.1), and (ST.2) can always be applied.

axioms of BJ + (AN!) $\vdash !^n c : !^{n-1} c : \dots : !c : c : A$ , where $(c, A) \in \text{CS}$ and $n \in \mathbb{N}$ (MP) if $T \vdash A$ and $T \vdash A \rightarrow B$ then $T \vdash B$ (CE) if $\vdash A$ then $\vdash P_{\geq 1} A$ (ST.1) if $T \vdash A \rightarrow CP_{\geq s - \frac{1}{k}}(B, C)$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \rightarrow CP_{\geq s}(B, C)$ (ST.2) if $T \vdash A \rightarrow D_{\geq r - \frac{1}{k}}(B, C)$ for every integer $k \geq \frac{1}{1+r}$ and $r > -1$ then $T \vdash A \rightarrow D_{\geq r}(B, C)$
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Fig. 1. System BJ<sub>CS</sub>

## 2.2 Semantics

To introduce semantics for BJ<sub>CS</sub>, we begin with the notion of a basic evaluation, which is the cornerstone for many interpretations of justification logic [5, 15]. In the following we use  $\mathcal{P}(X)$  to denote the power set of a set  $X$ .

**Definition 1 (Basic Evaluation).** *Let CS be a constant specification. A basic evaluation for CS, or a basic CS-evaluation, is a function  $*$  that maps atomic propositions to truth values and maps justification terms to subsets of Fml, i.e.*

$$* : \text{Prop} \rightarrow \{\text{T}, \text{F}\} \quad \text{and} \quad * : \text{Tm} \rightarrow \mathcal{P}(\text{Fml}),$$

such that for  $u, v \in \text{Tm}$ , for  $c \in \text{Con}$  and  $A, B \in \text{Fml}$  we have:

1.  $(A \rightarrow B \in u^* \text{ and } A \in v^*) \implies B \in (u \cdot v)^*$
2. if  $(c, A) \in \text{CS}$  then for all  $n \in \mathbb{N}$  we have<sup>3</sup>:

$$!^{n-1} c : !^{n-2} c : \dots : !c : c : A \in (!^n c)^*$$

We usually write  $t^*$  and  $p^*$  instead of  $*(t)$  and  $*(p)$ , respectively.

**Definition 2 (Algebra over a Set).** *Let  $W$  be a non-empty set and let  $H$  be a non-empty subset of  $\mathcal{P}(W)$ . We call  $H$  an algebra over  $W$  iff the following hold:*

- $W \in H$
- $U, V \in H \implies U \cup V \in H$
- $U \in H \implies W \setminus U \in H$

**Definition 3 (Finitely Additive Measure).** *Let  $H$  be an algebra over  $W$  and  $\mu : H \rightarrow [0, 1]$ . We call  $\mu$  a finitely additive measure iff the following hold:*

1.  $\mu(W) = 1$

<sup>3</sup> We agree to the convention that the formula  $!^{n-1} c : !^{n-2} c : \dots : !c : c : A$  represents the formula  $A$  for  $n = 0$ .

2. for all  $U, V \in H$ :

$$U \cap V = \emptyset \implies \mu(U \cup V) = \mu(U) + \mu(V)$$

**Definition 4 (Probability Space).** A probability space is a triple

$$P = \langle W, H, \mu \rangle,$$

where:

- $W$  is a non-empty set
- $H$  is an algebra over  $W$
- $\mu : H \rightarrow [0, 1]$  is a finitely additive measure

**Definition 5 (Model).** Let  $CS$  be a constant specification. A  $\text{BJ}_{CS}$ -model is a quintuple  $M = \langle U, W, H, \mu, * \rangle$  where:

1.  $U$  is a non-empty set of objects called worlds
2.  $W, H, \mu$  and  $*$  are functions, which have  $U$  as their domain, such that for every  $w \in U$ :
  - $\langle W(w), H(w), \mu(w) \rangle$  is a probability space with  $W(w) \subseteq U$
  - $*_w$  is a basic  $CS$ -evaluation<sup>4</sup>

The ternary satisfaction relation  $\models$  is defined between models, worlds, and formulas. We will use  $\mu_w$  for  $\mu(w)$ ,  $p_w^*$  for  $p^{*w}$ , and  $t_w^*$  for  $t^{*w}$ .

**Definition 6 (Truth in a  $\text{BJ}_{CS}$ -model).** Let  $CS$  be a constant specification and let  $M = \langle U, W, H, \mu, * \rangle$  be a  $\text{BJ}_{CS}$ -model. We define by simultaneous induction

1. what it means for a formula to hold in  $M$  at a world  $w \in U$  and
2. what it means for a formula to be measurable in  $M$  at a world  $w \in U$

as follows:

- $M, w \models p$  iff  $p_w^* = \top$  for  $p \in \text{Prop}$ ;
- $M, w \not\models \perp$ ;
- $M, w \models A \rightarrow B$  iff  $M, w \not\models A$  or  $M, w \models B$ ;
- $M, w \models t : A$  iff  $A \in t_w^*$ ;
- $M, w \models \text{CP}_{\geq s}(A, B)$  iff  $A \wedge B$  and  $B$  are measurable at  $w$  and either  $\mu_w([B]) = 0$ , or  $\mu_w([B]) > 0$  and  $\frac{\mu_w([A \wedge B])}{\mu_w([B])} \geq s$ ;
- $M, w \models \text{D}_{\geq s}(A, B)$  iff  $A$  and  $B$  are measurable at  $w$  and either  $\mu_w([B]) = 0$  and  $1 - \mu_w([A]) \geq s$ , or  $\mu_w([B]) > 0$  and  $\frac{\mu_w([A \wedge B])}{\mu_w([B])} - \mu_w([A]) \geq s$ .

We say a formula  $B$  is measurable in  $M$  at a world  $w \in U$  if the set

$$[B]_{M,w} := \{x \in W(w) \mid M, x \models B\}$$

is an element of  $H(w)$ .

<sup>4</sup> We will usually write  $*_w$  instead of  $*(w)$ .

**Definition 7 (Measurable Model).** Let  $CS$  be a constant specification and let  $M = \langle U, W, H, \mu, * \rangle$  be a  $BJ_{CS}$ -model.  $M$  is called measurable iff every formula  $A$  is measurable at each  $w \in U$ .  $BJ_{CS, Meas}$  denotes the class of measurable  $BJ_{CS}$  models.

Finally, we call a model Bayesian if terms of the form  $j_A$  represent Bayesian evidence.

**Definition 8.** A  $BJ_{CS}$ -model  $M = \langle U, W, H, \mu, * \rangle$  is called Bayesian model if at each  $w \in U$ ,

$$M, w \models D_{>0}(A, B) \quad \text{iff} \quad M, w \models j_B : A.$$

The class of Bayesian  $BJ_{CS}$ -models is denoted by  $BJ_{CS, Bayes}$ . The class of Bayesian measurable  $BJ_{CS}$ -models is denoted by  $BJ_{CS, Meas, Bayes}$ .

For a model  $M = \langle U, W, H, \mu, * \rangle$ ,  $M \models A$  means that  $M, w \models A$  for all  $w \in U$ . Let  $T \subseteq \text{Fml}$ . Then  $M \models T$  means that  $M \models A$  for all  $A \in T$ . Further  $T \models A$  means that for all  $M \in BJ_{CS, Meas, Bayes}$ ,  $M \models T$  implies  $M \models A$ .

To be precise we should write  $T \vdash_{CS} A$  and  $T \models_{CS} A$  instead of  $T \vdash A$  and  $T \models A$ , respectively, since these two notions depend on a given constant specification  $CS$ . However,  $CS$  will always be clear from the context and we thus omit it.

### 3 Soundness and Completeness for Bayesian Justification Logic

Soundness of  $BJ_{CS}$  can be proved by induction on the depth of derivations. To establish completeness, we make use of a canonical model construction. For lack of space, however, we cannot give a detailed completeness proof here. We will only present a series of definitions and lemmas (without proofs) that leads to the completeness result.

**Theorem 1 (Soundness).** Let  $CS$  be a constant specification. The axiomatic system  $BJ_{CS}$  is sound with respect to the class of  $BJ_{CS, Meas, Bayes}$ -models, i.e., for any formula  $A$  and any set  $T \subseteq \text{Fml}$  we have

$$T \vdash A \quad \implies \quad T \models A.$$

Now we define the notion of a  $BJ_{CS}$ -consistent sets.

**Definition 9 ( $BJ_{CS}$ -Consistent Sets).** Let  $CS$  be any constant specification and let  $T$  be a set of formulas.

- $T$  is said to be  $BJ_{CS}$ -consistent if and only if  $T \not\vdash_{BJ_{CS}} \perp$ . Otherwise  $T$  is said to be  $BJ_{CS}$ -inconsistent.
- $T$  is said to be maximal if and only if for every  $A \in \text{Fml}$  either  $A \in T$  or  $\neg A \in T$ .
- $T$  is said to be maximal  $BJ_{CS}$ -consistent if and only if it is maximal and  $BJ_{CS}$ -consistent.

We have the following deduction theorem for  $\text{BJ}_{\text{CS}}$ . The proof is similar to the one given in [13, 22].

**Theorem 2 (Deduction Theorem for  $\text{BJ}_{\text{CS}}$ ).** *Let  $T$  be a set of formulas and  $A$  and  $B$  be formulas. We have*

$$T, A \vdash B \quad \text{iff} \quad T \vdash A \rightarrow B.$$

The deduction theorem makes it possible to establish the following property of consistent sets of formulas, see [13, Lemma 27].

**Lemma 1.** *Let  $\text{CS}$  be a constant specification and let  $T$  be a  $\text{BJ}_{\text{CS}}$ -consistent set of formulas.*

1. *If  $\neg(B \rightarrow \text{CP}_{\geq s}(A, C)) \in T$  for  $s > 0$ , then there is some integer  $n \geq \frac{1}{s}$  such that  $T, \neg(B \rightarrow \text{CP}_{\geq s - \frac{1}{n}}(A, C))$  is  $\text{BJ}_{\text{CS}}$ -consistent.*
2. *If  $\neg(B \rightarrow \text{D}_{\geq r}(A, C)) \in T$  for  $r > -1$ , then there is some integer  $n \geq \frac{1}{r+1}$  such that  $T, \neg(B \rightarrow \text{D}_{\geq r - \frac{1}{n}}(A, C))$  is  $\text{BJ}_{\text{CS}}$ -consistent.*

The Lindenbaum lemma for probabilistic justification logics has been established in [13]. The proof for  $\text{BJ}_{\text{CS}}$  is similar.

**Lemma 2 (Lindenbaum).** *Let  $\text{CS}$  be a constant specification. Every  $\text{BJ}_{\text{CS}}$ -consistent set of formulas can be extended to a maximal  $\text{BJ}_{\text{CS}}$ -consistent set.*

**Definition 10 (Canonical Model).** *Let  $\text{CS}$  be a constant specification. The canonical model for  $\text{BJ}_{\text{CS}}$  is given by the quintuple  $M = \langle U, W, H, \mu, * \rangle$ , defined as follows:*

- $U = \{w \mid w \text{ is a maximal } \text{BJ}_{\text{CS}}\text{-consistent set of formulas}\}$
- for every  $w \in U$  the probability space  $\langle W(w), H(w), \mu(w) \rangle$  is defined as follows:
  1.  $W(w) = U$
  2.  $H(w) = \{(A)_M \mid A \in \text{Fml}\}$  where  $(A)_M = \{x \mid x \in U, A \in x\}$
  3. for all  $A \in \text{Fml}$ ,  $\mu(w)((A)_M) = \sup_s \{P_{\geq s} A \in w\}$
- for every  $w \in W$  the basic  $\text{CS}$ -evaluation  $*_w$  is defined as follows:
  1. for all  $p \in \text{Prop}$ :

$$p_w^* = \begin{cases} \text{T} & \text{if } p \in w \\ \text{F} & \text{if } \neg p \in w \end{cases}$$

2. for all  $t \in \text{Tm}$ :

$$t_w^* = \{A \mid t : A \in w\}$$

**Lemma 3.** *Let  $\text{CS}$  be a constant specification. The canonical model for  $\text{BJ}_{\text{CS}}$  is a  $\text{BJ}_{\text{CS}}$ -model.*

The following lemma is proved by induction on the complexity of the formula  $A$  where we make use of a complexity measure such that the complexity of  $\text{CP}_{\geq s}(B, C)$  and  $\text{D}_{\geq s}(B, C)$  is greater than the complexity of  $B \wedge C$ .

**Lemma 4.** *Let  $M = \langle U, W, H, \mu, * \rangle$  be the canonical model for  $\text{BJ}_{\text{CS}}$ . Then we have*

$$(\forall A \in \text{Fml})(\forall w \in U)[[A]_{M,w} = (A)_M].$$

From Lemma 4 we get the following corollary.

**Corollary 1.** *Let  $\text{CS}$  be any constant specification. The canonical model for  $\text{BJ}_{\text{CS}}$  is a  $\text{BJ}_{\text{CS},\text{Meas}}$ -model.*

Making use of the properties of maximal consistent sets, we can establish the truth lemma.

**Lemma 5 (Truth Lemma).** *Let  $\text{CS}$  be a constant specification and let  $M = \langle U, W, H, \mu, * \rangle$  be the canonical model for  $\text{BJ}_{\text{CS}}$ . For every  $A \in \text{Fml}$  and any  $w \in U$  we have:*

$$A \in w \iff M, w \models A.$$

Using the truth lemma we find that the canonical model satisfies the condition for Bayesian models, i.e. we have the following corollary.

**Corollary 2.** *Let  $\text{CS}$  be any constant specification. The canonical model for  $\text{BJ}_{\text{CS}}$  is a  $\text{BJ}_{\text{CS},\text{Meas},\text{Bayes}}$ -model.*

Finally, we get the completeness theorem as usual.

**Theorem 3 (Strong Completeness for BJ).** *Let  $\text{CS}$  be a constant specification, let  $T \subseteq \text{Fml}$  and let  $A \in \text{Fml}$ . Then we have:*

$$T \models A \implies T \vdash A.$$

## 4 Transitivity

It is well known that Bayesian confirmation is not transitive, i.e., the following principle is not valid

$$P(B|A) > P(B) \text{ and } P(C|B) > P(C) \implies P(C|A) > P(C) . \quad (2)$$

We refer to, e.g., [24, 27] for examples where transitivity fails.

It turns out, however, that there are conditions under which (2) holds. Shogenji [27] introduces the following condition, called *screening-off condition*,

$$P(C|A \wedge B) = P(C|B) \quad \text{and} \quad P(C|A \wedge \neg B) = P(C|\neg B) \quad (3)$$

and shows that transitivity holds under it. Intuitively, (3) means that once truth or falsity of  $B$  is known,  $A$  is irrelevant to the probability of  $C$ . In other words,  $A$  affects the probability of  $C$  only indirectly through its impact on  $B$  [24].

Roche [23] presents the following weakening of (3)

$$P(C|A \wedge B) \geq P(C|B) \quad \text{and} \quad P(C|A \wedge \neg B) \geq P(C|\neg B) . \quad (4)$$

and shows that transitivity also holds under this weaker condition.

We are now going to formalize this result in Bayesian justification logic. We show that we can represent (4) in BJ and that this condition entails transitivity of Bayesian justifications.

**Theorem 4.** *Let  $A$ ,  $B$ , and  $C$  be formulas of  $\text{Fml}$ . Let  $T$  be the set of formulas that consists of:*

1.  $\text{CP}_{=r}(C, B) \rightarrow \text{CP}_{\geq r}(C, A \wedge B)$  for all  $r \in S$ ,
2.  $\text{CP}_{=r}(C, \neg B) \rightarrow \text{CP}_{\geq r}(C, A \wedge \neg B)$  for all  $r \in S$ ,
3.  $\text{P}_{\neq 0}A$ ,  $\text{P}_{\neq 0}(A \wedge B)$ ,  $\text{P}_{\neq 0}(A \wedge \neg B)$ ,  $\text{P}_{\neq 0}B$ , and  $\text{P}_{\neq 0}\neg B$ .

*Then we have that*

$$T \vdash j_A : B \wedge j_B : C \rightarrow j_A : C .$$

Let  $M$  be any  $\text{BJ}_{\text{CS, Meas, Bayes}}$ -model such that  $M \models T$ . We observe that since  $M$  satisfies all formulas in  $T$ , the model  $M$  also satisfies condition (4). Thus we can show that  $M \models j_A : B \wedge j_B : C \rightarrow j_A : C$  by essentially following the original proof that transitivity holds under (4) given in [23]. The theorem follows by strong completeness of  $\text{BJ}_{\text{CS}}$ .

## 5 Conclusion

In this paper we have introduced  $\text{BJ}_{\text{CS}}$ , a family of justification logics that feature Bayesian confirmations. Because the language of Bayesian justification logics includes both probability operators and explicit justifications, we were able to define a class of models that satisfies condition (1). Hence  $\text{BJ}_{\text{CS}}$  not only includes justification terms built up from variables and constants, i.e. terms that represent assumptions and logical axioms, but also terms that represent Bayesian confirmations. In particular, a formula  $j_A : B$ , i.e.  $j_A$  justifies  $B$ , can be read as evidence  $A$  confirms  $B$  in the sense of *increase in firmness*.

We have established soundness and completeness of  $\text{BJ}_{\text{CS}}$  with respect to Bayesian models. Further we have shown that we can formalize the screening-off condition and that this condition entails transitivity of confirmation in Bayesian models.

Future work includes studying the computational properties of Bayesian justification logic, i.e., establishing decidability and complexity results, as well as developing a corresponding proof theory.

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