Cut elimination for the master modality*

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October 23, 2024

Abstract

In [11], we provided a method for eliminating cuts in non-wellfounded proofs with a local-progress condition, these being the simplest kind of non-wellfounded proofs. The method consisted of splitting the proof into nicely behaved fragments. This paper extends our method to proofs based on simple trace conditions. The main idea is to split the system with the trace condition into infinitely many local-progress calculi that together are equivalent to the original trace-based system. This provides a cut-elimination method using only basic tools of structural proof theory and corecursion, which is needed due to the non-wellfounded character of proofs. We will employ the method to obtain syntactic cut-elimination for K+, a system of modal logic with the master modality.

Introduction

Cut elimination in non-wellfounded/cyclic proof theory is currently an active topic of research and has been previously addressed by other researchers with different tools. For example via finite approximations in [1], via cyclic proofs in [2], via multicuts in [3], via runs in [4], via infinitary rewriting in [6], via ultrametric spaces in [7], [8] and [10], among others. First proof-theoretic results on establishing weakening admissibility in cyclic calculi for linear temporal logic can be found in [5]. The richness of methods is a witness of the hardness of the problem. We believe this hardness arises, among other things, from two fundamental facts:

1. In principle non-wellfounded proof do not have a notion of height associated to them. This means that we cannot do recursion over them.
2. Verifying that a tree is a proof in the finitary setting just requires to check a “local” condition globally, in particular that every node is the conclusion of a rule instance. In the non-wellfounded setting a global condition on the branches is added, proving that this condition is preserved after the process of cut elimination is the principal headache.

*Research supported by the Swiss National Science Foundation project 200021_214820.

In [11] we defined a method to establish cut elimination and, more generally, to make proof translations between local progress sequent calculi. These are sequent calculi allowing non-wellfounded proofs such that any infinite branch goes through some rules infinitely many times. The method is based in splitting the proof in adequate parts, each of which being amenable to the methods of recursion or corecursion. We left open the question whether this method can be employed to provide cut elimination for other non-wellfounded systems with more complex branch conditions. In this paper we answer this question positively providing cut elimination for a non-wellfounded system for the master modality (K^+). Our main contribution is a method to split a non-wellfounded calculus in infinitely many local-progress calculi, i.e., calculi where the global condition becomes amenable to a proof-theoretic treatment via our previously developed tools.

Cut elimination for K^+ was already proven by Shamkanov in [10] by the method of continuous cut elimination via ultrametric spaces.

0 Preliminaries

Master modality

We use this section to introduce some concepts that are needed for the rest of the paper.

K^+ is a modal logic, we will work with its formulation in the modal language with connectives $\bot, \to$ and unary modalities $\Box, \Box^+$. The operator $\Box^+$ is what we call the master modality. We use usual Kripke models for the semantics of this logic. If $\mathbf{M} = (W, R, V)$ is a Kripke model, then the semantics for box and master formulas is as follows:

$$\mathbf{M}, w \vDash \Box\phi \text{ iff for any } v \text{ such that } wRv \text{ we have that } \mathbf{M}, v \vDash \phi,$$
$$\mathbf{M}, w \vDash \Box^+\phi \text{ iff for any } v \text{ such that } wR^+v \text{ we have that } \mathbf{M}, v \vDash \phi,$$

where R^+ is the transitive closure of R, i.e. wR^+v iff there is a (non-empty) sequence $w_0, \ldots, w_n$ such that $w = w_0 R \cdots R w_n = v$.

Given a multiset of formulas Γ we will write $\boxdot^+\Gamma$ to mean $\Gamma, \Box^+\Gamma$.

Definition 1. We define the Hilbert system HK^+ as the Hilbert system over the language of the master modality with axioms:

1. $\phi \to (\psi \to \phi)$;
2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$;
3. $((\phi \to \bot) \to \bot) \to \phi$;
4. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
5. $\Box^+(\phi \to \psi) \to (\Box^+\phi \to \Box^+\psi)$;
6. $\Box^+\phi \to \Box\phi$;

7. $\Box^+\phi \to \Box\Box^+\phi$;
8. $\Box\phi \to (\Box^+(\phi \to \Box\phi) \to \Box^+\phi)$.

And the rules of modus ponens: and necessitation for $\Box^+$:

$$\frac{\phi \qquad \phi \to \psi}{\psi}\,\mathrm{MP} \qquad \frac{\phi}{\Box^+\phi}\,\mathrm{Nec}$$

■

As mentioned in [9], HK^+ is sound and weakly complete with respect to the K^+ semantics.

Shamkanov's non-wellfounded sequent calculus for $\mathbf{K}^+$

In [9] Shamkanov provided a non-wellfounded Gentzen calculus for K^+, let us denote it as $\mathrm{G}^\infty\mathrm{K}^+$ and its version with cut as $\mathrm{G}^\infty\mathrm{K}^+ + \mathsf{Cut}$. In $\mathrm{G}^\infty\mathrm{K}^+$ sequents are ordered pairs (Γ, Δ) where Γ, Δ are finite multisets of formulas. These are usually denoted as $\Gamma \Rightarrow \Delta$. Proofs are non-wellfounded trees, i.e. trees that are allowed to have infinite length paths from the root (infinite branches). It is easy to show that $\mathrm{G}^\infty\mathrm{K}^+$ is sound and that if $\mathrm{HK}^+ \vdash \phi$ then $\mathrm{G}^\infty\mathrm{K}^+ + \mathrm{Cut} \vdash\ \Rightarrow \phi$ (by induction in the size of the Hilbert proof and some inversion principles of $\mathrm{G}^\infty\mathrm{K}^+ + \mathrm{Cut}$), so $\mathrm{G}^\infty\mathrm{K}^+ + \mathrm{Cut}$ is also weakly complete. Then soundness of $\mathrm{G}^\infty\mathrm{K}^+$ is clear, (weak) completness will be a consequence of the cut elimination procedure we are going to describe in the paper.[1]

As usual with non-wellfounded proofs a condition limiting the possible shape of the infinite branches needs to be added, for the sake of soundness. This condition is simplified by turning to another non-wellfounded Gentzen calculus, which we will denote as $\mathrm{G}^\infty_\ell\mathrm{K}^+$, whose sequents are *annotated*. Let us define it precisely.

An *annotation* is just a formula or the symbol $\circ$, meaning that no formula is annotated. Sequents in $\mathrm{G}^\infty_\ell\mathrm{K}^+$ are triples (Γ, s, Δ), where Γ, Δ are finite multisets of formulas and s is an annotation. An additional condition is imposed on sequents, if the annotation of the sequent is a formula ϕ then $\Box^+\phi$ must occur in Δ. A sequent (Γ, s, Δ) is denoted by $\Gamma \Rightarrow_s \Delta$. In the case the annotation is a formula ϕ, we say that the formula is in *focus* and if it is $\circ$, we say that the sequent is *unfocused*. Then the rules of $\mathrm{G}^\infty_\ell\mathrm{K}^+$ are

$$\frac{}{\Gamma, p \Rightarrow_s p, \Delta}\,\mathrm{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow_s \Delta}\,\mathrm{Ax}\text{-}\bot$$

$$\frac{\Gamma \Rightarrow_s \Delta, \phi \qquad \Gamma, \psi \Rightarrow_s \Delta}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\,{\to}\mathrm{L} \qquad \frac{\Gamma, \phi \Rightarrow_s \psi, \Delta}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\,{\to}\mathrm{R}$$

[1] Note that in [9] the weak completeness of $\mathrm{G}^\infty\mathrm{K}^+$ is already proven via refutation trees. Then one could eliminate cuts via a semantical argument, since (weak) completeness was shown for the cut-free version of the calculus.

$$\frac{\Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_{s} \Box\phi, \Delta} \Box$$

$$\frac{\Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_{s} \Box^{+}\phi, \Delta} \Box^{+}$$

The condition on infinite branches is then that any branch has a suffix that always has the same formula ϕ in focus.[2]

We are allowed to move from studying cut elimination in $\mathrm{G}^{\infty}\mathrm{K}^{+}$ to study cut elimination in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}$ since it can be shown that

$$\mathrm{G}^{\infty}\mathrm{K}^{+} \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+} \vdash \Gamma \Rightarrow_{\circ} \Delta.$$

and the same if we have the cut rule in both. For the right to left we just need to delete the annotations and notice that the branch condition of $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}$ implies the branch condition of $\mathrm{G}^{\infty}\mathrm{K}^{+}$. For the left to right notice that by the shape of the rules there is only one possible way of annotating a proof in $\mathrm{G}^{\infty}\mathrm{K}^{+}$ such that the root is annotated with $\circ$. It is straightforward to see that the branch condition of $\mathrm{G}^{\infty}\mathrm{K}^{+}$ will imply that the branch condition of $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}$ is fulfilled in the annotated version of the proof.

Cut elimination for local-progress systems

As we mentioned in the introduction, we want to use our previously defined tools of [11] to prove cut elimination. Let us briefly introduce these tools here.

We work with local-progress non-wellfounded sequent calculi, let us fix an arbitrary one and call it $\mathcal{C}$. As usual, $\mathcal{C}$ has a set $\mathcal{R}$ whose elements are sequent rules, which we call the *rules of* $\mathcal{C}$. A pre-proof in $\mathcal{C}$ is a (possibly non-wellfounded) tree whose nodes are labelled by an ordered pair consisting of a sequent and a rule from $\mathcal{R}$ such that for any node with sequent S and rule R if $S_0, \dots, S_{n-1}$ are the sequents of its successors (in order) then $(S_0, \dots, S_{n-1}, S)$ must be a rule instance of R.

In addition, $\mathcal{C}$ has a function L called the *local progress function.* Given a rule instance r with premises $S_0, \dots, S_{n-1}$ and conlusion S of some rule $R \in \mathcal{R}$, we have that $L_R(S_0, \dots, S_{n-1}, S) \subseteq \{0, \dots, n-1\}$. We interpret this function as follows, imagine we have a pre-proof in $\mathcal{C}$ and w is a node with n successors. The node w is labelled with the sequent S and rule R and each of the successors is labelled with the sequent $S_0, \dots, S_{n-1}$, respectively. Then we will say that from w to its i-th successor there is

[2] This particular system is not defined by Shamkanov but is easily infered from the systems he defines. In particular, in [9] he defines this system with extra annotations since the purpouse is to prove a realization theorem of justification logic. In [10] he defines a system of K^{+} with the same annotations but the sequents have an extra (possibly infinite) set of formulas in order to establish a connection with an ω-rule system for K^{+}. In any case, the definition of the system here is to have the exact and precise definition for our purposes, but it is not original in any sense.

progress iff $i \in L_R(S_0, \dots, S_{n-1}, S)$. A *proof* in $\mathcal{C}$ is a pre-proof such that in any infinite branch there is infinitely often progress from a node in the branch to its successor in the branch.

When defining local progress system it will be usual to describe the rules as

$$\frac{\mathcal{S}_0 \qquad \cdots \qquad \mathcal{S}_{n-1}}{\mathcal{S}}\,\mathrm{R}$$

where $S_0, \dots, S_{n-1}, S$ are sequent-schemes. This defines the rule R whose rules instances are the tuples $(S_0, \dots, S_{n-1}, S)$ where S_i is an instantiation of $\mathcal{S}_i$ and S is an instantiation of $\mathcal{S}$. In addition we will say that progress only occurs at the $i_0, \dots, i_k$-th premises of R to mean that $L_R(S_0, \dots, S_{n-1}, S) = \{i_0, \dots, i_k\}$, where $(S_0, \dots, S_{n-1}, S)$ is any rule instance of R. In case we do not mention anything about progress in a rule R or we say that in R there is no progress, we mean that $L_R(r) = \varnothing$ for any rule instance r of R. [3]

Any proof in a local-progress system can be divided into (possibly infinitely many) finite trees, creating a partition of the nodes of the proof. An element of this partition is called a *local fragment* of the proof, the idea is that two nodes are in the same local fragment iff there is a (non-directed) path in the tree that allows to go from one node to the other without encountering progress. In particular, note that inside a local fragment no progress can occur and that all progress in the proof occurs from a leaf of a local fragment to the root of another local fragment. The *main local fragment* of a proof is the fragment starting at the root. The *local height* of the proof will be the height of the main local fragment, for a proof π this height will be denoted as $|\pi|$.

The tools that we developed in [11] basically allowed us to do corecursion at the level of local fragments. In other words, in a corecursive step we will give not only the last node of the resulting proof after the corecursion, but the whole main local fragment. For details consult Section 3 in [11].

1 Definition of α-$\mathbf{G}_\ell^\infty\mathbf{K}_s^+$

For the Gentzen calculi we are going to define we need to consider the concept of proof with witnesses. Usualy a Gentzen calculi rule is of the shape

$$\frac{S_0 \qquad \cdots \qquad S_{n-1}}{S}\,\mathrm{R}$$

However, we are going to use rules of shape

$$\frac{\overline{\pi_0 \vdash S_0} \qquad \cdots \qquad \overline{\pi_{m-1} \vdash S_{m-1}} \qquad S_m \quad \cdots \quad S_{m+n-1}}{S}\,\mathrm{R}$$

where $\pi_0, \dots, \pi_{m-1}$ will be proofs of $S_0, \dots, S_{m-1}$, respectively, in previously defined Gentzen calculi. In the proof tree this will look as a node with the label $(S, R, \pi_0, \dots, \pi_{m-1})$ and n successors, labelled with sequents S_m to S_{m+n-1} respectively.

[3] In this paper we will work with a system where progress will occur at most in one premise of the rule instance.

In such a rule instance $\pi_0, \ldots, \pi_{m-1}$ are called *witnesses.* Given a proof in a system with this kind of rules we will sometimes call it a *proof with witnesses* and the witnesses are all the witnesses that occur at some of its rule instances. The *main global fragment* of such a proof is its structure without considering the witnesses.

When we talk about a subproof in a calculus of this kind we mean a proof generated by taking one of its nodes as the root. In particular, these nodes always belong to the main global fragment, so the witnesses are not subproofs.

Note that in a proof π we may have that a witness τ is a proof with witnesses so it also has witnesses. The witnesses of τ are not witnesses of π (i.e. being a witness is not transitive). In this setting we have to understand that a cut-free proof is not only a proof without instances of cut, it is a proof without instances of cut such that all its witnesses have no instances of cut either, neither the witnesses of witnesses,... and so on.

Definition 2. For each ordinal α and annotation s, we define the system $\alpha\text{-}\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ as the local progress system with rules:

$$\frac{}{\Gamma, p \Rightarrow_s p, \Delta}\ \mathrm{Ax} \qquad\qquad \frac{}{\Gamma, \bot \Rightarrow_s \Delta}\ \mathrm{Ax}\text{-}\bot$$

$$\frac{\Gamma \Rightarrow_s \Delta, \phi \qquad \Gamma, \psi \Rightarrow_s \Delta}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\ {\to}\mathrm{L} \qquad\qquad \frac{\Gamma, \phi \Rightarrow_s \psi, \Delta}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\ {\to}\mathrm{R}$$

$$\frac{\overline{\tau \vdash^{\beta} \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi}}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box\phi, \Delta}\ \Box$$

$$s = \phi\ \frac{\overline{\tau \vdash^{\beta} \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi} \qquad \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box^{+}\phi, \Delta}\ \Box^{+}_{f}$$

$$s \neq \phi\ \frac{\overline{\tau_0 \vdash^{\beta} \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi} \qquad \overline{\tau_1 \vdash^{\gamma} \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box^{+}\phi, \Delta}\ \Box^{+}_{u}$$

where $\beta, \gamma < \alpha$ and $\tau \vdash^{\alpha} \Gamma \Rightarrow_{s'} \Delta$ means that τ is a proof of $\Gamma \Rightarrow_{s'} \Delta$ in $\alpha\text{-}\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s'}$. We note that $\Gamma, \Delta, \Sigma, \Pi$ are finite multisets of formulas and, in particular, may be empty.

In instances of the rules $\to$ L, $\to$ R, $\Box$, $\Box^{+}_{f}$ and $\Box^{+}_{u}$, we define the *principal formula* to be the displayed formula in the conclusion. In the modal rules, the multisets Σ and Δ will be called the *weakening part* of the rule instance.

Progress occurs at the right premise of the $\Box^{+}_{f}$ rule. Notice that $\Box^{+}_{f}$ and $\Box^{+}_{u}$ have side conditions $s = \phi$ and $s \neq \phi$, respectively. This, together with the progress conditon implies that if $s = \circ$ the system is in fact finitary (every branch will be finite). ■

As we stated in the preliminaries, a proof in a local-progress calculus can be splitted into local fragments. In particular, each proof in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$, has a local fragment starting at the root and local fragments starting at each right premise of a $\Box^{+}_{f}$ rule instance.

The s in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ determines which annotation is in focus, so when the annotation in focus is changed we are forced to go to a witness with a different annotation (see the $\Box^{+}_{u}$ rule).

Definition 3: Adding cuts**.** We define the cut rule

$$\frac{\Gamma \Rightarrow_s \Delta, \chi \qquad \chi, \Gamma \Rightarrow_s \Delta}{\Gamma \Rightarrow_s \Delta}\ \mathsf{Cut}$$

which makes no progress.

For each α we define the systems:

1. We define α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{Cut}$ to be the systems as in Definition 2 with adding Cut to the list of rules (in particular, Cut may be used in the main global fragment, in witnesses, in witnesses of witnesses, and so on).
2. α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{wCut}$ are the system as in Definition 2 with allowing the witnesses to belong to β-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{Cut}$ for $\beta < \alpha$ and **not** adding the Cut rule.
3. α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{mCut}$ are the systems as in Definition 2 with the same witnesses **but** adding the Cut rule, i.e. Cut may occur in the main global fragment but the witnesses must be cut-free.

A proof in the second system is said to have *witnesses-cuts* only and a proof in the third system is said to have *main-cuts* only. ■

Definition 4. We say that π is a proof in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ iff there is an α such that π is a proof in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ and similarly iff we add Cut, mCut or wCut. If π is a proof in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{Cut}$ we will denote by $\|\pi\|$ the minimum ordinal α such that π is a proof in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{Cut}$. This ordinal is called the *(ordinal) height* of the proof. The (ordinal) height of a node in a proof is the height of the subproof at that node. ■

Given a cut in a proof we can talk about its *size* (which is the size of its cut formula) and its *height* (which is the Hessenberg sum of the ordinal height of its premises).

The next proposition claims trivial facts about the three possible ways of adding cut to systems α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ and establishes the connection to system $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}$.

Proposition 5. We have that

1. If π is a proof in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ then it is also a proof in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{wCut}$ and in α-$\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{mCut}$.

2. If π is a proof in α-$\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{wCut}$ or in α-$\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut}$ then it is a proof in α-$\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$.
3. $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta$ iff $\mathrm{G}_\ell^\infty\mathrm{K}^+ \vdash \Gamma \Rightarrow_s \Delta$ and similarly if we add Cut.

Proof. The first two claims are proven simultaneously by induction on α. The third claim is proven by induction on $\|\pi\|$ where for π in $\mathrm{G}_\ell^\infty\mathrm{K}^+$ (+Cut), $\|\pi\|$ refers to the the definition of height in [10]. $\square$

Our objective will be to show that if $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut} \vdash \Gamma \Rightarrow_s \Delta$ then $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta$. Broadly, we will prove this in three steps (see Theorem 25):

1. If $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut} \vdash \Gamma \Rightarrow_s \Delta$ then $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut} \vdash \Gamma \Rightarrow_s \Delta$;
2. If $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut} \vdash \Gamma \Rightarrow_s \Delta$ then $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{wCut} \vdash \Gamma \Rightarrow_s \Delta$, with the additional property that the proof in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{wCut}$ will only have finitely many cuts in each witness; and
3. If $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{wCut} \vdash \Gamma \Rightarrow_s \Delta$ with only finitely many cuts in each witness, then (using cut admissibility) $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta$.

2 Change of annotations

Sometimes we have a proof of a sequent $\Gamma \Rightarrow_s \Delta$ and we need a proof of $\Gamma \Rightarrow_{s'} \Delta$, i.e. we need to change the annotation of the sequent. As we will show in the following definition, it is possible to define such a translation of proofs. However, it may change the (ordinal) height of the proof.

Definition 6. Given π in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+$ we can define the proof $\pi^{s'}$ of the same sequent in $\mathrm{G}_\ell^\infty\mathrm{K}_{s'}^+$. If $s = s'$ we just return the same proof, otherwise we proceed by induction in the local height and cases in the last rule applied:

$$\frac{}{\Gamma, p \Rightarrow_s p, \Delta}\,\mathrm{Ax} \mapsto \frac{}{\Gamma, p \Rightarrow_{s'} p, \Delta}\,\mathrm{Ax}$$

$$\frac{}{\Gamma, \bot \Rightarrow_s \Delta}\,\mathrm{Ax}\text{-}\bot \mapsto \frac{}{\Gamma, \bot \Rightarrow_{s'} \Delta}\,\mathrm{Ax}\text{-}\bot$$

$$\frac{\begin{matrix}\pi_0 \\ \Gamma \Rightarrow_s \Delta, \phi\end{matrix} \qquad \begin{matrix}\pi_1 \\ \Gamma, \psi \Rightarrow_s \Delta\end{matrix}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\,{\to}\mathrm{L} \mapsto \frac{\begin{matrix}\pi_0^{s'} \\ \Gamma \Rightarrow_{s'} \Delta, \phi\end{matrix} \qquad \begin{matrix}\pi_1^{s'} \\ \Gamma, \psi \Rightarrow_{s'} \Delta\end{matrix}}{\Gamma, \phi \to \psi \Rightarrow_{s'} \Delta}\,{\to}\mathrm{L}$$

$$\frac{\begin{matrix}\pi_0 \\ \Gamma, \phi \Rightarrow_s \psi, \Delta\end{matrix}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\,{\to}\mathrm{R} \mapsto \frac{\begin{matrix}\pi_0^{s'} \\ \Gamma, \phi \Rightarrow_{s'} \psi, \Delta\end{matrix}}{\Gamma \Rightarrow_{s'} \phi \to \psi, \Delta}\,{\to}\mathrm{R}$$

$$\dfrac{\overline{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta}\,\Box \mapsto \dfrac{\overline{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_{s'} \Box\phi, \Delta}\,\Box$$

If π is:

$$\dfrac{\overline{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi} \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\,\Box^+_f$$

(so $s = \phi$ and then $s' \neq \phi$) then it maps to

$$\dfrac{\overline{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi} \qquad \overline{\pi_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_{s'} \Box^+\phi, \Delta}\,\Box^+_u$$

If π is:

$$\dfrac{\overline{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi} \qquad \overline{\tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\,\Box^+_u$$

then it maps to

$$\dfrac{\overline{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi} \qquad \overline{\tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_{s'} \Box^+\phi, \Delta}\,\Box^+_u$$

in case $s' \neq \phi$, or it maps to

$$\dfrac{\overline{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi} \qquad \begin{array}{c}\tau_1\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_{s'} \Box^+\phi, \Delta}\,\Box^+_f$$

in case $s' = \phi$.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_s \Delta, \chi\end{array} \qquad \begin{array}{c}\pi_1\\ \chi, \Gamma \Rightarrow_s \Delta\end{array}}{\Gamma \Rightarrow_s \Delta}\,\mathrm{Cut} \mapsto \dfrac{\begin{array}{c}\pi_0^{s'}\\ \Gamma \Rightarrow_{s'} \Delta, \chi\end{array} \qquad \begin{array}{c}\pi_1^{s'}\\ \chi, \Gamma \Rightarrow_{s'} \Delta\end{array}}{\Gamma \Rightarrow_{s'} \Delta}\,\mathrm{Cut}$$

∎

After a change of annotation the height of the proof may be altered. In the next lemma we prove some bounds to its change.

Lemma 7. Let ϕ be a formula and s be an annotation. We have that

1. If π is a proof in $\mathrm{G}_\ell^\infty\mathrm{K}_\phi^+$, then $\|\pi^s\| \leq \|\pi\| + 1$.
2. If π is a proof in $\mathrm{G}_\ell^\infty\mathrm{K}_\circ^+$, then $\|\pi^s\| \leq \|\pi\|$.

Proof. By induction in the local height. In 1. the possible increase occurs in the $\Box_f^+$-case since it can turn a subproof π' with (possibly) $\|\pi'\| = \|\pi\|$ to a witness which will impose that the new height is strictly bigger ordinal. Notice that in 2. this is not possible since there are no instances of $\Box_f^+$ in the original proof, only of $\Box_u^+$. $\square$

In the future, we will need to apply this construction while we prove cut admissibility and cut elimination. However, in order to use this translation adequately in the proofs, we will need to have some additional properties. For example, if we apply this translation to a proof without cuts it must remain without cuts after the translation. We will now describe the conditions that we need for this translation and other auxiliary translations such as weakening, contraction and inversion. But first we need to define a special type of cut, local cuts:

Definition 8. Let π be a proof in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathrm{Cut}$. We say that π has *local cuts only* in case π is a proof in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathrm{mCut}$ and all the instances of cut occurs at the main local fragment (i.e. before any modal rule). $\blacksquare$

Definition 9. Let f be an n-ary function whose domain is some set of tuples of proofs and its range is some sets of proofs. We say that f preserves

1. *Ordinal height* iff given a tuple $(\pi_0, \ldots, \pi_{n-1})$ in the domain of f the height of $f(\pi_0, \ldots, \pi_{n-1})$ is smaller or equal than the maximum of heights of the π_i's.
2. *Local height* iff given a tuple $(\pi_0, \ldots, \pi_{n-1})$ in the domain of f the height of $f(\pi_0, \ldots, \pi_{n-1})$ is smaller or equal than the maximum of local heights of the π_i's.
3. *Size of cuts* iff given a tuple $(\pi_0, \ldots, \pi_{n-1})$ in the domain of f such that each π_i has all its cuts are of size smaller than n, then $f(\pi_0, \ldots, \pi_{n-1})$ has also all its cuts of size smaller than n.[4] In particular, if the π_i's are cut-free (i.e. the sizes of cuts are < 0) then $f(\pi_0, \ldots, \pi_{n-1})$ is also cut-free.
4. *Freeness of cuts in the main local fragment* iff given a tuple $(\pi_0, \ldots, \pi_{n-1})$ in the domain of f such that each π_i without cuts in its main local fragment, then $f(\pi_0, \ldots, \pi_{n-1})$ does not have cuts in its main local fragment neither.
5. *Locality of cuts* iff given a tuple $(\pi_0, \ldots, \pi_{n-1})$ in the domain of f such that each π_i has local cuts only, then $f(\pi_0, \ldots, \pi_{n-1})$ also has local cuts only.
6. *Locality of cuts in witnesses* iff given a tuple $(\pi_0, \ldots, \pi_{n-1})$ in the domain of f such that each π_i has witnesses with local cuts only, then the witnesses of $f(\pi_0, \ldots, \pi_{n-1})$ have local cuts only.

If f preserves local height, sizes of cuts, freeness of cuts in the main local fragment and locality of cuts we say that it is *weakly preserving.* If f presreves all the properties described above we will say that it is *strongly preserving.* $\blacksquare$

[4] With all its cuts here we mean the cuts in the main global fragment of the proof, in the witnesses, in the witnesses of witnesses, and so on.

The following observation is straightforward from the definition.

Proposition 10. Change of annotations, i.e. the function $\pi \mapsto \pi^s$, is weakly preserving.

3 Auxiliary functions

In this section we just state one big lemma with all the auxiliary functions that we will need (apart from change of annotations, see Section 2). All the auxiliary functions can be defined straightforwardly by recursion in the local height and all its properties can be shown straightforwardly by induction in the local height. If the reader wants to see explicit definitions they are encouraged to look in the appendix chapters A, B and C.

Lemma 11. Let p be a propositional variable, χ, χ_0, χ_1 be formulas and $\Gamma, \Gamma', \Delta, \Delta'$ be multisets of formulas. There are **strongly preserving** functions $\mathsf{wk}_{\Gamma';\Delta'}$, lctr_p, rctr_p, $\mathsf{rctr}_{\Box^+\chi}$, $\mathsf{inv}_\bot$, $\mathsf{linv}^0_{\chi_0\to\chi_1}$, $\mathsf{linv}^1_{\chi_0\to\chi_1}$, $\mathsf{rinv}_{\chi_0\to\chi_1}$ from proofs in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s + \mathsf{Cut}$ to proofs in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s + \mathsf{Cut}$ such that:

1. $\pi \vdash \Gamma \Rightarrow_s \Delta$ implies $\mathsf{wk}_{\Gamma';\Delta'}(\pi) \vdash \Gamma, \Gamma' \Rightarrow_s \Delta, \Delta'$.
2. $\pi \vdash \Gamma, p, p \Rightarrow_s \Delta$ implies $\mathsf{lctr}_p(\pi) \vdash \Gamma, p \Rightarrow_s \Delta$.
3. $\pi \vdash \Gamma \Rightarrow_s \Delta, p, p$ implies $\mathsf{rctr}_p(\pi) \vdash \Gamma \Rightarrow_s \Delta, p$.
4. $\pi \vdash \Gamma \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi$ implies $\mathsf{rctr}_{\Box^+\chi}(\pi) \vdash \Gamma \Rightarrow_s \Delta, \Box^+\chi$.
5. $\pi \vdash \Gamma \Rightarrow_s \Delta, \bot$ implies $\mathsf{ctr}_\bot(\pi) \vdash \Gamma \Rightarrow_s \Delta$.
6. $\pi \vdash \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta$ implies $\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi) \vdash \Gamma \Rightarrow_s \Delta, \chi_0$.
7. $\pi \vdash \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta$ implies $\mathsf{linv}^1_{\chi_0\to\chi_1}(\pi) \vdash \Gamma, \chi_1 \Rightarrow_s \Delta$.
8. $\pi \vdash \Gamma \Rightarrow_s \Delta, \chi_0 \to \chi_1$ implies $\mathsf{rinv}_{\chi_0\to\chi_1}(\pi) \vdash \Gamma, \chi_0 \Rightarrow_s \Delta, \chi_1$.

4 Pushing cuts

The first task to prove cut elimination will be to show that it is possible to push cuts to upper parts of the proof. We will do this in two stages: first pushing them outside the main local fragment recursively and then pushing them outside the main global fragment to witnesses corecursively.

4.1 Pushing a cuts outside main local fragment

Definition 12. Let $\pi \vdash \Gamma \Rightarrow_s \Delta, \chi$ and $\tau \vdash \chi, \Gamma \Rightarrow_s \Delta$. We define $\mathsf{cut}^\chi(\pi, \tau)$ to be the result of applying the rule Cut to π and τ.

Notice that it preserves locality of cuts. ∎

We start by pushing only one cut that we assume to be a top-most cut in the main local fragment, i.e. there are no cuts above it belonging to the main local fragment.

Lemma 13. There is a binary function $\mathsf{push\text{-}top}_\chi$ whose domain are the pairs (π, τ) of proofs such that

1. π and τ are proofs in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{Cut}$ with no cuts in their main local fragment, and
2. we have that $\pi \vdash \Gamma \Rightarrow_s \Delta, \chi$ and $\tau \vdash \chi, \Gamma \Rightarrow_s \Delta$ for some Γ, Δ,

and that returns a proof in $\mathsf{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{Cut}$ with the following properties:

1. $\mathsf{push\text{-}top}_\chi(\pi, \tau) \vdash \Gamma \Rightarrow_s \Delta$ and has no cuts in its main local fragment,
2. if the size of cuts in π, the size of cuts in τ and the size of the formula χ are strictly smaller than n, then the size of cuts in $\mathsf{push\text{-}top}_\chi(\pi, \tau)$ are strictly smaller than n, and
3. $\mathsf{push\text{-}top}$ preserves locality of cuts in witnesses.

Proof. We proceed by induction in the pair $(|\chi|, |\pi| + |\tau|)$ (i.e. the size of χ and the sum of the local heights of $|\pi|$ and $|\tau|$) ordered lexicographically. All the possible cut reductions the displayed in Appendix D, we are going to argue that the conditions 1 to 3 are fulfilled after each of the reductions, and we have to say what each cut_i is when it occurs in a cut reduction.

The case where the label is the cut formula and the axiomatic cases are straightforward using the properties of the auxiliary functions.

In the principal reduction case, cut_1 and cut_2 are just applications of the I.H. (Induction Hypothesis) using that the size of the cut formula is strictly smaller. No cut will be in the main local fragment and the condition on the size will be fulfilled since we use the I.H. with smaller cut formulas. If the original proof has witnesses with local cuts only, then so does the proof after the cut reduction just by using the I.H. In this argument, we have to use that the auxiliary functions used are strongly preserving.

For the commutative implication cases, each cut_i will just be an application of the I.H. using that the sum of local heights is smaller with the same cut formula χ. That the conditions are fulfilled can be seen by using that the auxiliary functions used are strongly preserving and employing the I.H. in a similar manner to previous case (although for the bound in the sizes of cuts we now use that the I.H. is used with χ, and not with a smaller cut formula).

We turn to the modal cases. If the cut formula is in the weakening part of a modal rule, all the desired properties are easily fulfilled. So we turn to the commutative cases. In these, cut_i will always be a proper application of the cut rule. We note that still the main local fragment will have no application of cut since all the cut_i appear outside the main local fragment. The condition on the sizes of cuts is easily fulfilled since the cut formulas of the cut_i are smaller or equal than the original cut formula (the cut formula will be of shape $\Box\chi_0$ or $\Box^+\chi_0$ and the cut_i's will have either the cut formula or χ_0 only). We also note that all the auxiliary functions used are at least weakly preserving, so using them will not break this property.

Lastly we argue for the locality of cuts in witnesses. We note one thing: when we apply an auxiliary function to a subproof of the main global fragment of the original proof, we want to use the preservation of locality of cuts in witnesses, while if we apply an auxiliary function to a witness, we want to use the preservation of locality of cuts (because the witness have the cuts local while the subproof will have the property that its witnesses have cuts local).[5] Strongly preserving auxiliary functions preserves both, but change of annotations only preserve locality of cuts. However, change of annotations is only applied to witnesses. Also, the cut_i's we add at witnesses are always in the main local fragment (since we add them at the bottom and we do not add any modal rule at witnesses).

Let us explicitely discuss what occurs at $\Box$-$\Box_f^+$ and $\Box_u^+$-$\Box_f^+$ reductions, since these cases are a little harder. In these cases some witnesses become subproofs after the cut reduction, π_0 in the first reduction and both π_0 and π_1 in the second. This is non-problematic for our reasoning since we know that after applying all the auxiliary functions to those witnesses they have local cuts only, and then all their witness have no cuts so in particular all the witnesses have local cuts only. In other words, we are adding the cuts in the main local fragment of the witnesses to the main global fragment of the proof after the cut reduction, thus enlarging the number of cuts in the main global fragment. This is non-problematic because the cuts added in this way are outside the main local fragment, which is what we want to have cut-free in this lemma. $\Box$

Lemma 14. There is a function $\mathsf{push\text{-}local}$ from proofs in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$ to proofs in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$ such that:

1. for any $\pi \vdash \Gamma \Rightarrow_s \Delta$ proof in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$ we have that $\mathsf{push\text{-}local}(\pi) \vdash \Gamma \Rightarrow_s \Delta$ with no cuts in its main local fragment,
2. $\mathsf{push\text{-}local}$ preserves sizes of cuts, and
3. $\mathsf{push\text{-}local}$ preserves locality of cuts in witnesses.

Proof. The proof is a simple induction in the number of cuts in the main local fragment of π, using $\mathsf{push\text{-}top}$. $\Box$

4.2 Pushing cuts outside main global fragment

Lemma 15. There is a function push from proofs in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$ to proofs in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{wCut}$ (so without cuts in its main global fragment) such that:

1. for any $\pi \vdash \Gamma \Rightarrow_s \Delta$ proof in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$, we have $\mathsf{push}(\pi) \vdash \Gamma \Rightarrow_s \Delta$,

[5] We remember that witnesses are not subproofs (neither the opposite).

2. push preserves sizes of cuts, and

3. push preserves locality of cuts in witnesses.

Proof. To obtain push it suffices to apply $\mathsf{push\text{-}local}$ corecursively through the main global fragment. By the first condition of $\mathsf{push\text{-}local}$ it is clear that the resulting proof will have no cuts in its main global fragment since all local fragments constituting the new proof are cut-free.

If we have a bound n on the size of cuts, by the conditions of $\mathsf{push\text{-}local}$, we have that each of the local fragments in the new proof will have the same bound for cuts. But note that if each local fragment of a proof has all the cut-sizes bounded by n, the all cut-sizes of the proof are bounded by n.

For the condition on witnesses we have the same, by $\mathsf{push\text{-}local}$ we know that if we start with a proof whose witnesses have local-cuts only then each local fragment in the resulting proof after the corecursion will have witnesses with local-cuts only. But a proof with such local fragments will be a proof whose witnesses have local-cuts only, as desired. $\square$

5 Cut admissibility

Our strategy to show cut elimination will require first to show that the rule Cut is admissible. Notice that for finitary proofs it is straightforward to prove cut elimination from cut admissibility by an induction in the height of the proof. Our cut elimination will follow a similar approach, but it is more involved since our notion of height is not so straightforward (we do not decrease height by going to a child node but only by going to a witness).

With cut admissibility in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ we mean that for any Γ, Δ, χ if $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta, \chi$ and $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \chi, \Gamma \Rightarrow_s \Delta$ then $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta$.

If we talk about cut admissibility for χ we mean the statement that for any Γ, Δ if $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta, \chi$ and $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \chi, \Gamma \Rightarrow_s \Delta$ then $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta$, and if we talk about cut admissibility for χ with cuts of height smaller than α we mean the statement that for any $\Gamma, \Delta, \pi, \tau$ if $\pi \vdash \Gamma \Rightarrow_s \Delta, \chi$, $\tau \vdash \chi, \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ and $\|\pi\| \oplus \|\tau\| < \alpha$ (where $\oplus$ is the Hessenberg sum of ordinals) then $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta$.

5.1 Atomic

Lemma 16. If $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta, p$ and $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash p, \Gamma \Rightarrow_s \Delta$, then $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta$.

Proof. We prove this by induction in $|\pi| + |\tau|$ (i.e. the sum of local heights). The possible cut reductions are the axiomatic cases, the implication commutative cases and the weakening case in the modal cases. We note that in all these cases were a cut rule appears in the reduction it can be solved using the I.H. (i.e. the local height is smaller). $\square$

5.2 Box formulas

Lemma 17. If we have cut admissibility for cut formula χ and $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta, \Box\chi$ and $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Box\chi, \Gamma \Rightarrow_s \Delta$, then $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta$.

Proof. We prove this by induction in $|\pi| + |\tau|$ (i.e. the sum of local heights). The possible cut reductions are the axiomatic cases where the axiomatic character is in a side formula, the implication commutative cases, the weakening case and $\Box$-$\Box$, $\Box$-$\Box^{+}_{u}$ and $\Box$-$\Box^{+}_{f}$ in the modal cases. In the axiomatic and weakening cases, no cut remains after the reduction. In the implication commutative cases, we can use the I.H. when a cut_i appears after the reduction since the sum of local heights is smaller. Finally, for the $\Box$-$\Box$, $\Box$-$\Box^{+}_{u}$ and $\Box$-$\Box^{+}_{f}$ cases it suffices to use the assumption of cut admissiblity for χ. $\Box$

5.3 Master formula

Definition 18. We say that a proof is $(\Box^{+}\chi, \alpha)$-unblocked iff all its cuts have cut formula χ or $\Box^{+}\chi$ and in case the cut formula is $\Box^{+}\chi$ then the height of the cut is $\leq \alpha$ and the subproofs at the premises are proofs in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ (i.e. they have not cuts). ■

Lemma 19. Assume we have cut admissibility for χ (with cuts of any height) and also for $\Box^{+}\chi$ with cuts of height smaller than α. If $\pi \vdash \Gamma \Rightarrow_s \Delta, \Box^{+}\chi$ and $\tau \vdash \Box^{+}\chi, \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ such that $\|\pi\| \oplus \|\tau\| \leq \alpha$, then there is a $\rho \vdash \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s} + \mathsf{mCut}$ that is $(\Box^{+}\chi, \alpha)$-unblocked and has no cuts in its main local fragment.

Proof. By induction in the sum of the local height of π and τ. The possible cut reductions are the axiomatic cases where the label is the cut formula, the axiomatic character is in a side formula, the commutative implication cases and the modal cases of weakening or $\Box^{+}_{u}$-$\Box$, $\Box^{+}_{u}$-$\Box^{+}_{u}$ and $\Box^{+}_{u}$-$\Box^{+}_{f}$. In the reduction rules where the label is the cut formula, axiomatic and modal weakening no cuts remain, so clearly the desired result holds.

In the commutative implication reductions it suffices to substitute each cut_i with an application of the I.H.

Finally, we treat the $\Box^{+}_{u}$-$\Box$, $\Box^{+}_{u}$-$\Box^{+}_{u}$ and $\Box^{+}_{u}$-$\Box^{+}_{f}$ reductions. By inspection we see that all the main local fragments have no cuts. Finally, let us see how to interpret cut_i in each case to get a proof with main cuts only and the unblocked condition.

$\Box^{+}_{u}$-$\Box$. In this case cut_1 is admissibility of cuts with cut formula $\Box^{+}\chi$ and height smaller than α and cut_2 is admissibility of cuts with cut formula χ. We are allowed to used the first admissibility for the cut_1 since

$$\|\mathsf{wk}(\pi_1)^{\circ}\| \oplus \|\mathsf{wk}(\tau_0)\| \leq (\|\mathsf{wk}(\pi_1)\| + 1) \oplus \|\mathsf{wk}(\tau_0)\| \leq \\ (\|\pi_1\| + 1) \oplus \|\tau_0\| \leq \|\pi\| \oplus \|\tau_0\| < \|\pi\| \oplus \|\tau\| = \alpha$$

thanks to $\|\pi_1\| < \|\pi\|, \|\tau_0\| < \|\tau\|$ being witnesses. This gives a proof without cuts so it has main cuts only and it is unblocked, as desired.

$\Box_u^+$-$\Box_u$. In this case $\mathsf{cut}_1, \mathsf{cut}_3$ are by admissibility of cuts with cut formula $\Box^+\chi$ and height smaller than α and $\mathsf{cut}_2, \mathsf{cut}_4$ is admissibility of cuts with cut formula χ. We are allowed to used the first admissibility for cut_1 and cut_3 since

$$\|\mathsf{wk}(\pi_1)^\circ\| \oplus \|\mathsf{wk}(\tau_0)\| \leq (\|\mathsf{wk}(\pi_1)\| + 1) \oplus \|\mathsf{wk}(\tau_0)\| \leq \\ (\|\pi_1\| + 1) \oplus \|\tau_0\| \leq \|\pi\| \oplus \|\tau_0\| < \|\pi\| \oplus \|\tau\| = \alpha$$

$$\|\mathsf{wk}(\pi_1)^\phi\| \oplus \|\mathsf{wk}(\tau_1)\| \leq (\|\mathsf{wk}(\pi_1)\| + 1) \oplus \|\mathsf{wk}(\tau_1)\| \leq \\ (\|\pi_1\| + 1) \oplus \|\tau_1\| \leq \|\pi\| \oplus \|\tau_1\| < \|\pi\| \oplus \|\tau\| = \alpha$$

thanks to $\|\pi_1\| < \|\pi\|, \|\tau_0\|, \|\tau_1\| < \|\tau\|$ being witnesses. This gives a proof without cuts so it has main cuts only and it is unblocked, as desired.

$\Box_u^+$-$\Box_f$. In this case cut_1 is by admissibility of cuts with cut formula $\Box^+\chi$ and height smaller than α and cut_2 is by admissibility of cuts with cut formula χ and $\mathsf{cut}_3, \mathsf{cut}_4$ are standard cuts. We are allowed to used the first admissibility for cut_1 since

$$\|\mathsf{wk}(\pi_1)^\circ\| \oplus \|\mathsf{wk}(\tau_0)\| \leq (\|\mathsf{wk}(\pi_1)\| + 1) \oplus \|\mathsf{wk}(\tau_0)\| \leq \\ (\|\pi_1\| + 1) \oplus \|\tau_0\| \leq \|\pi\| \oplus \|\tau_0\| < \|\pi\| \oplus \|\tau\| = \alpha$$

thanks to $\|\pi_1\| < \|\pi\|, \|\tau_0\| < \|\tau\|$ being witnesses. There are two cuts after the reduction, one with cut formula χ (cut_4) and other with cut formula $\Box^+\chi$ (cut_3). cut_3 has the premises with no cuts, and its height is

$$\|\mathsf{wk}(\pi_1)^\phi\| \oplus \|\mathsf{wk}(\tau_1)\| \leq (\|\mathsf{wk}(\pi_1)\| + 1) \oplus \|\mathsf{wk}(\tau_1)\| \leq \\ (\|\pi_1\| + 1) \oplus \|\tau_1\| \leq \|\pi\| \oplus \|\tau_1\| \leq \|\pi\| \oplus \|\tau\| = \alpha$$

thanks to $\|\pi_1\| < \|\pi\|$ being a witness and $\|\tau_1\| \leq \|\tau\|$ being a subproof. This means that the resulting proof has only main cuts and it is $(\Box^+\chi, \alpha)$-unblocked. $\Box$

Lemma 20. Assume we have cut admissibility for χ (with cuts of any height) and also for $\Box^+\chi$ with cuts of height smaller than α. Let $\pi \vdash \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut}$ and assume it is $(\Box^+\chi, \alpha)$-unblocked. Then, there is a $\rho \vdash \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut}$ that is $(\Box^+\chi, \alpha)$-unblocked and has no cuts with cut formula $\Box^+\chi$ in its main local fragment.

Proof. This is a simple induction using Lemma 19 in the number of cuts with cut formula $\Box^+\chi$ in its main local fragment. The assumption that π is $(\Box^+\chi, \alpha)$-unblocked is of fundamental importance since it allows use to use Lemma 19 to any cut at the main local fragment with cut formula $\Box^+\chi$. $\Box$

Lemma 21. Assume we have cut admissibility for χ (with cuts of any height) and also for $\Box^+\chi$ with cuts of height smaller than α. Then $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta, \Box^+\chi$ and $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Box^+\chi, \Gamma \Rightarrow_s \Delta$, implies $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut} \vdash \Gamma \Rightarrow_s \Delta$ such that all the cuts have χ as cut formula.

Proof. This is just a corecursion for a local progressing system (i.e. at each corecursive step we provide a whole local fragment, not just a node) using Lemma 20. $\square$

Lemma 22. Assume we have cut admissibility for cuts of size smaller than n and let $|\Box^+\chi| = n$. Then $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta, \Box^+\chi$ and $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Box^+\chi, \Gamma \Rightarrow_s \Delta$ implies $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta$.

Proof. Let $\pi \vdash \Gamma \Rightarrow_s \Delta, \Box^+\chi$ and $\tau \vdash \Box^+\chi, \Gamma \Rightarrow_s \Delta$. We proceed by induction in $\|\pi\| \oplus \|\tau\|$, so assume we have cut admissibility for $\Box^+\chi$ with cuts of height smaller than $\|\pi\| \oplus \|\tau\|$. We also have cut admisibility for χ thanks to the assumptions. Then by Lemma 21 we can get a proof ρ_0 of the same sequent in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{mCut}$ whose only cut formula is χ.

Then we can use Lemma 15 to obtain a proof ρ_1 with witness cuts only and all of them of size smaller than n and each ocurring at the main local fragment of some witness (so each witness have a finite amount of cuts). By an induction in the number of local cuts in the witness and using the hypothesis of cut admissibility for size $< n$ we can change each witness to a cut-free witness, obtaining the desired proof. $\square$

5.4 General case

Theorem 23: Cut admissibility**.** Let $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta, \chi$ and $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \chi, \Gamma \Rightarrow_s \Delta$. Then, we have that $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta$.

Proof. This is a simple induction in the size of χ using the lemmas previously proved in this section. In case χ is an implication, i.e. of shape $\chi_0 \to \chi_1$, first we need to apply the inversion to get formulas of smaller size (and weakening to make the sequents match). $\square$

With cut admissibility we can show cut elimination for proofs with finitely many cuts. However, in the non-wellfounded setting it is not straightforward that this gives us cut elimination for any proof, since our proofs may have infinitely many cuts. The purpose of the next section will be to show cut elimination in full generality for $\mathrm{G}_\ell^\infty\mathrm{K}_s^+$ using cut admissibility, thus providing and example of how to get cut elimination from admissibility in the non-wellfounded setting. Before finalizing this section we show a corollary which we will use during cut elimination.

Corollary 24. Suppose that $\pi \vdash \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ + \mathsf{Cut}$. Assume that either:

1. π has finitely many instances of cut, or
2. π has local cuts only.

Then $\mathrm{G}_\ell^\infty\mathrm{K}_s^+ \vdash \Gamma \Rightarrow_s \Delta$.

Proof. In case π has local cuts only then it must be the case that π has finitely many instances of cut, since the main local fragment is always finite. Then, in both cases we can assume that π has finitely many instances of cut. The result is then proven by an induction in the number of cuts using Theorem 23. $\square$

6 Cut elimination

We finish the paper by proving the promised result: cut elimination for $G^{\infty}_{\ell}K^{+}_{s}$. Note that by the translations defined in Section 1 between $G^{\infty}_{\ell}K^{+}$ and $G^{\infty}_{\ell}K^{+}_{s}$ this cut elimination will prove cut elimination for $G^{\infty}_{\ell}K^{+}$. In fact, since in reality the difference between $G^{\infty}_{\ell}K^{+}_{s}$ and $G^{\infty}_{\ell}K^{+}$ is just how we accomodate the information of proofs, it can be argued that this cut elimination is a cut elimination method for $G^{\infty}_{\ell}K^{+}$ and $G^{\infty}_{\ell}K^{+}_{s}$ just provides a way to define the necessary corecursive functions easier.

Theorem 25: Cut Elimination. If $G^{\infty}_{\ell}K^{+}_{s} + \mathsf{Cut} \vdash \Gamma \Rightarrow_s \Delta$, then $G^{\infty}_{\ell}K^{+}_{s} \vdash \Gamma \Rightarrow_s \Delta$.

Proof. Let $\pi \vdash \Gamma \Rightarrow_s \Delta$ in $G^{\infty}_{\ell}K^{+}_{s} + \mathsf{Cut}$. We proceed by induction on $\|\pi\|$. Note that for any witness τ of π we have that $\|\tau\| < \|\pi\|$. So by induction hypothesis we can get a τ' proof in $G^{\infty}_{\ell}K^{+}_{s}$ proving the same sequent as τ. Let π_1 be the result of replacing each of its witnesses τ by τ', we notice that π_1 is a proof in $G^{\infty}_{\ell}K^{+}_{s} + \mathsf{mCut}$ proving the same sequent as π (changing witnesses do not alter the conclusion, but we may have altered the height of the proof so from now own the I.H. cannot be used anymore).

Let $\pi_2 := \mathsf{push}(\pi_1)$, by Lemma 15 (using that a proof with main cuts only have witnesses with local cuts only) we have that π_2 is proof in $G^{\infty}_{\ell}K^{+}_{s} + \mathsf{wCut}$ proving the same sequent as π and such that all its witnesses have local cuts only.

Let ι be a witness of π_2, by Corollary 24 there is an ι' proof in $G^{\infty}_{\ell}K^{+}_{s}$ proving the same sequent as ι. Let π_3 be the result of replacing each witness ι of π_2 by ι'. Then π_3 proves the same sequent as π, since a change of witnesses do not change the conclusion. In addition since π_2 have witness-cuts only and π_3 has the same main global fragment as π_2 but with no cuts in the witnesses we can conclude that π_3 is cut free. $\square$

Conclusion and future work

We proved cut-elimination for a non-wellfounded calculus of K^{+}, using only basic techniques of structural proof theory (such as ordinal recursion) and corecursion. The method is mainly based in splitting the proofs nicely, taming the global branch condition in the process. This proof also works for similar systems to the master modality, such as common knowledge in which the $\square$ and $\square^{+}$ rules change to (for the system $G^{\infty}_{\ell}K^{+}$):

$$\frac{\Gamma, \Pi, \mathsf{C}\Pi \Rightarrow_{\circ} \phi}{\Sigma, \square_i\Gamma, \mathsf{C}\Pi \Rightarrow_s \square_i\phi, \Delta}\ \square_i \qquad \frac{(\Gamma_i, \Pi, \mathsf{C}\Pi \Rightarrow_{\circ} \phi)_{i<n} \qquad (\Gamma_i, \Pi, \mathsf{C}\Pi \Rightarrow_{\phi} \mathsf{C}\phi)_{i<n}}{\Sigma, \square_0\Gamma_0, \ldots, \square_{n-1}\Gamma_{n-1}, \mathsf{C}\Pi \Rightarrow_s \mathsf{C}\phi, \Delta}\ C$$

where we assume we have n agents, $i \in \{0, \ldots, n-1\}$, $\square_i$ is the knowledge modality for agent i, and C is the common knowledge modality.

Future work goes along this line and we will study how far this method can go. Of particular interest it is the case of PDL (Propositional Dynamic Logic). A problem for applying this method will be to guarantee that while we push cuts outside the main global fragment each witness has only a finite amount of cuts.

Finally, another line of work is to explore how our method can be extended to more complex branch conditions such as the ones used to deal with the modal μ-calculus. In this direction there are two questions which we consider of interest to explore:

1. How can we slice non-wellfounded proofs to ease cut eliminition?
2. Which methods can be use to show cut elimination from cut admissibility for non-wellfounded proofs?

A Weakening

Definition 26. Given $\pi \vdash \Gamma \Rightarrow_s \Delta$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{wk}_{\Gamma';\Delta'}(\pi)$ proof of $\Gamma, \Gamma' \Rightarrow_s \Delta, \Delta'$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ as:

$$\dfrac{}{\Gamma, p \Rightarrow_s p, \Delta}\,\mathrm{Ax} \mapsto \dfrac{}{\Gamma, \Gamma', p \Rightarrow_s p, \Delta, \Delta'}\,\mathrm{Ax}$$

$$\dfrac{}{\Gamma, \bot \Rightarrow_s \Delta}\,\mathrm{Ax}\text{-}\bot \mapsto \dfrac{}{\Gamma, \Gamma' \bot \Rightarrow_s \Delta, \Delta'}\,\mathrm{Ax}\text{-}\bot$$

$$\dfrac{\begin{matrix}\pi_0 \\ \Gamma \Rightarrow_s \Delta, \phi\end{matrix} \qquad \begin{matrix}\pi_1 \\ \Gamma, \psi \Rightarrow_s \Delta\end{matrix}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\,{\to}\mathrm{L} \mapsto \dfrac{\begin{matrix}\mathsf{wk}_{\Gamma';\Delta'}(\pi_0) \\ \Gamma, \Gamma' \Rightarrow_s \Delta, \Delta', \phi\end{matrix} \qquad \begin{matrix}\mathsf{wk}_{\Gamma';\Delta'}(\pi_1) \\ \Gamma, \Gamma', \psi \Rightarrow_s \Delta, \Delta'\end{matrix}}{\Gamma, \Gamma', \phi \to \psi \Rightarrow_s \Delta, \Delta'}\,{\to}\mathrm{L}$$

$$\dfrac{\begin{matrix}\pi_0 \\ \Gamma, \phi \Rightarrow_s \psi, \Delta\end{matrix}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\,{\to}\mathrm{R} \mapsto \dfrac{\begin{matrix}\mathsf{wk}_{\Gamma';\Delta'}(\pi_0) \\ \Gamma, \Gamma', \phi \Rightarrow_s \psi, \Delta, \Delta'\end{matrix}}{\Gamma, \Gamma' \Rightarrow_s \phi \to \psi, \Delta, \Delta'}\,{\to}\mathrm{R}$$

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta}\,\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Gamma', \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, \Delta'}\,\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{matrix}\pi_0 \\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{matrix}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\,\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash^{\beta} \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi\end{array}}{\Sigma, \Gamma', \Box\Gamma, \Box^{+}\Pi \Rightarrow_{s} \Box^{+}\phi, \Delta, \Delta'}\ \Box^{+}_{f}$$

If π is:

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \tau_1 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_{s} \Box^{+}\phi, \Delta}\ \Box^{+}_{u}$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \tau_1 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}{\Sigma, \Gamma', \Box\Gamma, \Box^{+}\Pi \Rightarrow_{s'} \Box^{+}\phi, \Delta, \Delta'}\ \Box^{+}_{u}$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_{s} \Delta, \chi\end{array} \qquad \begin{array}{c}\pi_1\\ \chi, \Gamma \Rightarrow_{s} \Delta\end{array}}{\Gamma \Rightarrow_{s} \Delta}\ \text{Cut} \mapsto \dfrac{\begin{array}{c}\mathsf{wk}_{\Gamma';\Delta'}(\pi_0)\\ \Gamma, \Gamma' \Rightarrow_{s} \Delta, \Delta', \chi\end{array} \qquad \begin{array}{c}\mathsf{wk}_{\Gamma';\Delta'}(\pi_1)\\ \chi, \Gamma, \Gamma' \Rightarrow_{s} \Delta, \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow_{s} \Delta, \Delta'}\ \text{Cut}$$

■

Lemma 27. The function $\mathsf{wk}_{\Gamma';\Delta'}$ is strongly preserving.

B Contraction

B.1 Contraction of propositional variable in the left

Definition 28. Given $\pi \vdash \Gamma, p, p \Rightarrow_s \Delta$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{lctr}_p(\pi)$ proof of $\Gamma, p \Rightarrow_s \Delta$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ as:

$$\dfrac{}{\Gamma, p, p \Rightarrow_{s} p, \Delta}\ \text{Ax} \mapsto \dfrac{}{\Gamma, p \Rightarrow_{s} p, \Delta}\ \text{Ax}$$

$$\dfrac{}{\Gamma, p, p, q \Rightarrow_{s} q, \Delta}\ \text{Ax} \mapsto \dfrac{}{\Gamma, p, q \Rightarrow_{s} q, \Delta}\ \text{Ax}$$

for $q \neq p$.

$$\dfrac{}{\Gamma, p, p, \bot \Rightarrow_{s} \Delta}\ \text{Ax-}\bot \mapsto \dfrac{}{\Gamma, p, \bot \Rightarrow_{s} \Delta}\ \text{Ax-}\bot$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, p, p \Rightarrow_{s} \Delta, \phi\end{array} \qquad \begin{array}{c}\pi_1\\ \Gamma, p, p, \psi \Rightarrow_{s} \Delta\end{array}}{\Gamma, p, p, \phi \to \psi \Rightarrow_{s} \Delta}\ {\to}\text{L} \mapsto \dfrac{\begin{array}{c}\mathsf{lctr}_p(\pi_0)\\ \Gamma, p \Rightarrow_{s} \Delta, \phi\end{array} \qquad \begin{array}{c}\mathsf{lctr}_p(\pi_1)\\ \Gamma, p, \psi \Rightarrow_{s} \Delta\end{array}}{\Gamma, p, \phi \to \psi \Rightarrow_{s} \Delta}\ {\to}\text{L}$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, p, p, \phi \Rightarrow_s \psi, \Delta\end{array}}{\Gamma, p, p \Rightarrow_s \phi \to \psi, \Delta}\to\mathrm{R} \mapsto \dfrac{\begin{array}{c}\mathsf{lctr}_p(\pi_0)\\ \Gamma, p, \phi \Rightarrow_s \psi, \Delta\end{array}}{\Gamma, p \Rightarrow_s \phi \to \psi, \Delta}\to\mathrm{R}$$

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, p, p, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta}\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, p, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta}\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, p, p, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, p, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\Box^+_f$$

If π is:

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, p, p, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, p, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\Box^+_u$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, p, p \Rightarrow_s \Delta, \chi\end{array} \qquad \begin{array}{c}\pi_1\\ \chi, \Gamma, p, p \Rightarrow_s \Delta\end{array}}{\Gamma, p, p \Rightarrow_s \Delta}\mathrm{Cut} \mapsto \dfrac{\begin{array}{c}\mathsf{lctr}_p(\pi_0)\\ \Gamma, p \Rightarrow_s \Delta, \chi\end{array} \qquad \begin{array}{c}\mathsf{lctr}_p(\pi_1)\\ \chi, \Gamma, p \Rightarrow_s \Delta\end{array}}{\Gamma, p \Rightarrow_s \Delta}\mathrm{Cut}$$

■

Lemma 29. The function lctr_p is strongly preserving.

B.2 Contraction of propositional variable in the right

Definition 30. Given $\pi \vdash \Gamma \Rightarrow_s \Delta, p, p$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{lctr}_p(\pi)$ proof of $\Gamma \Rightarrow_s \Delta, p$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ as:

$$\dfrac{}{\Gamma, p \Rightarrow_s p, p, \Delta}\mathrm{Ax} \mapsto \dfrac{}{\Gamma, p \Rightarrow_s p, \Delta}\mathrm{Ax}$$

$$\dfrac{}{\Gamma, q \Rightarrow_s q, p, p, \Delta}\,\mathrm{Ax} \mapsto \dfrac{}{\Gamma, q \Rightarrow_s q, p, \Delta}\,\mathrm{Ax}$$

for $q \neq p$.

$$\dfrac{}{\Gamma, \bot \Rightarrow_s p, p, \Delta}\,\mathrm{Ax}\text{-}\bot \mapsto \dfrac{}{\Gamma, \bot \Rightarrow_s p, \Delta}\,\mathrm{Ax}\text{-}\bot$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_s \Delta, p, p, \phi\end{array} \qquad \begin{array}{c}\pi_1\\ \Gamma, \psi \Rightarrow_s \Delta, p, p\end{array}}{\Gamma, \phi \to \psi \Rightarrow_s p, p, \Delta}\,{\to}\mathrm{L} \mapsto \dfrac{\begin{array}{c}\mathsf{rctr}_p(\pi_0)\\ \Gamma \Rightarrow_s \Delta, p, \phi\end{array} \qquad \begin{array}{c}\mathsf{rctr}_p(\pi_1)\\ \Gamma, \psi \Rightarrow_s \Delta, p\end{array}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, p}\,{\to}\mathrm{L}$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \phi \Rightarrow_s \psi, \Delta, p, p\end{array}}{\Gamma, \Rightarrow_s \phi \to \psi, \Delta, p, p}\,{\to}\mathrm{R} \mapsto \dfrac{\begin{array}{c}\mathsf{rctr}_p(\pi_0)\\ \Gamma, \phi \Rightarrow_s \psi, \Delta, p\end{array}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, p}\,{\to}\mathrm{R}$$

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, p, p}\,\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, p}\,\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, p, p}\,\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, p}\,\Box^+_f$$

If π is:

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, p, p}\,\Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, p}\,\Box^+_u$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_s \Delta, p, p, \chi\end{array} \qquad \begin{array}{c}\pi_1\\ \chi, \Gamma \Rightarrow_s \Delta, p, p\end{array}}{\Gamma \Rightarrow_s \Delta, p, p}\,\mathrm{Cut} \mapsto \dfrac{\begin{array}{c}\mathsf{rctr}_p(\pi_0)\\ \Gamma \Rightarrow_s \Delta, p, \chi\end{array} \qquad \begin{array}{c}\mathsf{rctr}_p(\pi_1)\\ \chi, \Gamma \Rightarrow_s \Delta, p\end{array}}{\Gamma \Rightarrow_s \Delta, p}\,\mathrm{Cut}$$

$\blacksquare$

Lemma 31. The function rctr_p is strongly preserving.

B.3 Contraction of boxed formula in the right

Definition 32. Given $\pi \vdash \Gamma \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{rctr}_p(\pi)$ proof of $\Gamma \Rightarrow_s \Delta, \Box^+\chi$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ as:

$$\dfrac{}{\Gamma, p \Rightarrow_s p, \Box^+\chi, \Box^+\chi, \Delta}\,\mathrm{Ax} \mapsto \dfrac{}{\Gamma, p \Rightarrow_s p, \Box^+\chi, \Delta}\,\mathrm{Ax}$$

$$\dfrac{}{\Gamma, \bot \Rightarrow_s \Box^+\chi, \Box^+\chi, \Delta}\,\mathrm{Ax}\text{-}\bot \mapsto \dfrac{}{\Gamma, \bot \Rightarrow_s \Box^+\chi, \Delta}\,\mathrm{Ax}\text{-}\bot$$

If π is

$$\dfrac{\overset{\pi_0}{\Gamma \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi, \phi} \qquad \overset{\pi_1}{\Gamma, \psi \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi}}{\Gamma, \phi \to \psi \Rightarrow_s \Box^+\chi, \Box^+\chi, \Delta}\,{\to}\mathrm{L}$$

then it maps to

$$\dfrac{\overset{\mathsf{rctr}_{\Box^+\chi}(\pi_0)}{\Gamma \Rightarrow_s \Delta, \Box^+\chi, \phi} \qquad \overset{\mathsf{rctr}_{\Box^+\chi}(\pi_1)}{\Gamma, \psi \Rightarrow_s \Delta, \Box^+\chi}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, \Box^+\chi}\,{\to}\mathrm{L}$$

If π is

$$\dfrac{\overset{\pi_0}{\Gamma, \phi \Rightarrow_s \psi, \Delta, \Box^+\chi, \Box^+\chi}}{\Gamma, \Rightarrow_s \phi \to \psi, \Delta, \Box^+\chi, \Box^+\chi}\,{\to}\mathrm{R} \mapsto \dfrac{\overset{\mathsf{rctr}_{\Box^+\chi}(\pi_0)}{\Gamma, \phi \Rightarrow_s \psi, \Delta, \Box^+\chi}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, \Box^+\chi}\,{\to}\mathrm{R}$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, \Box^+\chi, \Box^+\chi}\,\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, \Box^+\chi}\,\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \overset{\pi_0}{\Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \Box^+\chi, \Box^+\chi}\,\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \overset{\pi_0}{\Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \Box^+\chi}\,\Box^+_f$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \chi \qquad \overset{\pi_0}{\Gamma, \boxdot^+\Pi \Rightarrow_\chi \Box^+\chi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\chi, \Delta, \Box^+\chi}\,\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \chi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\chi\end{array}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\chi, \Delta}\ \Box^+_f$$

If π is:

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \Box^+\chi, \Box^+\chi}\ \Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \Box^+\chi}\ \Box^+_u$$

If π is:

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \chi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\chi \Box^+\chi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\chi, \Delta, \Box^+\chi}\ \Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \chi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\chi \Box^+\chi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\chi, \Delta}\ \Box^+_u$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi, \phi\end{array} \qquad \begin{array}{c}\pi_1\\ \phi, \Gamma \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi\end{array}}{\Gamma \Rightarrow_s \Delta, \Box^+\chi, \Box^+\chi}\ \mathrm{Cut}$$

maps to

$$\dfrac{\begin{array}{c}\mathsf{rctr}_{\Box^+\chi}(\pi_0)\\ \Gamma \Rightarrow_s \Delta, \Box^+\chi, \phi\end{array} \qquad \begin{array}{c}\mathsf{rctr}_{\Box^+\chi}(\pi_1)\\ \phi, \Gamma \Rightarrow_s \Delta, \Box^+\chi\end{array}}{\Gamma \Rightarrow_s \Delta, \Box^+\chi}\ \mathrm{Cut}$$

■

Lemma 33. The function $\mathsf{rctr}_{\Box^+\chi}$ is strongly preserving.

C Inversion

C.1 Inversion of $\bot$

Definition 34. Given $\pi \vdash \Gamma \Rightarrow_s \Delta, \bot$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{inv}_{\bot}(\pi)$ proof of $\Gamma \Rightarrow_s \Delta$ in $\mathrm{G}^{\infty}_{\ell}\mathrm{K}^{+}_{s}$ as:

$$\dfrac{}{\Gamma, p \Rightarrow_s p, \Delta, \bot}\,\mathrm{Ax} \mapsto \dfrac{}{\Gamma, p \Rightarrow_s p, \Delta}\,\mathrm{Ax}$$

$$\dfrac{}{\Gamma, \bot \Rightarrow_s \Delta, \bot}\,\mathrm{Ax}\text{-}\bot \mapsto \dfrac{}{\Gamma, p, \bot \Rightarrow_s \Delta}\,\mathrm{Ax}\text{-}\bot$$

$$\dfrac{\begin{array}{cc}\pi_0 & \pi_1\\ \Gamma \Rightarrow_s \Delta, \bot, \phi & \Gamma, \psi \Rightarrow_s \Delta, \bot\end{array}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, \bot}\,{\to}\mathrm{L} \mapsto \dfrac{\begin{array}{cc}\mathsf{inv}_{\bot}(\pi_0) & \mathsf{inv}_{\bot}(\pi_1)\\ \Gamma \Rightarrow_s \Delta, \phi & \Gamma, \psi \Rightarrow_s \Delta\end{array}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\,{\to}\mathrm{L}$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \phi \Rightarrow_s \psi, \Delta, \bot\end{array}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, \bot}\,{\to}\mathrm{R} \mapsto \dfrac{\begin{array}{c}\mathsf{inv}_{\bot}(\pi_0)\\ \Gamma, \phi \Rightarrow_s \psi, \Delta\end{array}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\,{\to}\mathrm{R}$$

$$\dfrac{\tau \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box\phi, \Delta, \bot}\,\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box\phi, \Delta}\,\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi\end{array}}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box^{+}\phi, \Delta, \bot}\,\Box^{+}_{f}$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi\end{array}}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box^{+}\phi, \Delta}\,\Box^{+}_{f}$$

If π is:

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \tau_1 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box^{+}\phi, \Delta, \bot}\,\Box^{+}_{u}$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\circ} \phi \qquad \tau_1 \vdash \Gamma, \boxdot^{+}\Pi \Rightarrow_{\phi} \Box^{+}\phi}{\Sigma, \Box\Gamma, \Box^{+}\Pi \Rightarrow_s \Box^{+}\phi, \Delta}\,\Box^{+}_{u}$$

$$\dfrac{\begin{array}{cc}\pi_0 & \pi_1\\ \Gamma \Rightarrow_s \Delta, \bot, \chi & \chi, \Gamma \Rightarrow_s \Delta, \bot\end{array}}{\Gamma \Rightarrow_s \Delta, \bot}\,\mathrm{Cut} \mapsto \dfrac{\begin{array}{cc}\mathsf{inv}_{\bot}(\pi_0) & \mathsf{inv}_{\bot}(\pi_1)\\ \Gamma \Rightarrow_s \Delta, \chi & \chi, \Gamma \Rightarrow_s \Delta\end{array}}{\Gamma, \Rightarrow_s \Delta}\,\mathrm{Cut}$$

∎

Lemma 35. The function $\mathsf{inv}_\bot$ is strongly preserving.

C.2 Inversion of $\to$ L

C.2.1 Left inversion

Definition 36. Given $\pi \vdash \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi)$ proof of $\Gamma \Rightarrow_s \Delta, \chi_0$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ as:

$$\dfrac{}{\Gamma, \chi_0 \to \chi_1, p \Rightarrow_s p, \Delta}\,\mathrm{Ax} \mapsto \dfrac{}{\Gamma, p \Rightarrow_s p, \Delta, \chi_0}\,\mathrm{Ax}$$

$$\dfrac{}{\Gamma, \chi_0 \to \chi_1, \bot \Rightarrow_s \Delta}\,\mathrm{Ax}\text{-}\bot \mapsto \dfrac{}{\Gamma, \bot \Rightarrow_s \Delta, \chi_0}\,\mathrm{Ax}\text{-}\bot$$

If π is of shape:

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta, \phi\end{array} \qquad \begin{array}{c}\pi_1\\ \Gamma, \chi_0 \to \chi_1, \psi \Rightarrow_s \Delta\end{array}}{\Gamma, \chi_0 \to \chi_1, \phi \to \psi \Rightarrow_s \Delta}\,\to\mathrm{L}$$

we transform it to

$$\dfrac{\begin{array}{c}\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi_0)\\ \Gamma \Rightarrow_s \Delta, \chi_0, \phi\end{array} \qquad \begin{array}{c}\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi_1)\\ \Gamma, \psi \Rightarrow_s \Delta, \chi_0\end{array}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, \chi_0}\,\to\mathrm{L}$$

If π is of shape:

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_s \Delta, \chi_0\end{array} \qquad \begin{array}{c}\pi_1\\ \Gamma, \chi_1 \Rightarrow_s \Delta\end{array}}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta}\,\to\mathrm{L}$$

the desired proof is π_0.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \chi_0 \to \chi_1, \phi \Rightarrow_s \psi, \Delta\end{array}}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \phi \to \psi, \Delta}\,\to\mathrm{R} \mapsto \dfrac{\begin{array}{c}\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi_0)\\ \Gamma, \phi \Rightarrow_s \psi, \Delta, \chi_0\end{array}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, \chi_0}\,\to\mathrm{R}$$

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \chi_0 \to \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta}\,\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, \chi_0}\,\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{array}{c}\pi_0\\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{array}}{\Sigma, \chi_0 \to \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\,\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \overset{\pi_0}{\Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \chi_0}\ \Box^+_f$$

If π is

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \chi_0 \to \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta}\ \Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \chi_0}\ \Box^+_u$$

If π is

$$\dfrac{\overset{\pi_0}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta, \phi} \qquad \overset{\pi_1}{\phi, \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta}}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta}\ \mathrm{Cut}$$

then it maps to

$$\dfrac{\overset{\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi_0)}{\Gamma \Rightarrow_s \Delta, \chi_0, \phi} \qquad \overset{\mathsf{linv}^0_{\chi_0\to\chi_1}(\pi_1)}{\phi, \Gamma \Rightarrow_s \Delta, \chi_0}}{\Gamma, \Rightarrow_s \Delta, \chi_0}\ \mathrm{Cut}$$

■

Lemma 37. The function $\mathsf{linv}^0_{\chi_0\to\chi_1}$ is strongly preserving.

C.2.2 Right inversion

Definition 38. Given $\pi \vdash \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{linv}^1_{\chi_0\to\chi_1}(\pi)$ proof of $\Gamma, \chi_1 \Rightarrow_s \Delta$ in $\mathrm{G}^\infty_\ell\mathrm{K}^+_s$ as:

$$\dfrac{}{\Gamma, \chi_0 \to \chi_1, p \Rightarrow_s p, \Delta}\ \mathrm{Ax} \mapsto \dfrac{}{\Gamma, \chi_1, p \Rightarrow_s p, \Delta}\ \mathrm{Ax}$$

$$\dfrac{}{\Gamma, \chi_0 \to \chi_1, \bot \Rightarrow_s \Delta}\ \mathrm{Ax}\text{-}\bot \mapsto \dfrac{}{\Gamma, \chi_1, \bot \Rightarrow_s \Delta}\ \mathrm{Ax}\text{-}\bot$$

If π is of shape:

$$\dfrac{\overset{\pi_0}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta, \phi} \qquad \overset{\pi_1}{\Gamma, \chi_0 \to \chi_1, \psi \Rightarrow_s \Delta}}{\Gamma, \chi_0 \to \chi_1, \phi \to \psi \Rightarrow_s \Delta}\ {\to}\mathrm{L}$$

we transform it to

$$\dfrac{\overset{\mathsf{linv}^1_{\chi_0\to\chi_1}(\pi_0)}{\Gamma, \chi_1 \Rightarrow_s \Delta, \phi} \qquad \overset{\mathsf{linv}^1_{\chi_0\to\chi_1}(\pi_1)}{\Gamma, \chi_1, \psi \Rightarrow_s \Delta}}{\Gamma, \chi_1, \phi \to \psi \Rightarrow_s \Delta}\ {\to}\mathrm{L}$$

If π is of shape:

$$\dfrac{\genfrac{}{}{0pt}{}{\pi_0}{\Gamma \Rightarrow_s \Delta, \chi_0} \qquad \genfrac{}{}{0pt}{}{\pi_1}{\Gamma, \chi_1 \Rightarrow_s \Delta}}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta} \to\mathrm{L}$$

the desired proof is π_1.

$$\dfrac{\genfrac{}{}{0pt}{}{\pi_0}{\Gamma, \chi_0 \to \chi_1, \phi \Rightarrow_s \psi, \Delta}}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \phi \to \psi, \Delta} \to\mathrm{R} \mapsto \dfrac{\genfrac{}{}{0pt}{}{\mathsf{linv}^1_{\chi_0 \to \chi_1}(\pi_0)}{\Gamma, \chi_1, \phi \Rightarrow_s \psi, \Delta}}{\Gamma, \chi_1 \Rightarrow_s \phi \to \psi, \Delta} \to\mathrm{R}$$

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \chi_0 \to \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta} \Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta} \Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \genfrac{}{}{0pt}{}{\pi_0}{\Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \chi_0 \to \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta} \Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \genfrac{}{}{0pt}{}{\pi_0}{\Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}}{\Sigma, \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta} \Box^+_f$$

If π is

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \chi_0 \to \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta} \Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \chi_1, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta} \Box^+_u$$

If π is

$$\dfrac{\genfrac{}{}{0pt}{}{\pi_0}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta, \phi} \qquad \genfrac{}{}{0pt}{}{\pi_1}{\phi, \Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta}}{\Gamma, \chi_0 \to \chi_1 \Rightarrow_s \Delta} \mathrm{Cut}$$

then it maps to

$$\dfrac{\genfrac{}{}{0pt}{}{\mathsf{linv}^1_{\chi_0 \to \chi_1}(\pi_0)}{\Gamma, \chi_1 \Rightarrow_s \Delta, \phi} \qquad \genfrac{}{}{0pt}{}{\mathsf{linv}^1_{\chi_0 \to \chi_1}(\pi_1)}{\phi, \Gamma, \chi_1 \Rightarrow_s \Delta}}{\Gamma, \chi_1 \Rightarrow_s \Delta} \mathrm{Cut}$$

■

Lemma 39. The function $\mathsf{linv}^1_{\chi_0 \to \chi_1}$ is strongly preserving.

C.3 Inversion of $\to$ R

Definition 40. Given $\pi \vdash \Gamma \Rightarrow_s \chi_0 \to \chi_1, \Delta$ in $G^{\infty}_{\ell}K^+_s$ we can define, by recursion in the local height and cases in the last rule apply, the proof $\mathsf{rinv}_{\chi_0 \to \chi_1}(\pi)$ proof of $\Gamma, \chi_0 \Rightarrow_s \Delta, \chi_1$ in $G^{\infty}_{\ell}K^+_s$ as:

$$\dfrac{}{\Gamma, p \Rightarrow_s p, \Delta, \chi_0 \to \chi_1}\text{Ax} \mapsto \dfrac{}{\Gamma, \chi_0, p \Rightarrow_s p, \Delta, \chi_1}\text{Ax}$$

$$\dfrac{}{\Gamma, \bot \Rightarrow_s \Delta, \chi_0 \to \chi_1}\text{Ax-}\bot \mapsto \dfrac{}{\Gamma, \chi_0, \bot \Rightarrow_s \Delta, \chi_1}\text{Ax-}\bot$$

If π is of shape:

$$\dfrac{\begin{matrix}\pi_0 \\ \Gamma \Rightarrow_s \Delta, \chi_0 \to \chi_1, \phi\end{matrix} \qquad \begin{matrix}\pi_1 \\ \Gamma, \psi \Rightarrow_s \Delta, \chi_0 \to \chi_1\end{matrix}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, , \chi_0 \to \chi_1}\to\text{L}$$

we transform it to

$$\dfrac{\begin{matrix}\mathsf{rinv}_{\chi_0 \to \chi_1}(\pi_0) \\ \Gamma, \chi_0 \Rightarrow_s \Delta, \chi_1, \phi\end{matrix} \qquad \begin{matrix}\mathsf{rinv}_{\chi_0 \to \chi_1}(\pi_1) \\ \Gamma, \chi_0, \psi \Rightarrow_s \Delta, \chi_1\end{matrix}}{\Gamma, \chi_0, \phi \to \psi \Rightarrow_s \Delta, \chi_1}\to\text{L}$$

$$\dfrac{\begin{matrix}\pi_0 \\ \Gamma, \phi \Rightarrow_s \psi, \Delta, \chi_0 \to \chi_1\end{matrix}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, \chi_0 \to \chi_1}\to\text{R} \mapsto \dfrac{\begin{matrix}\mathsf{rinv}_{\chi_0 \to \chi_1}(\pi_0) \\ \Gamma, \chi_0, \phi \Rightarrow_s \psi, \Delta, \chi_1\end{matrix}}{\Gamma, \chi_0 \Rightarrow_s \phi \to \psi, \Delta, \chi_1}\to\text{R}$$

$$\dfrac{\begin{matrix}\pi_0 \\ \Gamma, \chi_0 \Rightarrow_s \chi_1, \Delta\end{matrix}}{\Gamma \Rightarrow_s \chi_0 \to \chi_1, \Delta}\to\text{R}$$

the desired proof is π_0.

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, \chi_0 \to \chi_1}\Box \mapsto \dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi}{\Sigma, \chi_0, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box\phi, \Delta, \chi_1}\Box$$

If π is:

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{matrix}\pi_0 \\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{matrix}}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \chi_0 \to \chi_1}\Box^+_f$$

then it maps to

$$\dfrac{\tau \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \begin{matrix}\pi_0 \\ \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi\end{matrix}}{\Sigma, \chi_0, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \chi_1}\Box^+_f$$

If π is

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \chi_0 \to \chi_1}\ \Box^+_u$$

then it maps to

$$\dfrac{\tau_0 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\circ \phi \qquad \tau_1 \vdash \Gamma, \boxdot^+\Pi \Rightarrow_\phi \Box^+\phi}{\Sigma, \chi_0, \Box\Gamma, \Box^+\Pi \Rightarrow_s \Box^+\phi, \Delta, \chi_1}\ \Box^+_u$$

If π is

$$\dfrac{\begin{matrix}\pi_0 \\ \Gamma \Rightarrow_s \Delta, , \chi_0 \to \chi_1, \phi\end{matrix} \qquad \begin{matrix}\pi_1 \\ \phi, \Gamma \Rightarrow_s \Delta, \chi_0 \to \chi_1\end{matrix}}{\Gamma \Rightarrow_s \Delta, \chi_0 \to \chi_1}\ \mathrm{Cut}$$

then it maps to

$$\dfrac{\begin{matrix}\mathsf{rinv}_{\chi_0 \to \chi_1}(\pi_0) \\ \Gamma, \chi_0 \Rightarrow_s \Delta, \chi_1, \phi\end{matrix} \qquad \begin{matrix}\mathsf{rinv}_{\chi_0 \to \chi_1}(\pi_1) \\ \phi, \Gamma, \chi_0 \Rightarrow_s \Delta, \chi_1\end{matrix}}{\Gamma, \chi_0 \Rightarrow_s \Delta, \chi_1}\ \mathrm{Cut}$$

■

Lemma 41. The function $\mathsf{rinv}_{\chi_0 \to \chi_1}$ is strongly preserving.

D Cut reductions

Here we display all the cut-reductions we are going to use. We will write cut_i with $i \in \mathbb{N}$ to have a way to refer to each instance of cut since when we use the cut-reductions in proofs, it may be the case that we leave cuts as cuts or that we replace them by some proof provided by cut-admissibility.

We notice that the cut reductions of this section provides all possible cut reductions when we have a $\pi \vdash \Gamma \Rightarrow_s \Delta, \chi$ and $\tau \vdash \chi, \Gamma \Rightarrow_s \Delta$ for an arbitrary χ (the cut formula). In case χ is of a particular shape (e.g. a propositional variable or a box formula) we will mention explicitely in the proof which are the cut reductions that appear. We are prepared to write all the cut reductions:

D.1 Labeling and cut formula coincides

Let $\chi = \Box^+\chi_0$ and $s = \chi_0$. Since $\chi, \Gamma \Rightarrow_s \Delta$ is the conclusion of τ and s is a formula, we know that $\Box^+ s = \Box^+\chi_0$ must appear in Δ.[6] Then $\Delta = \Delta_0, \chi$ and the conclusion of π would be $\Gamma \Rightarrow_s \Delta_0, \Box^+\chi_0, \chi$, i.e. $\Gamma \Rightarrow_s \Delta_0, \chi, \chi$.

We have the cut reduction

$$\begin{array}{c}\pi\\ \Gamma \Rightarrow_s \Delta_0, \chi, \chi\end{array} \quad \begin{array}{c}\tau\\ \chi, \Gamma \Rightarrow_s \Delta_0, \chi\end{array} \quad \mapsto \quad \begin{array}{c}\mathsf{rctr}_{\Box^+\chi_0}(\pi)\\ \Gamma \Rightarrow_s \Delta_0, \chi\end{array}$$

From now own we assume that either χ is not of shape $\Box^+\chi_0$ or if it is then $s \neq \chi_0$.

D.2 Axiomatic cases

Assume that either π or τ is axiomatic.

Axiomatic character is in side formula

$$\dfrac{}{\Gamma, p \Rightarrow_s p, \Delta, \chi} \quad \begin{array}{c}\tau\\ \chi, \Gamma, p \Rightarrow_s p, \Delta\end{array} \quad \mapsto \quad \dfrac{}{\Gamma, p \Rightarrow_s p, \Delta}$$

$$\begin{array}{c}\pi\\ \Gamma, p \Rightarrow_s p, \Delta, \chi\end{array} \quad \dfrac{}{\chi, \Gamma, p \Rightarrow_s p, \Delta} \quad \mapsto \quad \dfrac{}{\Gamma, p \Rightarrow_s p, \Delta}$$

$$\dfrac{}{\Gamma, \bot \Rightarrow_s \Delta, \chi} \quad \begin{array}{c}\tau\\ \chi, \Gamma, \bot \Rightarrow_s \Delta\end{array} \quad \mapsto \quad \dfrac{}{\Gamma, \bot \Rightarrow_s p, \Delta}$$

$$\begin{array}{c}\pi\\ \Gamma, \bot \Rightarrow_s \Delta, \chi\end{array} \quad \dfrac{}{\chi, \Gamma, \bot \Rightarrow_s \Delta} \quad \mapsto \quad \dfrac{}{\Gamma, \bot \Rightarrow_s p, \Delta}$$

Axiomatic character is in the cut formula

$$\dfrac{}{\Gamma, p \Rightarrow_s \Delta, p} \quad \begin{array}{c}\tau\\ p, \Gamma, p \Rightarrow_s \Delta\end{array} \quad \mapsto \quad \begin{array}{c}\mathrm{lctr}_p(\tau)\\ \Gamma, p \Rightarrow_s \Delta\end{array}$$

[6]We remember that, by definition of annotated sequent, if $\Gamma \Rightarrow_\phi \Delta$ then $\Box^+\phi$ must occur in Δ.

$$\begin{array}{c}\pi\\ \Gamma \Rightarrow_s p, \Delta, p\end{array} \quad \frac{}{p, \Gamma \Rightarrow_s p, \Delta} \mapsto \begin{array}{c}\mathrm{rctr}_p(\pi)\\ \Gamma \Rightarrow_s p, \Delta\end{array}$$

$$\begin{array}{c}\pi\\ \Gamma \Rightarrow_s \Delta, \bot\end{array} \quad \frac{}{\bot, \Gamma \Rightarrow_s \Delta} \mapsto \begin{array}{c}\mathrm{inv}_\bot(\pi)\\ \Gamma \Rightarrow_s \Delta\end{array}$$

From now own we assume that neither π nor τ is axiomatic, so π and τ have each a principal formula.

D.3 Principal reduction

Assume the cut formula is principal in π and τ. Then $\chi = \chi_0 \to \chi_1$ and the cut reduction has shape

$$\dfrac{\begin{array}{c}\pi_0\\ \chi_0, \Gamma \Rightarrow_s \Delta, \chi_1\end{array}}{\Gamma \Rightarrow_s \Delta, \chi_0 \to \chi_1}\to\mathrm{R} \quad \dfrac{\begin{array}{c}\tau_0\\ \Gamma \Rightarrow_s \Delta, \chi_0\end{array} \quad \begin{array}{c}\tau_1\\ \chi_1, \Gamma \Rightarrow_s \Delta\end{array}}{\chi_0 \to \chi_1, \Gamma \Rightarrow_s \Delta}\to\mathrm{L}$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\mathsf{wk}(\tau_0)\\ \Gamma \Rightarrow_s \Delta, \chi_1, \chi_0\end{array} \quad \begin{array}{c}\pi_0\\ \chi_0, \Gamma \Rightarrow_s \Delta, \chi_1\end{array}}{\Gamma \Rightarrow_s \Delta, \chi_1}\mathsf{cut}_1 \quad \begin{array}{c}\tau_1\\ \chi_1, \Gamma \Rightarrow_s \Delta\end{array}}{\Gamma \Rightarrow_s \Delta}\mathsf{cut}_2$$

D.4 Commutative implication cases

Assume the cut formula is not principal in π or τ and the principal formula where the cut formula is not principal is an implication. The cut reduction has 4 possible shapes:

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \phi \Rightarrow_s \psi, \Delta, \chi\end{array}}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, \chi}\to\mathrm{R} \quad \begin{array}{c}\tau\\ \chi, \Gamma \Rightarrow_s \phi \to \psi, \Delta\end{array}$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma, \phi \Rightarrow_s \psi, \Delta, \chi\end{array} \quad \begin{array}{c}\mathrm{rinv}_{\phi\to\psi}(\tau)\\ \chi, \Gamma, \phi \Rightarrow_s \psi, \Delta\end{array}}{\Gamma, \phi \Rightarrow_s \psi, \Delta}\mathsf{cut}_1}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\to R$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow_s \Delta, \phi, \chi\end{array} \quad \begin{array}{c}\pi_1\\ \Gamma, \psi \Rightarrow_s \Delta, \chi\end{array}}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, \chi}\to\mathrm{L} \quad \begin{array}{c}\tau\\ \chi, \Gamma, \phi \to \psi \Rightarrow_s \Delta\end{array}$$

$$\downarrow$$

$$
\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow_s \Delta, \phi, \chi} \qquad \overset{\mathsf{inv}^0_{\phi\to\psi}(\tau)}{\chi, \Gamma \Rightarrow_s \Delta, \phi}}{\Gamma \Rightarrow_s \Delta, \phi}\ \mathsf{cut}_1 \qquad \dfrac{\overset{\pi_1}{\Gamma, \psi \Rightarrow_s \Delta, \chi} \qquad \overset{\mathsf{inv}^1_{\phi\to\psi}(\tau)}{\chi, \Gamma, \psi \Rightarrow_s \Delta}}{\Gamma, \psi \Rightarrow_s \Delta}\ \mathsf{cut}_2}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\ {\to}\mathrm{L}
$$

$$
\overset{\pi}{\Gamma \Rightarrow_s \phi \to \psi, \Delta, \chi} \qquad \dfrac{\overset{\tau_0}{\chi, \Gamma, \phi \Rightarrow_s \psi, \Delta}}{\chi, \Gamma \Rightarrow_s \phi \to \psi, \Delta}\ {\to}\mathrm{R}
$$

$$
\downarrow
$$

$$
\dfrac{\dfrac{\overset{\mathrm{rinv}_{\phi\to\psi}(\pi)}{\Gamma, \phi \Rightarrow_s \psi, \Delta, \chi} \qquad \overset{\tau_0}{\chi, \Gamma, \phi \Rightarrow_s \psi, \Delta}}{\Gamma, \phi \Rightarrow_s \psi, \Delta}\ \mathsf{cut}_1}{\Gamma \Rightarrow_s \phi \to \psi, \Delta}\ \to R
$$

$$
\overset{\pi}{\Gamma, \phi \to \psi \Rightarrow_s \Delta, \chi} \qquad \dfrac{\overset{\tau_0}{\chi, \Gamma \Rightarrow_s \Delta, \phi} \qquad \overset{\tau_1}{\chi, \Gamma, \psi \Rightarrow_s \Delta}}{\chi, \Gamma, \phi \to \psi \Rightarrow_s \Delta}\ {\to}\mathrm{L}
$$

$$
\downarrow
$$

$$
\dfrac{\dfrac{\overset{\mathsf{inv}^0_{\phi\to\psi}(\pi)}{\Gamma \Rightarrow_s \Delta, \phi, \chi} \qquad \overset{\tau_0}{\chi, \Gamma \Rightarrow_s \Delta, \phi}}{\Gamma \Rightarrow_s \Delta, \phi}\ \mathsf{cut}_1 \qquad \dfrac{\overset{\mathsf{inv}^1_{\phi\to\psi}(\pi)}{\Gamma, \psi \Rightarrow_s \Delta, \chi} \qquad \overset{\tau_1}{\chi, \Gamma, \psi \Rightarrow_s \Delta}}{\Gamma, \psi \Rightarrow_s \Delta}\ \mathsf{cut}_2}{\Gamma, \phi \to \psi \Rightarrow_s \Delta}\ {\to}\mathrm{L}
$$

D.5 Modal cases

Assume the cut formula is not principal in π or τ and where the cut formula is not principal, the principal formula is of shape $\Box\phi$ or $\Box^+\phi$. So where the cut formula is not principal the last rule is either $\Box, \Box^+_f$ or $\Box^+_u$.

D.5.1 Weakening

If the cut formula belongs to the weakening part where the cut-formula is not principal, then it is trivial to eliminate the cut: just keep that proof changing the weakening part to not include the cut-formula.

D.5.2 Commutative

We can assume that the cut formula is not principal with last rule either $\Box, \Box^+_f$ or $\Box^+_u$ occurs at τ since if it occurs at π it would be in the weakening part. In addition, to not occur at the weakening part of the LHS of τ the cut formula must be of shape $\Box\chi_0$ or $\Box^+\chi_0$. In π we have two options, either the cut-formula is principal or not.

If it were not principal, then the principal formula of π is of shape $\phi \to \psi$, $\Box\phi$ or $\Box^+\phi$. The first case was already covered in Subsection D.4 and the other two will provoke that the cut formula belongs to the weakening part of a modal rule, which we covered in Subsubsection D.5.1. So we can safely assume that χ is principal in π and in addition, since if $\chi = \Box^+\chi_0$ then $s \neq \chi_0$ (by Subsection D.1), we know that the last rule applied at π is either $\Box$ or $\Box^+_u$. This leaves six cases depending on the last rule of π and the last rule of τ.

In all the subsequent cases we are going to have $\Sigma_0, \Sigma_1, \Gamma_0, \Gamma_1, \Delta_0, \Delta_1$ such that $\Sigma_0, \Box\Gamma_0, \Box^+\Delta_0 = \Sigma_1, \Box\Gamma_1, \Box^+\Delta_1$. In all of these we define $\Gamma_2 = \Gamma_0, (\Gamma_1 \setminus \Gamma_0) = \Gamma_1, (\Gamma_0 \setminus \Gamma_1)$, $\Delta_2 = \Delta_0, (\Delta_1 \setminus \Delta_0) = \Delta_1, (\Delta_0 \setminus \Delta_1)$ and Σ_2 as the only multiset such that $\Sigma_2, \Box\Gamma_2, \Box^+\Delta_2 = \Sigma_0, \Box\Gamma_0, \Box^+\Delta_0 = \Sigma_1, \Box\Gamma_1, \Box^+\Delta_1$.

$\Box$-$\Box$ reduction.

$$\dfrac{\overline{\pi_0 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\circ \chi_0}}{\Sigma_0, \Box\Gamma_0, \Box^+\Pi_0 \Rightarrow_s \Box\phi, \Delta, \Box\chi_0}\,\Box \qquad \dfrac{\overline{\tau_0 \vdash \chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\circ \phi}}{\Box\chi_0, \Sigma_1, \Box\Gamma_1, \Box^+\Pi_1 \Rightarrow_s \Box\phi, \Delta}\,\Box$$

$$\longmapsto$$

$$\dfrac{\overline{\mathsf{cut}_1(\mathsf{wk}(\pi_0), \mathsf{wk}(\tau_0)) \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \phi}}{\Sigma_2, \Box\Gamma_2, \Box^+\Pi_2 \Rightarrow_s \Box\phi, \Delta}\,\Box$$

$\Box$-$\Box_u^+$ reduction.

$$\dfrac{\overline{\pi_0 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\circ \chi_0}}{\Sigma_0, \Box\Gamma_0, \Box^+\Pi_0 \Rightarrow_s \Box^+\phi, \Delta, \Box\chi_0}\,\Box \qquad \dfrac{\overline{\tau_0 \vdash \chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\circ \phi} \qquad \overline{\tau_1 \vdash \chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\phi \Box^+\phi}}{\Box\chi_0, \Sigma_1, \Box\Gamma_1, \Box^+\Pi_1 \Rightarrow_s \Box^+\phi, \Delta}\,\Box_u^+$$

$$\longmapsto$$

$$\dfrac{\overline{\mathsf{cut}_1(\mathsf{wk}(\pi_0), \mathsf{wk}(\tau_0)) \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \phi} \qquad \overline{\mathsf{cut}_2(\mathsf{wk}(\pi_0)^\phi, \mathsf{wk}(\tau_1)) \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \Box^+\phi}}{\Sigma_2, \Box\Gamma_2, \Box^+\Pi_2 \Rightarrow_s \Box^+\phi, \Delta}\,\Box_u^+$$

$\Box$-$\Box_f^+$ reduction.

$$\dfrac{\overline{\pi_0 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\circ \chi_0}}{\Sigma_0, \Box\Gamma_0, \Box^+\Pi_0 \Rightarrow_s \Box^+\phi, \Delta, \Box\chi_0}\,\Box \qquad \dfrac{\overline{\tau_0 \vdash \chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\circ \phi} \qquad \begin{matrix}\tau_1 \\ \chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\phi \Box^+\phi\end{matrix}}{\Box\chi_0, \Sigma_1, \Box\Gamma_1, \Box^+\Pi_1 \Rightarrow_s \Box^+\phi, \Delta}\,\Box_u^+$$

$$\longmapsto$$

$$\dfrac{\overline{\mathsf{cut}_1(\mathsf{wk}(\pi_0), \mathsf{wk}(\tau_0)) \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \phi} \qquad \dfrac{\begin{matrix}\mathsf{wk}(\pi_0)^\phi \\ \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi, \chi_0\end{matrix} \qquad \begin{matrix}\mathsf{wk}(\tau_1) \\ \chi_0, \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi\end{matrix}}{\Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi}\,\mathsf{cut}_2}{\Sigma_2, \Box\Gamma_2, \Box^+\Pi_2 \Rightarrow_s \Box^+\phi, \Delta}\,\Box_f^+$$

$\Box_u^+$-$\Box$ reduction.

$$\dfrac{\dfrac{}{\pi_0 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\circ \chi_0} \quad \dfrac{}{\pi_1 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\phi \Box^+\chi_0}}{\Sigma_0, \Box\Gamma_0, \Box^+\Pi_0 \Rightarrow_s \Box\phi, \Delta, \Box^+\chi_0}\,\Box_u^+ \quad \dfrac{\dfrac{}{\tau_0 \vdash \boxdot^+\chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\circ \phi}}{\Box^+\chi_0, \Sigma_1, \Box\Gamma_1, \Box^+\Pi_1 \Rightarrow_s \Box\phi, \Delta}\,\Box$$

$$\longmapsto$$

$$\dfrac{\dfrac{}{\rho_0 \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \phi}}{\Box\Gamma_2, \Box^+\Pi_2 \Rightarrow_\circ \Box\phi, \Delta}\,\Box$$

where $\rho_0 = \mathsf{cut}_2(\mathsf{wk}(\pi_0), \mathsf{cut}_1(\mathsf{wk}(\pi_1)^\circ, \mathsf{wk}(\tau_0)))$.

$\Box_u^+$-$\Box_u^+$ reduction.

$$\dfrac{\dfrac{}{\pi_0 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\circ \chi_0} \quad \dfrac{}{\pi_1 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_{\chi_0} \Box^+\chi_0}}{\Sigma_0, \Box\Gamma_0, \Box^+\Pi_0 \Rightarrow_s \Box^+\phi, \Delta, \Box^+\chi_0}\,\Box_u^+ \quad \dfrac{\dfrac{}{\tau_0 \vdash \boxdot^+\chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\circ \phi} \quad \dfrac{}{\tau_1 \vdash \boxdot^+\chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\phi \Box^+\phi}}{\Box^+\chi_0, \Sigma_1, \Box\Gamma_1, \Box^+\Pi_1 \Rightarrow_s \Box^+\phi, \Delta}\,\Box_u^+$$

$$\longmapsto$$

$$\dfrac{\dfrac{}{\rho_0 \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \phi} \quad \dfrac{}{\rho_1 \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi}}{\Sigma_2, \Box\Gamma_2, \Box^+\Pi_2 \Rightarrow_s \Box^+\phi}\,\Box_u^+$$

where $\rho_0 = \mathsf{cut}_2(\mathsf{wk}(\pi_0), \mathsf{cut}_1(\mathsf{wk}(\pi_1)^\circ, \mathsf{wk}(\tau_0)))$ and $\rho_1 = \mathsf{cut}_4(\mathsf{wk}(\pi_0)^\phi, \mathsf{cut}_3(\mathsf{wk}(\pi_1)^\phi, \mathsf{wk}(\tau_1)))$

$\Box^+_u$-$\Box^+_f$ reduction.

$$\dfrac{\dfrac{}{\pi_0 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_\circ \chi_0} \qquad \dfrac{}{\pi_1 \vdash \Gamma_0, \boxdot^+\Pi_0 \Rightarrow_{\chi_0} \Box^+\chi_0}}{\Sigma_0, \Box\Gamma_0, \Box^+\Pi_0 \Rightarrow_s \Box^+\phi, \Delta, \Box^+\chi_0}\,\Box^+_u \qquad \dfrac{\dfrac{}{\tau_0 \vdash \boxdot^+\chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\circ \phi} \qquad \begin{array}{c}\tau_1\\ \boxdot^+\chi_0, \Gamma_1, \boxdot^+\Pi_1 \Rightarrow_\phi \Box^+\phi\end{array}}{\Box^+\chi_0, \Sigma_1, \Box\Gamma_1, \Box^+\Pi_1 \Rightarrow_s \Box^+\phi, \Delta}\,\Box^+_f$$

$$\longmapsto$$

$$\dfrac{\dfrac{}{\rho_0 \vdash \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\circ \phi} \qquad \dfrac{\begin{array}{c}\mathsf{wk}(\pi_0)^\phi\\ \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi, \chi_0\end{array} \qquad \dfrac{\begin{array}{c}\mathsf{wk}(\pi_1)^\phi\\ \chi_0, \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi, \Box^+\chi_0\end{array} \qquad \begin{array}{c}\mathsf{wk}(\tau_1)\\ \boxdot^+\chi_0, \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi\end{array}}{\chi_0, \Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi}\,\mathsf{cut}_3}{\Gamma_2, \boxdot^+\Pi_2 \Rightarrow_\phi \Box^+\phi}\,\mathsf{cut}_4}{\Sigma_2, \Box\Gamma_2, \Box^+\Pi_2 \Rightarrow_s \Box^+\phi}\,\Box^+_f$$

where $\rho_0 = \mathsf{cut}_2(\mathsf{wk}(\pi_0), \mathsf{cut}_1(\mathsf{wk}(\pi_1)^\circ, \mathsf{wk}(\tau_0)))$.

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