Chapter 1
Justification Logics with Probability Operators*

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Abstract In this chapter we present a formal system that results from the combination of two well known formalisms for knowledge representation: probabilistic logic and justification logic. This framework, called probabilistic justification logic, allows the analysis of epistemic situations with incomplete information. We present two sound and strongly complete probabilistic justification logics, which are defined by adding probability operators to the minimal justification logic J. The first logic does not allow nesting of the probability operators and can be used to express statements like "t is a justification for A with probability at least 30%". The second logic allows iterations of the probability operators and can be used to express statements like "I am uncertain for the fact that t is a justification for a coin being counterfeit" or to describe more complex epistemic situations like Kyburg's Lottery Paradox. We also present tight complexity bounds for the satisfiability problem in the aforementioned logics which are obtained with the help of the theory of linear programming and by applying a tableau procedure. Finally, we present two more extensions of the logic J.

1.1 Introduction

In this section we briefly discuss how the frameworks of justification logic and probabilistic logic were introduced. We also explain why the system of probabilistic

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justification logic is necessary and discuss some related work. For all the aforementioned systems we present examples that illustrate their expressive power.

Justification Logic

The description of knowledge as "justified true belief" is usually attributed to Plato. While traditional modal epistemic logic [7] uses formulas of the form $\Box\alpha$ to express that an agent believes/knows α, the language of justification logic [2] 'unfolds' the $\Box$-modality into a family of so-called *justification terms*, which are used to represent evidence for the agent's belief/knowledge. Hence, instead of $\Box\alpha$, justification logic includes formulas of the form $t : \alpha$ (which are usually called justification assertions) meaning

the agent believes α for reason t.

Artemov developed the first justification logic, the Logic of Proofs (usually abbreviated to LP), to provide intuitionistic logic with a classical provability semantics [3, 4, 28, 29]. There, justification terms represent formal proofs in Peano Arithmetic. However, justification terms can be used to represent evidence of a more informal nature. This more general reading of terms lead to the development of justification logics for various purposes and applications [9–11, 26, 27].

Melvin Fitting [15] introduced the use of Kripke models in justification logic in order to place justification logics in the broader family of modal logics. However, semantics to justification logics can also be given by so-called basic modular models. Artemov [5] initially proposed these models to provide an ontologically transparent semantics for justifications. Kuznets and Studer [26] further developed basic modular models so that they can provide semantics to many different justification logics. Note that basic modular models are mathematically equivalent to appropriate adaptations of Mkrtychev models [35] which were introduced earlier.

It is interesting that a famous correspondence between modal logics and justification logics has been established: the so called *realization theorem*. Artemov [4] proved that any theorem in LP can be translated into a theorem in the modal logic $\mathsf{S4}$ by replacing any justification term by the modal operator $\Box$ and that any theorem in $\mathsf{S4}$ can be translated into a theorem in LP by replacing any occurrence of a $\Box$ by an appropriate justification term. So, we say that LP realizes $\mathsf{S4}$, or that LP is the explicit counterpart of $\mathsf{S4}$. In the same way explicit counterparts for many famous modal logics were found [8]. For example, the minimal modal logic K corresponds to the basic justification logic J.

An overview on decidability and complexity results for justification logic can be found in Kuznets' PhD thesis [25]. The problem of decidability for systems with negative introspection has been solved later in [40].

Probabilistic Logic

Chapter **??** describes several probability logics motivated by Nilsson's paper [36]. The logic LPP_2 adds (non-nested) probability operators of the form $P_{\geq s}$ (where s is a rational number) to the language of classical propositional logic, and the logic LPP_1 allows iterations (nesting) of the probability operators. In both logics the formula $P_{\geq s}\alpha$ intuitively means "the probability of truthfulness of the classical propositional formula α is at least s". Having iterations of the probability operators, we can describe more complicated situations, e.g., if c is a coin and p denotes the event "c lands tails", then, $P_{\geq 0.8}P_{\geq 0.6}p$, which reads as "it is at least 80% certain that the probability of c landing tails is at least 60%", can express our uncertainty about the fact of c being counterfeit.

Already in Boole's "Laws of Thought" a procedure for reducing sets of probabilistic constraints to systems of linear (in)equalities was provided. That problem, denoted PSAT, can be stated in the following form: "assume that we are given a formula in conjunctive normal form and a probability for each clause. Is there a probability distribution (over the set of all possible truth assignments of the variables appearing in the clauses) that satisfies all the clauses?" The papers [13, 17] proved NP-completeness of PSAT for logics without iterations of probability operators, while [12] mentioned (without giving a complete formal proof) that complexity bounds for the satisfiability problem in a probabilistic logic that allows nesting of the probability operators (like in LPP_1) can be obtained by employing an algorithm based on a tableau construction as in classical modal logic [20].

Probabilistic Justification Logic

In everyday life we often have to deal with incomplete evidence which naturally leads to vague justifications. So, it seems necessary to combine reasoning about knowledge and uncertainty. Let us consider the following example.

Motivating Example
Anna receives a phone call from Bethany. Bethany tells Anna that tax rates will increase. Anna reads in the New York Times that tax rates will increase. Anna considers the New York Times to be a much more reliable source than Bethany.

In order to describe the situation in the Motivating Example, Anna needs a framework that allows reasoning about justifications and uncertainty together. Anna for example needs to say that "the probability of the fact that tax rates will increase, because Bethany said so, is 30%" or that "the probability of the fact that tax rates will increase, because it is written in the New York times, is 80%". This kind of statements can be nicely expressed in probabilistic justification logic, which is a framework that allows reasoning about the probabilities of justified statements.

In Section 1.3 we describe the probabilistic justification logic PJ [22], which is a probabilistic logic over the basic justification logic J that allows formulas of the form "$P_{\geq s}(t : \alpha)$", meaning that

the probability that t justifies α is at least s.

For instance, we can study the formula:

$$P_{\geq r}(u : (\alpha \to \beta)) \to \big(P_{\geq s}(v : \alpha) \to P_{\geq r \cdot s}(u \cdot v : \beta)\big) , \tag{1.1}$$

which states that the probability of the conclusion of an application axiom is greater than or equal to the product of the probabilities of its premises. We will see later that this, of course, only holds in models where the premises are independent.

The semantics of PJ consists of a set of possible worlds, each a model of justification logic, and a probability measure $\mu(\cdot)$ on sets of possible worlds. We assign a probability to a formula α of justification logic as follows. We first determine the set $[\alpha]$ of possible worlds that satisfy α. Then we obtain the probability of α as $\mu([\alpha])$, i.e. by applying the measure function to the set $[\alpha]$. Hence our logic relies on the usual model of probability. This makes it possible, e.g., to explore the role of independence and to investigate formulas like (1.1) in full generality.

In Section 1.4 we present the logic PPJ [23] that allows iterations of the probability operators and also application of justification terms to probability operators and vice versa (as we will see later this is the property that makes finding complexity bounds for the satisfiability problem in PPJ a challenging task). So, continuing our example with the counterfeit coin c, if t is some explicit reason to believe that c is counterfeit, then in PPJ we could have the formula $P_{\geq 0.8}(t : P_{\geq 0.6}p)$, meaning "I am uncertain for a particular justification of c being counterfeit, e.g. because this coin looks similar to a counterfeit coin I have seen some time ago". A more interesting and complicated application of PPJ is that it can be used to analyze Kyburg's famous lottery paradox [30].

We conclude that probabilistic justification logics are intended for comparing different sources of information. Thus the key idea behind the introduction of logics PJ and PPJ is that:

$$\begin{array}{l}\text{different kinds of evidence for } \alpha \\ \text{lead to different degrees of belief in } \alpha.\end{array} \tag{1.2}$$

Also, in Section 1.6 we explain how tight complexity bounds for the probabilistic justification logics PJ and PPJ can be obtained. In the case of PJ the complexity bounds are obtained via a small model property and in the case of PPJ the bounds are obtained by applying a tableau procedure [21].

In the last Section of this chapter, 1.7, we briefly discuss two more extensions of the basic justification logic J.

Related Work

In the past 5 years there have been several attempts to develop a framework that models uncertain reasoning in justification logic. Very closely related to our approach are Milnikel's proposal [34] for a system with uncertain justifications, Ghari's fuzzy justification logics [18, 19], the possibilistic justification logic introduced by Su, Fan and Liau in [14, 41] and a justification logic based on both probabilistic and possibilistic logic proposed by Lurie [33].

Milnikel introduces formulas of the form $t :_q \alpha$, which correspond to our

$$P_{\geq q}(t : \alpha).$$

However, there are two important differences between our work and Milnikel's.

The first difference concerns the semantics. Whereas our models are probability spaces, Milnikel uses a variation of Kripke-Fitting models. He also uses a special kind of semantics that allow him to avoid infinitary rules. To every world w, term t and formula α, Milnikel assigns an interval $E(w,t,\alpha)$ that is equal to $[0,r)$ or $[0,r]$ for some rational r from $[0,1]$. Then the justification assertion $t :_q \alpha$ is true at a world w iff $q \in E(w,t,\alpha)$ and also α is true in all worlds accessible from w.

The second difference concerns independence. As will be evident later, in our setting formula (1.1) holds only in models that satisfy certain conditions. However Milnikel accepts this formula as an axiom, which implies that he assumes that various pieces of evidence are independent.

Ghari combines justification logic with uncertain reasoning in a different way. He presents several justification logics where the classical base is replaced with Hájek's rational Pavelka logic. He proves that all the principles that hold in Milnikel's logic of uncertain justifications also hold in his framework and also presents an analysis of the famous sorites paradox.

The logic of Su, Fan and Liau includes formulas $t :_r A$ to express that *according to evidence t, A is believed with certainty at least r*. However, the following principle holds in their logic:

$$s :_r A \wedge t :_q A \rightarrow s :_{\max(r,q)} A.$$

Hence all justifications for a belief yield the same (strongest) certainty, which is not in accordance with our guiding idea (1.2).

Lurie introduced the justification logic pr-LP which incorporates the essential features of both a fuzzy logic and a probabilistic logic. He shows that any model for pr-LP is a model for Kolmogorov probability and also discusses several arguments which show that pr-LP is the most appropriate model for evidentialist justification.

The combination of evidenced-based reasoning and reasoning under uncertainty has also been studied by Artemov in [6] and Schechter in [39]. Schechter combined features from justification logics and logics of plausibility based beliefs to build a normal modal logic of explicit beliefs, where each agent can explicitly state which is their justification for believing in a given sentence. Artemov studied a justification logic to formalize aggregated probabilistic evidence. His approach can handle conflicting and inconsistent data and positive and negative evidence for the same proposition as well.

1.2 The Basic Justification Logic J

In this section we present the syntax and semantics and recall some fundamental properties of the minimal justification logic J.

Justification terms are built from countably many constants and countably many variables according to the following grammar:

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid\, !t\ ,$$

where c is a constant and x is a variable. Tm denotes the set of all terms and Con denotes the set of all constants. For $c \in \mathsf{Con}$ and $n \in \omega$ we define

$$!^0 c := c \qquad \text{and} \qquad !^{n+1} c := !\,(!^n c)\ .$$

The operators $\cdot$ and $+$ are assumed to be left-associative. The intended meaning of the connectives used to construct terms will be clear when we present the deductive system for J.

Let Prop be a countable set of propositional letters. Formulas of the language $\mathscr{L}_\mathsf{J}$ (justification formulas) are built according to the following grammar:

$$\alpha ::= p \mid t : \alpha \mid \neg\alpha \mid \alpha \wedge \alpha\ ,$$

where $t \in \mathsf{Tm}$ and $p \in \mathsf{Prop}$.

The deductive system for J is the Hilbert system presented in Table 1.1. Axiom (J) is also called the *application axiom* and is the explicit version of the distribution axiom in modal logic. It states that we can combine a justification for $\alpha \to \beta$ and a justification for α in order to obtain a justification for β. Axiom $(+)$, which is also called the *monotonicity axiom*, states that if s or t is a justification for α then the term $s + t$ is also a justification for α. This operator can model monotone reasoning like proofs in some formal system of mathematics: if I already have a proof t for a formula α, then t remains a proof for α if a few more lines are added to it. Rule (AN!) states that any constant can be used to justify any axiom and also that we can use the operator ! to express positive introspection: if c justifies axiom instance α, then $!c$ justifies $c : \alpha$, $!!c$ justifies $!c : c : \alpha$ and so on. The previous situation is the explicit analogue of the positive iteration of modalities in traditional modal logic: I know α, I know that I know α and so on. The operator ! is also called proof checker or proof verifier. This is because we can think that α is a problem given to a student, c is the solution (or the proof) given by the student and $!c$ is the verification of correctness for the proof given by the tutor. So justification logic can model the following situation:

Student: I have a proof for α (i.e. $c : \alpha$).

Tutor: I can verify your proof for α (i.e. $!c : c : \alpha$).

In justification logic it is common to assume that only some axioms are justified by constants (see the notion of *constant specification* in [2]). However, in our approach we assume that every constant justifies every axiom (this assumption corresponds to the notion of a *total constant specification* [2]).

Axioms:	
(P)	finite set of axiom schemata axiomatizing classical propositional logic in the language $\mathscr{L}_{\mathsf{J}}$
(J)	$\vdash s : (\alpha \to \beta) \to (t : \alpha \to s \cdot t : \beta)$
$(+)$	$\vdash (s : \alpha \vee t : \alpha) \to s + t : \alpha$
Rules:	
(MP)	if $T \vdash \alpha$ and $T \vdash \alpha \to \beta$ then $T \vdash \beta$
$(\mathsf{AN!})$	$\vdash !^n c : !^{n-1} c : \cdots : !c : c : \alpha$, where c is a constant, α is an instance of (P), (J) or $(+)$ and $n \in \omega$

Table 1.1 The Deductive System J

In order to illustrate the usage of axioms and rules in J we present the following example:

Example 1.1. Let $a, b \in \mathsf{Con}$, $\alpha, \beta \in \mathscr{L}_{\mathsf{J}}$ and x, y be variables. Then we have the following:

$$\vdash_{\mathsf{J}} (x : \alpha \vee y : \beta) \to a \cdot x + b \cdot y : (\alpha \vee \beta).$$

Proof. Since $\alpha \to \alpha \vee \beta$ and $\beta \to \alpha \vee \beta$ are instances of (P), we can use $(\mathsf{AN!})$ to obtain

$$\vdash_{\mathsf{J}} a : (\alpha \to \alpha \vee \beta)$$

and

$$\vdash_{\mathsf{J}} b : (\beta \to \alpha \vee \beta).$$

Using (J) and (MP) we obtain

$$\vdash_{\mathsf{J}} x : \alpha \to a \cdot x : (\alpha \vee \beta)$$

and

$$\vdash_{\mathsf{J}} y : \beta \to b \cdot y : (\alpha \vee \beta).$$

Using $(+)$ and propositional reasoning we obtain

$$\vdash_{\mathsf{J}} x : \alpha \to a \cdot x + b \cdot y : (\alpha \vee \beta)$$

and

$$\vdash_{\mathsf{J}} y : \beta \to a \cdot x + b \cdot y : (\alpha \vee \beta).$$

We can now obtain the desired result by applying propositional reasoning. ■

Logic J also enjoys the *internalization property*, which is presented in the following theorem. Internalization states that the logic internalizes its own notion of proof. The version without premises is an explicit form of the necessitation rule of modal logic.

Theorem 1.1 (Internalization, [26]). *For any formulas* $\alpha,\ \beta_1,\dots,\beta_n \in \mathscr{L}_\mathsf{J}$ *and terms* $t_1,\dots,t_n \in \mathsf{Tm}$*, if*

$$\beta_1,\ \dots\ ,\ \beta_n \vdash_\mathsf{J} \alpha$$

then there exists a term t such that

$$t_1 : \beta_1,\ \dots\ ,\ t_n : \beta_n \vdash_\mathsf{J} t : \alpha\ . \qquad \blacksquare$$

The models for J which we are going to use are called M-models and were introduced by Mkrtychev [35] for the logic LP. Later Kuznets [24] adapted these models for other justification logics (including J) and proved the corresponding soundness and completeness theorems. An M-model consists of an evaluation for propositional atoms and an evidence function that assigns justifications to formulas. Formally, we have the following:

Definition 1.1 (M-Model). An M-model is a pair $\langle v,\mathscr{E}\rangle$, where $v : \mathsf{Prop} \to \{\mathsf{T},\mathsf{F}\}$ and $\mathscr{E} : \mathsf{Tm} \to \mathbb{P}(\mathscr{L}_\mathsf{J})$ such that for every $s,t \in \mathsf{Tm}$, for $c \in \mathsf{Con}$ and $\alpha,\beta \in \mathscr{L}_\mathsf{J}$, for γ being an axiom instance of J and $n \in \omega$ we have

1. $\big(\alpha \to \beta \in \mathscr{E}(s)$ and $\alpha \in \mathscr{E}(t)\big) \Longrightarrow \beta \in \mathscr{E}(s \cdot t)$;
2. $\mathscr{E}(s) \cup \mathscr{E}(t) \subseteq \mathscr{E}(s+t)$;
3. $!^{n-1}c : !^{n-2}c : \cdots :!c : c : \gamma \in \mathscr{E}(!^n c)$. $\blacksquare$

Definition 1.2 (Truth in an M-model). We define what it means for an $\mathscr{L}_\mathsf{J}$-formula to hold in the M-model $M = \langle v,\mathscr{E}\rangle$ inductively as follows (the connectives $\neg$ and $\wedge$ are treated classically):

$$M \models p \Longleftrightarrow v(p) = \mathsf{T} \qquad \text{for } p \in \mathsf{Prop}\ ;$$
$$M \models t : \alpha \Longleftrightarrow \alpha \in \mathscr{E}(t)\ . \qquad \blacksquare$$

Last but not least, we have soundness and completeness of J with respect to M-models [5, 26] and some tight complexity bounds for the satisfiability problem[2].

Theorem 1.2 (Soundness and Completeness of J). *Let* $\alpha \in \mathscr{L}_\mathsf{J}$*. Then we have:*

$$\vdash_\mathsf{J} \alpha \quad \Longleftrightarrow \quad \models_\mathsf{M} \alpha\ ,$$

where $\models_\mathsf{M} \alpha$ *means that* α *holds in any* M*-model.* $\blacksquare$

Theorem 1.3. *The* J*-satisfiability problem is* Σ_2^p*-complete.* $\blacksquare$

[2] the satisfiability problem for some logic L is defined as usual: given a formula A in the language of L, is there an L-model for this formula?

The upper bound in the previous theorem was shown in [24] and the lower bound in [1]. It is interesting that the complexity gap between the satisfiability problems in the minimal modal logic K and its explicit analogue, which is the justification logic J, is huge: the former is PSPACE-complete [20] while the latter is only Σ_2^{p}-complete.

1.3 Non-Iterated Probabilistic Justification Logic

In this section we present the syntax, semantics and some interesting properties of the non-iterated probabilistic logic PJ, which is defined over the basic justification logic J.

Syntax and Semantics

Let $\mathbb{S}$ denote the set of all rational numbers from the interval $[0,1]$. The formulas of the language $\mathscr{L}_{\mathsf{PJ}}$ are built according to the following grammar:

$$A ::= P_{\geq s}\alpha \mid \neg A \mid A \wedge A$$

where $s \in \mathbb{S}$, and $\alpha \in \mathscr{L}_{\mathsf{J}}$. The intended meaning of the formula $P_{\geq s}\alpha$ is that "the probability of truthfulness for the justification formula α" is at least s.

We also use the following syntactical abbreviations:

$$P_{<s}\alpha \equiv \neg P_{\geq s}\alpha \qquad P_{>s}\alpha \equiv \neg P_{\leq s}\alpha$$
$$P_{\leq s}\alpha \equiv P_{\geq 1-s}\neg\alpha \qquad P_{=s}\alpha \equiv P_{\geq s}\alpha \wedge P_{\leq s}\alpha\ .$$

We use capital Latin letters like $A,\ B,\ C,\ \ldots$ for members of $\mathscr{L}_{\mathsf{PJ}}$ possibly primed or with subscripts.

The system PJ is the deductive system presented in Table 1.2. Axiom (NN) corresponds to the fact that the probability of truthfulness for every formula is at least 0 (the acronym (NN) stands for non-negative). Observe that by substituting $\neg A$ for A in (NN), we have $P_{\geq 0}\neg A$, which by our syntactical abbreviations is $P_{\leq 1}A$. Hence axiom (NN) also corresponds to the fact that the probability of truthfulness for every formula is at most 1. Axioms $(\mathsf{L1})$ and $(\mathsf{L2})$ describe some properties of inequalities (the L in $(\mathsf{L1})$ and $(\mathsf{L2})$ stands for less). Axioms $(\mathsf{Add1})$ and $(\mathsf{Add2})$ correspond to the additivity of probabilities for disjoint events (the Add in $(\mathsf{Add1})$ and $(\mathsf{Add2})$ stands for additivity).

Rule (PN) is the probabilistic analogue of the necessitation rule in modal logics (hence the acronym (PN) stands for probabilistic necessitation): if a justification formula is valid, then it has probability 1. Rule (ST) intuitively states that if the probability of a formula is arbitrary close to s, then it is at least s. Observe that the rule (ST) is infinitary in the sense that it has an infinite number of premises. The acronym (ST) stands for strengthening, since the statement of the result is stronger than the statement of the premises. The reader should pay attention to the premises of the 3 rules. Whereas for (MP) and (ST), the formula in the premises can have a proof of any length, in the case of (PN) the formula has to be a theorem. This is an important restriction. Without it we are unable to prove the Deduction Theorem in PJ, and as a consequence we do not have strong completeness either.

A proof of an $\mathscr{L}_{\mathsf{PJ}}$-formula A from a set T of $\mathscr{L}_{\mathsf{PJ}}$-formulas is a sequence of formulas A_k indexed by countable ordinal numbers such that the last formula is A, and each formula in the sequence is an axiom, or a formula from T, or it is derived from the preceding formulas by a PJ-rule of inference.

Axioms:	
(P)	finitely many axiom schemata axiomatizing classical propositional logic in the language $\mathscr{L}_{\mathsf{PJ}}$
(NN)	$\vdash P_{\geq 0}A$
(L1)	$\vdash P_{\leq r}A \to P_{<s}A$, where $s > r$
(L2)	$\vdash P_{<s}A \to P_{\leq s}A$
(Add1)	$\vdash P_{\geq r}A \wedge P_{\geq s}B \wedge P_{\geq 1}\neg(A \wedge B) \to P_{\geq \min(1,r+s)}(A \vee B)$
(Add2)	$\vdash P_{\leq r}A \wedge P_{<s}B \to P_{<r+s}(A \vee B)$, where $r+s \leq 1$
Rules:	
(MP)	if $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(PN)	if $\vdash_{\mathsf{J}} \alpha$ then $\vdash_{\mathsf{PJ}} P_{\geq 1}\alpha$
(ST)	if $T \vdash A \to P_{\geq s-\frac{1}{k}}\alpha$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to P_{\geq s}\alpha$

Table 1.2 System PJ

A model for PJ is a probability space. The universe of the probability space is a set of models for the logic J. In order to determine the probability of a justification formula α in such a probability space we have to find the measure of the set containing all the M-models that satisfy α. The following definitions formalize the notion of a PJ-model and the notion of satisfiability in a PJ-model (the notions of algebras, finitely additive measures and probability spaces are introduced in Definition **??**).

Definition 1.3 (PJ-Model). A model for PJ or simply a PJ-*model* is a structure $M = \langle W, H, \mu, v \rangle$ where:

- $\langle W, H, \mu \rangle$ is a probability space ;
- v is a function from W to the set of all M-models, i.e. $v(w)$ is an M-model for each world $w \in W$. We will usually write v_w instead of $v(w)$.
- for every $\alpha \in \mathscr{L}_{\mathsf{J}}$, $[\alpha]_M \in H$ holds, where

$$[\alpha]_M = \{w \in W \mid v_w \models \alpha\} .$$

We will omit the subscript M, i.e. we will simply write $[\alpha]$, if M is clear from the context. ■

Definition 1.4 (Truth in a PJ-model). Let $M = \langle W, H, \mu, v\rangle$ be a PJ-model. We define what it means for an $\mathscr{L}_{\mathsf{PJ}}$-formula to hold in M inductively as follows:

$$\begin{aligned} M \models P_{\geq s}\alpha &\Longleftrightarrow \mu([\alpha]_M) \geq s \ ; \\ M \models \neg A &\Longleftrightarrow M \not\models A \ ; \\ M \models A \wedge B &\Longleftrightarrow \big(M \models A \text{ and } M \models B\big) \ . \end{aligned}$$ ■

Properties

In this part we establish some useful properties about the logic PJ. The following lemma states that if $\alpha \to \beta$ is a theorem of J, then PJ proves that β is at least as probable as α. It is interesting to observe that this property resembles the distribution axiom in modal logic (and of course also the rule modus ponens). The lemma also states some monotonicity properties of inequalities that can be proved in PJ.

Lemma 1.1 ([22, 37]). *If* $\vdash_{\mathsf{J}} \alpha \to \beta$ *then* $\vdash_{\mathsf{PJ}} P_{\geq s}\alpha \to P_{\geq s}\beta$. ■

The next property we are going to present is a probabilistic version of internalization. Many forms of probabilistic internalization can be proved for the logic PJ. Theorem 1.4 states two of them. Item 1 of Theorem 1.4 states that if we have uncertainty for the conjunction of the premises, this uncertainty is passed to the result, whereas item 2 of Theorem 1.4 states that uncertainty in a single premise is again passed to the result.

Theorem 1.4 (Probabilistic Internalization, [22]). *For any* $\alpha,\ \beta_1,\ \dots\ ,\beta_n \in \mathscr{L}_{\mathsf{J}}$, $t_1,\ \dots\ ,t_n \in \mathsf{Tm}$ *and* $s \in \mathbb{S}$, *if:*

$$\beta_1,\ \dots\ ,\ \beta_n \vdash_{\mathsf{J}} \alpha$$

then there exists a term t such that:

1. $P_{\geq s}(t_1 : \beta_1 \wedge\ \dots\ \wedge\ t_n : \beta_n) \vdash_{\mathsf{PJ}} P_{\geq s}(t : \alpha)$;
2. *for every* $i \in \{1, \dots, n\}$*:*

$$\big\{\ P_{\geq 1}(t_j : \beta_j)\ \big|\ j \neq i\ \big\}, P_{\geq s}(t_i : \beta_i) \vdash_{\mathsf{PJ}} P_{\geq s}(t : \alpha)\ .$$ ■

Remark 1.1. If we consider the formulation of probabilistic internalization without premises, then we obtain that:

$$\vdash_{\mathsf{J}} \alpha \qquad \text{implies} \qquad \vdash_{\mathsf{PJ}} P_{\geq 1}(t : \alpha) \quad \text{for some term } t.$$

The above rule contains a combination of constructive and probabilistic necessitation. ■

We close this section by presenting a semantical characterization of independence in the system PJ.

The notion of independent set in a model is defined as usual.

Definition 1.5 (Independent Sets in a PJ-Model). Let $M = \langle W, H, \mu, v \rangle$ be a model for PJ and let $U, V \in H$. U and V will be called *independent* in M iff the following holds:

$$\mu(U \cap V) = \mu(U) \cdot \mu(V) . \qquad \blacksquare$$

Theorem 1.5. *Let $u, v \in \mathsf{Tm}$, let $\alpha, \beta \in \mathscr{L}_{\mathsf{J}}$ and let M be a PJ-model. Assume that $[u : (\alpha \to \beta)]_M$ and $[v : \alpha]_M$ are independent in M. Then for any $r, s \in \mathbb{S}$ we have:*

$$M \models P_{\geq r}(u : (\alpha \to \beta)) \to \big(P_{\geq s}(v : \alpha) \to P_{\geq r \cdot s}(u \cdot v : \beta)\big) . \qquad \blacksquare$$

So, as we promised in Section 1.1 we have shown that equation (1.1) holds in models where the premises are independent. It seems that a syntactical characterization of independence is impossible in PJ, unless we assume that all pieces of evidence are independent as in the work of Milnikel [34]. Of course this would imply that we have to abandon our natural semantics which is based on the standard model for probability.

1.4 Iterated Probabilistic Justification Logic

In this section, we present the syntax and semantics of the iterated probabilistic justification logic PPJ [23]. The logic PPJ follows the design of LPP_1 [37] and allows formulas of the form $t : (P_{\geq s}A)$ as well as $P_{\geq r}(P_{\geq s}A)$. This explains the name PPJ: the two P's refer to iterated P-operators. We also show how Kyburg's [30] famous lottery paradox can be formalised in the language of PPJ.

Syntax and Semantics

The language $\mathscr{L}_{\mathsf{PPJ}}$ is defined by the following grammar:

$$A ::= p \mid P_{\geq s}A \mid \neg A \mid A \wedge A \mid t : A$$

where $t \in \mathsf{Tm}$, $s \in \mathbb{S}$ and $p \in \mathsf{Prop}$. For the language $\mathscr{L}_{\mathsf{PPJ}}$ we assume the same abbreviations as for the language $\mathscr{L}_{\mathsf{PJ}}$.

This system is presented in Table 1.3. As we can see, the axiomatization for PPJ simply consists of the axiomatization for PJ and the axiomatization for J that we have already presented. There is only one difference: in PJ rule (PN) can only be applied if α is a theorem of J (and not PJ), whereas in the case of PPJ, (PN) can be applied if A is a theorem of PPJ.

Axioms:	
(P)	finitely many axiom schemata axiomatizing classical propositional logic in the language $\mathscr{L}_{\mathsf{PPJ}}$
(NN)	$\vdash P_{\geq 0}A$
(L1)	$\vdash P_{\leq r}A \to P_{<s}A$, where $s > r$
(L2)	$\vdash P_{<s}A \to P_{\leq s}A$
(Add1)	$\vdash P_{\geq r}A \wedge P_{\geq s}B \wedge P_{\geq 1}\neg(A \wedge B) \to P_{\geq \min(1,r+s)}(A \vee B)$
(Add2)	$\vdash P_{\leq r}A \wedge P_{<s}B \to P_{<r+s}(A \vee B)$, where $r+s \leq 1$
(J)	$\vdash s : (A \to B) \to (t : A \to s \cdot t : B)$
(+)	$\vdash (s : A \vee t : A) \to s+t : A$
Rules:	
(MP)	if $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(PN)	if $\vdash A$ then $\vdash P_{\geq 1}A$
(ST)	if $T \vdash A \to P_{\geq s-\frac{1}{k}}B$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to P_{\geq s}B$
(AN!)	$\vdash\ !^n c : !^{n-1}c : \cdots : !c : c : A$, where $c \in \mathsf{Con}$, A is an instance of some PPJ-axiom and $n \in \omega$

Table 1.3 The Deductive System PPJ

A PPJ-model is a combination of a PJ-model and a J-model. It consists of a probability space, where to every possible world an M-model as well as a probability space is assigned. This way we can deal with iterated probabilities and justifications over probabilities. The formal defining of a PPJ-model follows:

Definition 1.6 (PPJ-Model and Truth in a PPJ-Model). Let $M = \langle U, W, H, \mu, v\rangle$ where:

1. U is a non-empty set of objects called worlds;
2. W, H, μ and v are functions, which have U as their domain, such that for every $w \in U$:
   - $\langle W(w), H(w), \mu(w)\rangle$ is a probability space with $W(w) \subseteq U$;
   - v_w is an M-model[3].

Truth in M is defined as follows:

[3] We will usually write v_w instead of $v(w)$.

$$
\begin{aligned}
M,w \models p \quad &\iff \quad p^v_w = \mathsf{T} \quad \text{for } p \in \mathsf{Prop}\ ;\\
M,w \models P_{\geq s}B \quad &\iff \quad \Big([B]_{M,w} \in H(w) \text{ and } \mu(w)\big([B]_{M,w}\big) \geq s\Big)\\
&\qquad\quad \text{where } [B]_{M,w} = \{x \in W(w) \mid M,x \models B\}\ ;\\
M,w \models \neg B \quad &\iff \quad M,w \not\models B\ ;\\
M,w \models B \wedge C \quad &\iff \quad \big(M,w \models B \text{ and } M,w \models C\big)\ ;\\
M,w \models t : B \quad &\iff \quad B \in t^v_w\ .
\end{aligned}
$$

M is called a PPJ-model if for every $w \in U$ and for every $A \in \mathscr{L}_{\mathsf{PPJ}}$:

$$[A]_{M,w} \in H(w)\ . \qquad \blacksquare$$

Application to the Lottery Paradox

The situation described in the famous lottery paradox, defined by Kyburg, can be described as follows: consider a lottery containing 1000 tickets, where every ticket has exactly the same probability to win and exactly one ticket will win. We also assume that a proposition is believed if and only if its degree of belief is greater than 0.99. Under these assumptions it is rational to believe that the first ticket can not win, it is rational to believe that the second ticket can not win, etc. Since rational belief is closed under conjunction, we have the following paradoxical situation: it is rational to believe that no ticket wins and also that at least one ticket wins.

Using the system of PPJ we can analyze the lottery paradox as follows. Firstly we need to express in a PPJ-formula how we can move from degrees of belief to justifications (this principle is what Foley [16] calls *the Lockean thesis*). So, we assume that for every $t \in \mathsf{Tm}$ there exists a term $\mathsf{pb}(t)$ such that:

$$t : (P_{>0.99}A) \to \mathsf{pb}(t) : A\ . \tag{1.3}$$

Let w_i be the proposition *ticket i wins*. For each $1 \leq i \leq 1000$, there is a term t_i such that $t_i : \big(P_{=\frac{999}{1000}} \neg w_i\big)$ holds. Hence by statement (1.3) we get

$$\mathsf{pb}(t_i) : \neg w_i \quad \text{for each } 1 \leq i \leq 1000. \tag{1.4}$$

Now in PPJ we have that

$$s_1 : A \wedge s_2 : B \to \mathsf{Con}(s_1, s_2) : (A \wedge B) \tag{1.5}$$

is a valid principle (for a suitable term $\mathsf{Con}(s_1, s_2)$). Hence by statement (1.4) we conclude that

$$\text{there exists a term } t \text{ with } t : (\neg w_1 \wedge \cdots \wedge \neg w_{1000})\ , \tag{1.6}$$

which leads to a paradoxical situation since it is also believed that one of the tickets wins.

As we mentioned in Section 1.2, it is common in justification logic to assume that only some axioms are justified by constants. A similar restriction is possible in PPJ. So, in PPJ we can avoid the paradox by restricting the axioms that are justified so that (1.5) is valid only if $\mathsf{Con}(s_1, s_2)$ does not contain two different subterms of the form $\mathsf{pb}(t)$. Then the step from (1.4) to (1.6) is no longer possible and we can avoid the paradoxical belief.

Leitgeb, in his *Stability Theory of Belief* [32], presents a solution to the lottery paradox according to which *it is not permissible to apply the conjunction rule for beliefs across different contexts*. Our proposed restriction on the justifications for the axioms is one way to formalize Leitgeb's idea. Even if our analysis of the lottery paradox is not very deep, we feel that it is worth further employing probabilistic justification logic in the investigation of the lottery paradox. For example one interesting direction for further research is to try to interpret the above justifications t_i as stable sets in Leitgeb's sense.

1.5 Soundness and Strong Completeness

Soundness for both PJ and PPJ is proved by transfinite induction on the depth of the proof. In order to prove soundness of the rule (ST) we need the *Archimedean property* for the real numbers, i.e. that for any real number $\varepsilon > 0$ there exists an $n \in \omega$ such that $\frac{1}{n} < \varepsilon$.

The strong completeness theorems are obtained by applying the standard Henkin procedure, which can be summarized as follows:

- a version of the Lindenbaum lemma is proved, i.e. that every consistent set of formulas can be extended to a maximal consistent set
- a canonical model is defined, i.e. a Kripke structure where the worlds are the maximal consistent sets [4]
- the truth lemma is proved, i.e. it is shown that every consistent set is satisfied in some world of the canonical model.

The most difficult part is to prove an appropriate version of the Lindenbaum lemma. Let T be a consistent set. In a typical completeness proof the set T is extended to a maximal consistent set by adding for every formula A either A or $\neg A$. This maintains the consistency of T in deductive systems where the proofs have finite length. However in PJ and PPJ the proofs may have infinite length so it is possible that this simple construction leads to an inconsistent set. This problem is overcome as follows: when the negation of a possible result of the infinite rule (ST) is added, the negation of at least one premise of (ST) is added too. This way the consistency is maintained. So, we have:

[4] in the case of PJ the worlds of the canonical model are all the M-models, since no iteration of the probability operators is allowed.

Theorem 1.6 (Strong Completeness for PJ and PPJ). *Let $T \subseteq \mathscr{L}_{\mathsf{PJ}}$, $A \in \mathscr{L}_{\mathsf{PJ}}$, $R \subseteq \mathscr{L}_{\mathsf{PPJ}}$ and $B \in \mathscr{L}_{\mathsf{PPJ}}$. It holds:*

$$T \vdash_{\mathsf{PJ}} A \iff T \models_{\mathsf{PJ}} A \quad \textit{and}$$
$$R \vdash_{\mathsf{PPJ}} A \iff R \models_{\mathsf{PPJ}} B$$

where $T \models_{\mathsf{PJ}} A$ means that for any PJ-model M,

$$M \models T \text{ implies } M \models A$$

and $R \models_{\mathsf{PPJ}} B$ means that for any PPJ-model M and any world w,

$$M, w \models R \text{ implies } M, w \models B^5. \qquad \blacksquare$$

1.6 Complexity Bounds

In this section we present complexity bounds for the satisfiability problem in the logics PJ and PPJ, i.e. the decision problem that asks whether a model satisfying a given formula exists.

In the case of PJ we can reduce the satisfiability problem to the satisfiability of a set of linear (in)equalities. Then by using a well known theorem from linear algebra it can be shown that every satisfiable PJ-formula is satisfied in a small model M where the following holds:

- the number of worlds of M is bounded by the size of the formula
- the sizes (in binary) of the probabilities assigned to every world are bounded by the size of the formula
- the M-model assigned to each world can also “be described by the formula in a finite way”.

This small model can be guessed by a non-deterministic algorithm in polynomial time and the M-models assigned to the worlds can be guessed by a coNP-algorithm using a result from Kuznets [24]. The previous two algorithms give us:

Theorem 1.7 ([21]). *The satisfiability problem for PJ is Σ_2^{p}-complete.* ■

Obtaining complexity bounds for the satisfiability in PPJ is much more involved. Using a very similar algorithm as the one for modal logic [20] it can be shown that every PPJ-satisfiable formula is satisfied in a model that looks like a tree, whose size is bounded by the number of probability operators appearing in the formula. In order to check a formula for satisfiability, this tree can be traversed in a depth first manner using polynomial space. However, it has to be noted that the presence of formulas of the form $t : P_{\geq s}A$ and the semantics of the probability operators force us

[5] When we say that a model satisfies a set, we mean that the model satisfies every formula in the set.

to use more complex arguments than in the case of standard modal logic. So, one can prove that:

Theorem 1.8 ([21]). *The satisfiability problem for* PPJ *is* PSPACE*-complete.* ■

1.7 Extensions

The justification logic J can also been extended with operators for approximate conditional probabilities [38]. Formally, formulas of the form $\mathsf{CP}_{\approx r}(\alpha,\beta)$ are introduced with the following meaning:

the probability of α *under the condition* β *is approximately* r.

This makes it possible to express defeasible inferences for justification logic. For instance, we can express

if x justifies that Tweety is a bird, then *usually* $t(x)$ justifies that Tweety flies

as $\mathsf{CP}_{\approx 1}(t(x){:}\mathsf{flies}, x{:}\mathsf{bird})$.

Another probabilistic extension of justification logic is possible if we change to subset semantics for justification logic [31]. The main idea there is to interpret terms as sets of possible worlds. Then we define a formula $t{:}A$ to be true if and only if the interpretation of t is a subset of the set of worlds where A holds. Since terms are now interpreted as sets, which one can measure, it is possible to assign probabilities to terms (and not only to formulas as in PJ and PPJ). This makes it possible to represent uncertain justifications and probabilistic evidence. This model, in particular, subsumes Artemov's approach of aggregating probabilistic evidence [6].

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