Justification logic with approximate conditional probabilities

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Abstract

The importance of logics with approximate conditional probabilities is reflected by the fact that they can model non-monotonic reasoning. We introduce a new logic of this kind, CPJ, which extends justification logic and supports non-monotonic reasoning with and about evidences.

1 Introduction

Justification logic [1] is a variant of modal logic that 'unfolds' the $\Box$-modality into justification terms, i.e., justification logics replace modal formulas $\Box\alpha$ with formulas of the form $t{:}\alpha$ that mean t *is a justification for the agent's belief (or knowledge) in* α. This interpretation of justification logic has many applications and has been successfully employed to analyze many different epistemic situations including certain forms of defeasible knowledge [2, 3, 4, 5, 12].

In a general setting, justifications need not to be certain. Milnikel [14] was the first to approach this problem with his logic of uncertain justifications. Kokkinis et al. [8, 9, 10] study probabilistic justification logic, which provides a very general framework for uncertain reasoning with justifications that subsumes Milnikel's system.

In the present paper we extend probabilistic justification logic with operators for approximate conditional probabilities. Formally, we introduce formulas $\mathsf{CP}_{\approx r}(\alpha,\beta)$ meaning *the probability of* α *under the condition* β *is approximately* r. This makes it possible to express defeasible inferences for justification logic. For instance, we can express

if x justifies that Tweety is a bird, then *usually* $t(x)$ justifies that Tweety flies

as $\mathsf{CP}_{\approx 1}(t(x){:}\mathsf{flies}, x{:}\mathsf{bird})$.

Our paper builds on previous work on probabilistic logics and non-monotonic reasoning. Logics with probability operators are important in artificial intelligence and computer science in general [6, 15]. They are interpreted over Kripke-style models with probability measures over possible worlds. Ognjanović and Rašković [16] develop probability logics with infinitary rules to obtain strong completeness results. The recent [17] provides an overview over the topic of probability logics.

Kraus et al. [11] propose a hierarchy of non-monotonic reasoning systems. In particular, they introduce a core system P for default reasoning and establish that P is sound and complete with respect to preferential models. Lehmann and Magidor [13] propose a family of non-standard ($^*\mathbb{R}$) probabilistic models. A default $\alpha \rightarrowtail \beta$ holds in a model of this kind if either

*Supported by the Ministry of education, science and technological development grants 174026 and III44006.
†Supported by the SNSF project 200021_165549 *Justifications and non-classical reasoning.*

the probability of α is 0 or the conditional probability of β given α is infinitesimally close to 1. Using this interpretation, they show that system P is also sound and complete with respect to ${}^*\mathbb{R}$-probabilistic models. Rašković et al. [18] present a logic with approximate conditional probabilities, $\mathsf{LPP}^{\mathsf{S}}$, whose models are a subclass of non-standard ${}^*\mathbb{R}$-probabilistic models. They prove the following: for any finite default base Δ and for any default $\alpha \rightarrowtail \beta$

$$\Delta \vdash_{\mathsf{P}} \alpha \rightarrowtail \beta \quad \text{iff} \quad \Delta \vdash_{\mathsf{LPP}^{\mathsf{S}}} \alpha \rightarrowtail \beta.$$

We will introduce operators for approximate conditional probabilities to justification logic. This makes it possible to formalize non-monotonic reasoning with and about evidences.

2 Basic Justification Logic J

Let C be a countable set of constants, V a countable set of variables, and Prop a countable set of atomic propositions. Justification terms and formulas are given as follows:

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid\ !t \qquad \text{and} \qquad \alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid t : \alpha$$

where $c \in C$, $x \in V$, and $p \in \mathsf{Prop}$. We denote the set of all justification formulas by $\mathsf{Fml}_{\mathsf{J}}$. Other classical Boolean connectives, $\vee$, $\rightarrow$, $\leftrightarrow$, as well as $\bot$ and $\top$, are defined as usual.

The axioms of the logic J are following:

all propositional tautologies	$u : (\alpha \rightarrow \beta) \rightarrow (v : \alpha \rightarrow u \cdot v : \beta)$
$u : \alpha \rightarrow u + v : \alpha$	$v : \alpha \rightarrow u + v : \alpha$

A set $\mathsf{CS} \subseteq \{(c, \alpha) \mid c \in C, \alpha$ is an instance of any axiom of $\mathsf{J}\}$ is called *constant specification*. For a given constant specification CS, we define the Hilbert-style *deductive system* J_{CS} by adding the following two rules to the axioms of J:

1. For $(c, \alpha) \in \mathsf{CS}$, $n \in \mathbb{N}$, infer $!^n c :!^{n-1} c : \cdots :!c : c : \alpha$
2. From α and $\alpha \rightarrow \beta$ infer β

A *basic evaluation* for J_{CS}, where CS is any constant specification, is a function $*$ such that $* : \mathsf{Prop} \rightarrow \{\text{true}, \text{false}\}$ and $* : \mathsf{Term} \rightarrow \mathcal{P}(\mathsf{Fml}_{\mathsf{J}})$, and for $u, v \in \mathsf{Term}$, any constant c and $\alpha \in \mathsf{Fml}_{\mathsf{J}}$ we have:

1. if there is $\beta \in v^*$ with $\beta \rightarrow \alpha \in u^*$, then $\alpha \in (u \cdot v)^*$
2. $u^* \cup v^* \subseteq (u + v)^*$
3. if $(c, \alpha) \in \mathsf{CS}$, then $\alpha \in c^*$ and for each $n \in \mathbb{N}$, $!^n c :!^{n-1} c : \cdots :!c : c : \alpha \in (!^{n+1} c)^*$.

Instead of writing $*(t)$ and $*(p)$, we write t^* and p^* respectively. Now, we are ready to define the notion of truth under a basic evaluation. The binary relation $\Vdash$ is defined by:

$* \Vdash p \quad \text{iff} \quad p^* = \text{true}$	$* \Vdash \neg\alpha \quad \text{iff} \quad * \nVdash \alpha$
$* \Vdash \alpha \wedge \beta \quad \text{iff} \quad * \Vdash \alpha \text{ and } * \Vdash \beta$	$* \Vdash t : \alpha \quad \text{iff} \quad \alpha \in t^*$

3 The Logic CPJ

Consider a non-standard elementary extension ${}^*\mathbb{R}$ of the real numbers. An element ϵ of ${}^*\mathbb{R}$ is called an infinitesimal iff $|\epsilon| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Let S be the unit interval of the Hardy field

$\mathbb{Q}[\epsilon]$, which contains all rational functions of a fixed positive infinitesimal ϵ of ${}^*\mathbb{R}$, for details see, e.g., [7].

The set of probabilistic formulas, denoted by $\mathsf{Fml_P}$, is the smallest set that contains all the formulas of the form

$$\mathsf{CP}_{\geq s}(\alpha,\beta) \qquad \mathsf{CP}_{\leq s}(\alpha,\beta) \qquad \mathsf{CP}_{\approx r}(\alpha,\beta)$$

for $\alpha,\beta \in \mathsf{Fml_J}$, $s \in S$, and $r \in \mathbb{Q}\cap[0,1]$ and that is closed under negation and conjunction. We use $\varphi,\psi,\dots$ to denote $\mathsf{Fml_P}$-formulas. The set of all formulas, Fml, of the logic CPJ is defined by $\mathsf{Fml} = \mathsf{Fml_J} \cup \mathsf{Fml_P}$. Elements of Fml will be denoted by $\theta,\theta_1,\theta_2,\dots$. We use the following standard abbreviations, see [18]:

$$\mathsf{CP}_{<s}(\alpha,\beta) \qquad \mathsf{CP}_{>s}(\alpha,\beta) \qquad \mathsf{CP}_{=s}(\alpha,\beta) \qquad \mathsf{P}_{\rho s}\alpha \text{ with } \rho \in \{\geq,\leq,>,<,=,\approx\}.$$

The semantics for the logic CPJ is based on possible worlds models. Let CS be the constant specification. A $\mathsf{CPJ_{CS}}$-*model* (or just model) is a tuple $M = \langle W,H,\mu,*\rangle$ where:

- W is a non-empty set of objects called worlds
- H is an algebra of subsets of W
- μ is a finitely additive probability measure on H
- $*$ is a function from W to all basic J_{CS}-evaluations. We write $*_w$ for $*(w)$.

Let $M = \langle W,H,\mu,*\rangle$. We put $[\alpha]_M := \{w \in W \mid *_w \Vdash \alpha\}$. Whenever M is clear from the context, we will write $[\alpha]$ instead of $[\alpha]_M$.

A $\mathsf{CPJ_{CS}}$-model M is *measurable* if and only if $[\alpha]_M \in H$, for every $\alpha \in \mathsf{Fml_J}$. A $\mathsf{CPJ_{CS}}$-model M is *neat* if and only if the empty set has the zero probability and no other set has. The class of all measurable and neat $\mathsf{CPJ_{CS}}$ models is denoted by $\mathsf{CPJ_{CS,Meas,Neat}}$.

Let CS be any constant specification. The *satisfaction relation* $\models\ \subseteq \mathsf{CPJ_{CS,Meas,Neat}} \times \mathsf{Fml}$ is defined, for any $M \in \mathsf{CPJ_{CS,Meas,Neat}}$, as follows:

1. $M \models \alpha$ if for every $w \in W$, $*_w \Vdash \alpha$,
2. $M \models \mathsf{CP}_{\leq s}(\alpha,\beta)$ if either $\mu([\beta]) = 0$ and $s = 1$, or $\mu([\beta]) > 0$ and $\frac{\mu([\alpha\wedge\beta])}{\mu([\beta])} \leq s$,
3. $M \models \mathsf{CP}_{\geq s}(\alpha,\beta)$ if either $\mu([\beta]) = 0$, or $\mu([\beta]) > 0$ and $\frac{\mu([\alpha\wedge\beta])}{\mu([\beta])} \geq s$,
4. $M \models \mathsf{CP}_{\approx r}(\alpha,\beta)$ if either $\mu([\beta]) = 0$ and $r = 1$, or $\mu([\beta]) > 0$ and for each $n \in \mathbb{N}$, $\frac{\mu([\alpha\wedge\beta])}{\mu([\beta])} \in [\max\{0, r-\frac{1}{n}\}, min\{1, r+\frac{1}{n}\}]$,
5. $M \models \neg\varphi$ iff it is not the case that $M \models \varphi$,
6. $M \models \varphi\wedge\psi$ iff $M \models \varphi$ and $M \models \psi$.

We assume that the conditional probability is by default 1, whenever the condition has the probability 0, which explains the formulation of case 3 in the above definition.

We introduce the following axiom system for $\mathsf{CPJ_{CS}}$, where we set $f(s,t) := \min\{1, s+t\}$, $r^- := \mathbb{Q}\cap[0,r)$, and $r^+ := \mathbb{Q}\cap(r,1]$:

Axiom schemes

1) all $\mathsf{J_{CS}}$-provable formulas	2) all $\mathsf{Fml_P}$-instances of classical tautologies
3) $\mathsf{CP}_{\geq 0}(\alpha,\beta)$	4) $\mathsf{CP}_{\leq s}(\alpha,\beta) \to \mathsf{CP}_{<t}(\alpha,\beta),\ t > s$
5) $\mathsf{CP}_{<s}(\alpha,\beta) \to \mathsf{CP}_{\leq s}(\alpha,\beta)$	6) $\mathsf{P}_{\geq 1}(\alpha\leftrightarrow\beta) \to (\mathsf{P}_{=s}\alpha \to \mathsf{P}_{=s}\beta)$
7) $\mathsf{P}_{\leq s}\alpha \leftrightarrow \mathsf{P}_{\geq 1-s}\neg\alpha$	8) $(\mathsf{P}_{=s}\alpha \wedge \mathsf{P}_{=t}\beta \wedge \mathsf{P}_{\geq 1}\neg(\alpha\wedge\beta)) \to \mathsf{P}_{=f(s,t)}(\alpha\vee\beta)$
9) $\mathsf{P}_{=0}\beta \to \mathsf{CP}_{=1}(\alpha,\beta)$	10) $(\mathsf{P}_{=t}\beta \wedge \mathsf{P}_{=s}(\alpha\wedge\beta)) \to \mathsf{CP}_{=\frac{s}{t}}(\alpha,\beta),\ t \neq 0$
11) $\mathsf{CP}_{\approx r}(\alpha,\beta) \to \mathsf{CP}_{\geq r_1}(\alpha,\beta),\ r_1 \in r^-$	12) $\mathsf{CP}_{\approx r}(\alpha,\beta) \to \mathsf{CP}_{\leq r_1}(\alpha,\beta),\ r_1 \in r^+$.

Inference Rules

1. From θ_1 and $\theta_1 \to \theta_2$ infer θ_2.
2. From α infer $\mathsf{P}_{\geq 1}\alpha$.
3. From the set of premises $\{\varphi \to \mathsf{P}_{\neq s}\alpha \mid s \in S\}$ infer $\varphi \to \bot$.
4. Let $r \in \mathbb{Q} \cap [0,1]$. From the two sets of premises $\{\varphi \to \mathsf{CP}_{\geq r-\frac{1}{n}}(\alpha,\beta) \mid n \geq \frac{1}{r}, n \in \mathbb{N}\}$ and $\{\varphi \to \mathsf{CP}_{\leq r+\frac{1}{n}}(\alpha,\beta) \mid n \geq \frac{1}{1-r}, n \in \mathbb{N}\}$ infer $\varphi \to \mathsf{CP}_{\approx r}(\alpha,\beta)$.

Axiom 3, putting $\top$ instead of β, says that the probability of each formula being satisfied in some set of worlds is at least 0, and we can easily infer (using $\neg\alpha$ instead of α) that the upper bound is 1, i.e. $\mathsf{P}_{\leq 1}\alpha$. Axioms 4 and 5 say that we can weaken the degree of confidence of truth, while Axiom 6 says that equivalent formulas have the same probability. Axiom 8 corresponds to finite addivity of a measure. Axiom 9 ensures that the conditional probability is equal to 1 whenever the condition has probability 0. Axiom 10 is the formula that states the standard definition of the conditional probability. Finally, the Axioms 11 and 12 (together with Inference Rule 4) give us the relationship between the conditional probability infinitesimally close to the some rational number $r \in [0,1]$ and the standard conditional probability.

Note that there are two bottom elements in Fml, namely $\bot_{\mathsf{J}} \in \mathsf{Fml}_{\mathsf{J}}$ and $\bot_{\mathsf{P}} \in \mathsf{Fml}_{\mathsf{P}}$. Accordingly we say that a set T of Fml-formulas is CS-*consistent* if $T \not\vdash_{\mathsf{CS}} \bot_{\mathsf{J}}$ and $T \not\vdash_{\mathsf{CS}} \bot_{\mathsf{P}}$.

Similar to [18], we can establish an extended completeness result.

Theorem 1. *Let CS be any constant specification. A set T of Fml-formulas is CS-consistent if and only if T has a $\mathsf{CPJ}_{\mathsf{CS},\mathsf{Meas},\mathsf{Neat}}$-model, i.e., there exists a $\mathsf{CPJ}_{\mathsf{CS},\mathsf{Meas},\mathsf{Neat}}$-model M with $M \models \theta$ for each $\theta \in T$.*

4 Conclusion

We extended probabilistic justification logic with operators for approximate conditional probabilities, which makes it possible to express defaults in justification logic. In particular:

$$\mathsf{CP}_{\approx 1}(t(x){:}\mathsf{flies}, x{:}\mathsf{bird}) \tag{1}$$

means *if x justifies that Tweety is a bird, then usually $t(x)$ justifies that Tweety flies*;

$$\mathsf{CP}_{\approx 1}(\neg t(x){:}\mathsf{flies}, x{:}\mathsf{penguin}) \tag{2}$$

means *if x justifies that Tweety is a penguin, then usually it is not the case that $t(x)$ justifies that Tweety flies*;

$$\mathsf{CP}_{\approx 1}(x{:}\mathsf{bird}, x{:}\mathsf{penguin}) \tag{3}$$

means *if x justifies that Tweety is a penguin, then usually x also justifies that Tweety is a bird.*

Similar to [13, 18], it is possible to show that (the corresponding translations) of the axioms and rules of system P are sound with respect to CPJ. In particular we can apply the rule of cautious monotonicity to (2) and (3) in order to infer

$$\mathsf{CP}_{\approx 1}(\neg t(x){:}\mathsf{flies}, x{:}\mathsf{penguin} \wedge x{:}\mathsf{bird}),$$

which is consistent with (1).

Besides the possibility of expressing defaults, CPJ also features non-monotonic versions of classical operations on justifications. Let us consider the sum operator with its defining axiom

$$u : \alpha \vee v : \alpha \to u + v : \alpha. \tag{4}$$

This axiom states that justifications are monotone: if u justifies α, then the combination of u with v still justifies α. Often the sum operation is motivated as follows. Think of u and v as two volumes of book collection and $u + v$ as the set of those two volumes. Imagine that volume u contains a justification for a proposition α, i.e., $u : \alpha$ is the case. Then the larger set $u + v$ also contains a justification for α, i.e., $u + v : \alpha$. This idea is reflected in the provability semantics of justification logic where the sum operation is interpreted as proof concatenation, which, of course, is monotone.

This motivational example can also be read in another way. It is possible that the second volume v contains a retraction of α, i.e., it withdraws the justification given for α in volume u. To model situations of this kind, one could introduce a non-monotonic sum operation, $\tilde{+}$, with

$$\mathsf{CP}_{\approx 1}(u \mathbin{\tilde{+}} v{:}\alpha, u{:}\alpha) \quad \text{and} \quad \mathsf{CP}_{\approx 1}(u \mathbin{\tilde{+}} v{:}\alpha, v{:}\alpha).$$

Using the (Or) rule of system P we get $\mathsf{CP}_{\approx 1}(u \mathbin{\tilde{+}} v{:}\alpha, u{:}\alpha \vee v{:}\alpha)$, which is a non-monotonic version of (4).

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