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Possible World Semantics of Justification Logic: A Survey

Abstract. In this paper, we present neighborhood semantics for various justification logics and establish the completeness theorem. We then compare various possible world semantics of the logic of proofs, including Fitting models, modular models, subset models, and relational models, with the neighborhood models. Finally, we present topological semantics for the logic of proofs and show the completeness theorem.

Keywords: justification logic; possible world semantics; neighborhood semantics; topological semantics

1. Introduction

Justification logics are modal-like logics that provide a framework for reasoning about epistemic justifications (see [4, 5, 26]). The language of justification logics extends the language of propositional logic by proof terms and expressions of the form $[t]\varphi$, with the intended meaning “ t is a justification for φ ” or “ t is a proof for φ ”. The *logic of proofs* LP is the first introduced logic in the family of justification logics due to Artemov ([1, 2]). The logic of proofs is a justification counterpart of modal logic S4. Other modal logics have justification counterparts too (cf. [9, 10, 15, 17, 20, 32, 39]).

Using Artemov’s *Realization Theorem* (see [2]), the justification modalities $[\cdot]$ are translated into $\Box$ modality in modal logic and vice versa. Usually, the dual of the justification assertions are not mentioned in the language of justification logics (exceptions are [12, 18, 23]). One

way to consider a justification counterpart for the $\Diamond$ modality is to define it as the dual of the justification modalities. This is the way we define $\langle \cdot \rangle$ in our definition $\langle t \rangle \varphi := \neg [t] \neg \varphi$ (see also [12, 18] for similar definitions). It is worth remarking that in [23] the authors suggest an independent treatment of the $\Diamond$ modality in justification logics. They present constructive justification logics with two distinct justification modalities: one involving justification terms with expressions of the form $t : \varphi$, where t is a justification term, as a counterpart of the formula $\Box \varphi$ (following our notation $t : \varphi$ is denoted by $[t] \varphi$); the other involving *satisfiers* with expressions of the form $\mu : \varphi$, where μ is a satisfier, as a counterpart of the formula $\Diamond \varphi$. Thus satisfiers realize the intuitionistic $\Diamond$ modality and are not the dual of the justification modalities, i.e., formulas involving satisfiers are not interdefinable by formulas involving justification terms. The intended interpretation of $t : \varphi$ and $\mu : \varphi$ are respectively “ t is a proof of φ ” and “ μ is a model of φ .” Since our base logic is the classical logic, we prefer to define $\langle \cdot \rangle$ as the dual of $[\cdot]$.

We now review some works in the literature on semantics of propositional justification logics. Various possible world semantics have been proposed for classical justification logics:¹ Fitting models ([14]), modular models ([3, 24]), subset models ([27]), relational models ([41, 42, 43]), topological semantics ([6]), and neighborhood semantics ([12, 18, 38, 44, 46]). We should also mention that there are possible world models for non-classical justification logics. For instance, some fuzzy, intermediate, and intuitionistic justification logics are provided with possible world semantics [16, 29, 30, 35, 36], and in addition relevant justification logic is accompanied by a semantics based on a combination of Routley-Meyer possible world models and a suitable evidence function [19, 40, 44, 45, 46]. In this paper, we focus on the following possible world semantics: Fitting models, modular models, subset models, relational models, topological models, and neighborhood models. It is worth noting that Mkrtychev, Fitting, modular, relational, and subset models (or their variants) are used as the semantics of various justification logics in the literature (we only cite the original papers here). However, as far as the authors know, there is no independent presentation of the relational, neighborhood, and topological models for the classical logic of proofs LP.

¹ Outside of the scope of this research are other kinds of semantics: arithmetical semantics ([2, 25]), Mkrtychev models ([31]), game semantics ([37]), algebraic semantics ([8, 22, 34, 36]), etc.

We first present neighborhood models for various justification logics (including the logic of proofs), and we show the completeness theorem by a canonical model construction using maximal consistent sets. By introducing a notion of equivalence between possible world semantics, we compare various possible world semantics of the logic of proofs. This notion of equivalence enables us to reduce a completeness result from one class of possible world models to a completeness result for another class of models. We remark that Fitting models and modular models are equivalent to each other, as shown in [3, 24], and in addition the relational models presented in this paper are equivalent to subset models (indeed, they are notational variants of each other). In fact, the relational models presented here for the logic of proofs are different from those originally given in [41, 42, 43].

We finally present topological semantics for the logic of proofs. The topological semantics is only presented in [6] for a justification logic with no operations on justifications. Here we extend the topological semantics to the full language of the logic of proofs. The completeness of LP with respect to topological models is proved by means of the completeness of LP with respect to Fitting models.

2. Language

Let **Const** be a countable set of justification constants, **Var** a countable set of justification variables, and **Prop** a countable set of atomic propositions.

Justification terms and formulas are constructed by the following mutual grammar:

$$\begin{aligned} t &::= c \mid x \mid t \cdot t \mid t + t \mid !t \mid ?t \mid \bar{?}t, \\ \varphi &::= p \mid \perp \mid \varphi \rightarrow \varphi \mid [t]\varphi, \end{aligned}$$

where $c \in \mathbf{Const}$, $x \in \mathbf{Var}$, and $p \in \mathbf{Prop}$. The set of justification terms and the set of formulas are denoted by **Tm** and **Fm** respectively. Formulas of the form $[t]\varphi$ are called *justification assertions*.

We use the following usual abbreviations:

$$\begin{aligned} \neg\varphi &:= \varphi \rightarrow \perp, & \top &:= \neg\perp, \\ \varphi \vee \psi &:= \neg\varphi \rightarrow \psi, & \varphi \wedge \psi &:= \neg(\neg\varphi \vee \neg\psi), \\ \varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi), & \langle t \rangle \varphi &:= \neg[t]\neg\varphi. \end{aligned}$$

Formulas of the form $\langle t \rangle \varphi$ are the dual of the justification assertions. Associativity and precedence of connectives, as well as the corresponding omission of brackets, are handled in the usual manner.

Justification assertions have several readings depending on the semantics given to them. In general, $[t]\varphi$ says “ t is a justification for φ ”, and in an epistemic reading, it says that “ t is a reason for believing (or knowing) φ ”.² Accordingly, $\langle t \rangle \varphi$ generally says “ t is compatible with φ ”, and in an epistemic reading it says “the reason t is compatible with the agent’s belief (or knowledge) in φ ”.

3. Axiomatic systems

We now begin with describing the axiom schemes and rules of justification logics. The set of axiom schemes of the basic justification logic J is $PC \cup \{jK, j+\}$, where PC is a finite set of axiom schemes for classical propositional calculus, and jK and $j+$ denote the following axiom schemes:

$$\begin{aligned} jK. \quad & [t](\varphi \rightarrow \psi) \rightarrow ([s]\varphi \rightarrow [t \cdot s]\psi) & \text{(application)} \\ j+. \quad & [t]\varphi \vee [s]\varphi \rightarrow [t + s]\varphi. & \text{(sum)} \end{aligned}$$

The only rule of inference is Modus Ponens

$$\frac{\vdash \varphi \quad \vdash \varphi \rightarrow \psi}{\vdash \psi} \text{ (MP).}$$

Other justification logics are obtained by adding any combination of the following axiom schemes to the basic justification logic J :

$$\begin{aligned} jT. \quad & [t]\varphi \rightarrow \varphi & \text{(reflexivity)} \\ jD. \quad & \neg[t]\perp & \text{(consistency)} \\ jD^\diamond. \quad & [t]\varphi \rightarrow \langle t \rangle \varphi & \text{(no conflicts)} \\ j4. \quad & [t]\varphi \rightarrow [!t][t]\varphi & \text{(positive introspection)} \\ jB. \quad & \neg\varphi \rightarrow [\bar{?}t]\neg[t]\varphi & \text{(weak negative introspection)} \\ jB^\diamond. \quad & \varphi \rightarrow [\bar{?}t]\langle t \rangle \varphi & \text{(weak negative introspection}^\diamond\text{)} \\ j5. \quad & \neg[t]\varphi \rightarrow [?t]\neg[t]\varphi & \text{(negative introspection)} \end{aligned}$$

² For an arithmetical semantics see [2, 21], for a game semantics see [37], and for a deontic reading see [12, 13, 18].

$j5^\circ. \langle t \rangle \varphi \rightarrow [?t]\langle t \rangle \varphi.$ (negative introspection $^\circ$)

The name of each justification logic is indicated by the list of its axioms. For example, the set of axioms of the justification logic $\text{JT}5^\circ\text{B}$ is $PC \cup \{jK, j+\} \cup \{jT, j5^\circ, jB\}$. The logic $\text{JT}4$, with the set of axioms $PC \cup \{jK, j+\} \cup \{jT, j4\}$, is usually called the logic of proofs LP . In what follows, JL denotes any of the justification logics. Given an axiom jL , by $jL \in \text{JL}$ we mean that jL is an axiom scheme of the logic JL .

Note that axioms jB° and $j5^\circ$ are respectively weaker than jB and $j5$, namely $\varphi \rightarrow [\bar{?}t]\langle t \rangle \varphi$ is derivable in JB and $\langle t \rangle \varphi \rightarrow [?t]\langle t \rangle \varphi$ is derivable in J5 . Further, in [11, 12] it is shown that axiom jD° is weaker than jD .

Given a justification logic JL , let $\text{TCS}(\text{JL})$ denote the set of all formulas of the form $[c_{j_n}] \dots [c_{j_1}] \varphi$ such that $n \geq 1$, $c_{j_n}, \dots, c_{j_1} \in \text{Const}$, and φ is an axiom instance of JL . The set $\text{TCS}(\text{JL})$ is called the *total constant specification* for JL .

A *constant specification* CS for JL is any subset of $\text{TCS}(\text{JL})$ which is *downward closed* in the sense that whenever $[c_{j_n}][c_{j_{n-1}}] \dots [c_{j_1}] \varphi \in \text{CS}$, for $n > 1$, then $[c_{j_{n-1}}] \dots [c_{j_1}] \varphi \in \text{CS}$.

A constant specification CS for JL is *axiomatically appropriate* provided for every axiom instance φ of JL there is a constant $c \in \text{Const}$ such that $[c]\varphi \in \text{CS}$, and moreover CS is *upward closed*: if $[c_{j_n}] \dots [c_{j_1}] \varphi \in \text{CS}$, then there is a constant c such that $[c][c_{j_n}] \dots [c_{j_1}] \varphi \in \text{CS}$.

Note that $\text{TCS}(\text{JL})$ is obviously an axiomatically appropriate constant specification for JL .

Given a constant specification CS for JL , the justification logic JL_{CS} is defined as follows. The set of axioms of JL_{CS} consists of the set of axioms of JL and all formulas of CS , and the only rule of inference is (MP). Whenever we write JL_{CS} , it means that CS is a constant specification for JL . By $\vdash_{\text{JL}_{\text{CS}}}$ we denote derivability in JL_{CS} . The definition of derivation from a set of premises T is as follows:

$T \vdash_{\text{JL}_{\text{CS}}} \varphi$ iff there exist $\psi_1, \dots, \psi_n \in T$ s.t. $\vdash_{\text{JL}_{\text{CS}}} (\psi_1 \wedge \dots \wedge \psi_n) \rightarrow \varphi$.

Next, we aim to show that all justification logics JL_{CS} are consistent. To this end, we employ the *forgetful projection*, which is a translation from the language of justification logic to the language of modal logic obtained from substituting $\Box\varphi$ for all occurrences of $[t]\varphi$ in a justification formula (see e.g. [2]).

LEMMA 3.1 (Consistency). *Each justification logic JL_{CS} is consistent.*

PROOF. We shall show that each justification logic $\mathbf{JL}_{CS}$ is conservative over a modal logic $\mathbf{ML}$. Given a justification formula φ , the forgetful projection of φ , denoted by φ° , is a modal formula defined inductively as follows: $(p)^\circ := p$, $(\perp)^\circ := \perp$, $(\varphi \rightarrow \psi)^\circ := \varphi^\circ \rightarrow \psi^\circ$, $([t]\varphi)^\circ := \Box\varphi^\circ$. Now it is easy to show that if $\vdash_{\mathbf{JL}_{CS}} \varphi$, then $\vdash_{\mathbf{ML}} (\varphi)^\circ$, where $\mathbf{ML}$ is the modal counterpart of $\mathbf{JL}_{CS}$.³ Having proved this fact, the consistency of $\mathbf{JL}_{CS}$ follows immediately from that of its modal logic counterpart. $\dashv$

4. Neighborhood semantics

In this section, we present neighborhood semantics for all of the justification logics introduced in Section 3.

DEFINITION 4.1. [Quasi neighborhood model] A *quasi neighborhood model* is a tuple $\mathcal{M} = (W, W_N, N, V)$ where

1. W and W_N are sets of worlds (or states) and normal worlds, respectively, such that $W_N \neq \emptyset$ and $W_N \subseteq W$,
2. $N : W_N \times \mathbf{Tm} \rightarrow \wp(\wp(W))$ is a neighborhood function,
3. V is a valuation function such that $V : W_N \rightarrow \wp(\mathbf{Prop})$ and $V : W \setminus W_N \rightarrow \wp(\mathbf{Fm})$.

DEFINITION 4.2. [Truth] Let $\mathcal{M} = (W, W_N, N, V)$ be a quasi neighborhood model. We now define when a formula φ is true at a pointed model $(\mathcal{M}, w)$ for $w \in W$ (if so, we write $(\mathcal{M}, w) \models \varphi$). Truth at non-normal worlds $w \in W \setminus W_N$ are defined as follows:

$$(\mathcal{M}, w) \models \varphi \quad \text{iff} \quad \varphi \in V(w).$$

Truth at normal worlds $w \in W_N$ is inductively defined as follows:

$$\begin{aligned} &(\mathcal{M}, w) \not\models \perp, \\ &(\mathcal{M}, w) \models p \text{ iff } p \in V(w), \text{ for } p \in \mathbf{Prop}, \\ &(\mathcal{M}, w) \models \varphi \rightarrow \psi \text{ iff } (\mathcal{M}, w) \models \varphi \text{ implies } (\mathcal{M}, w) \models \psi, \\ &(\mathcal{M}, w) \models [t]\varphi \text{ iff } \|\varphi\|_{\mathcal{M}} \in N(w, t), \end{aligned}$$

where $\|\varphi\|_{\mathcal{M}} := \{w \in W \mid (\mathcal{M}, w) \models \varphi\}$ is the *truth set* of φ in $\mathcal{M}$.⁴

³ For example, $\mathbf{K}$, $\mathbf{KT4}$, and $\mathbf{KD}$ are the modal counterparts of $\mathbf{J}_{CS}$, $\mathbf{JT4}_{CS}$, and $\mathbf{JD}^\circ_{CS}$ respectively, and the modal logic $\mathbf{KT45}$ is the modal counterpart of both $\mathbf{JT45}_{CS}$ and $\mathbf{JT45}^\circ_{CS}$, etc. For more details refer to [15, 17, 20]

⁴ Usually $\|\varphi\|_{\mathcal{M}}$ is interpreted as the *proposition* expressed by φ in $\mathcal{M}$ or the *extension* of φ .

DEFINITION 4.3. [Neighborhood model] A *neighborhood model* $\mathcal{M} = (W, W_N, N, V)$ for JL_{CS} is a quasi neighborhood model such that the neighborhood function $N : W_N \times \text{Tm} \rightarrow \wp(\wp(W))$ satisfies the following conditions. For all terms $s, t \in \text{Tm}$, all formulas $\varphi, \psi \in \text{Fm}$, all worlds $w \in W_N$:

1. If $[t]\varphi \in \text{CS}$, then $\|\varphi\|_{\mathcal{M}} \in N(w, t)$. (constant-specification- N)
2. If $\|\varphi \rightarrow \psi\|_{\mathcal{M}} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}} \in N(w, s)$, then $\|\psi\|_{\mathcal{M}} \in N(w, t \cdot s)$. (application- N)
3. $N(w, t) \cup N(w, s) \subseteq N(w, t + s)$. (sum- N)
4. Case $jT \in \text{JL}$:
 $w \in \cap N(w, t)$. (reflexivity- N)
5. Case $jD \in \text{JL}$:
 $\|\perp\|_{\mathcal{M}} \notin N(w, t)$. (consistency- N)
6. Case $jD^\circ \in \text{JL}$:
 if $\|\varphi\|_{\mathcal{M}} \in N(w, t)$, then $\|\neg\varphi\|_{\mathcal{M}} \notin N(w, t)$. (noc- N)
7. Case $j4 \in \text{JL}$:
 if $\|\varphi\|_{\mathcal{M}} \in N(w, t)$, then $\|[t]\varphi\|_{\mathcal{M}} \in N(w, !t)$. (pos-introsp- N)
8. Case $jB \in \text{JL}$:
 if $w \notin \|\varphi\|_{\mathcal{M}}$, then $\|\neg[t]\varphi\|_{\mathcal{M}} \in N(w, \bar{?}t)$. (weak-neg-introsp- N)
9. Case $jB^\circ \in \text{JL}$:
 if $w \in \|\varphi\|_{\mathcal{M}}$, then $\|\langle t \rangle \varphi\|_{\mathcal{M}} \in N(w, \bar{?}t)$. (weak-neg-introsp- N°)
10. Case $j5 \in \text{JL}$:
 if $\|\varphi\|_{\mathcal{M}} \notin N(w, t)$, then $\|\neg[t]\varphi\|_{\mathcal{M}} \in N(w, ?t)$. (neg-introsp- N)
11. Case $j5^\circ \in \text{JL}$:
 if $\|\neg\varphi\|_{\mathcal{M}} \notin N(w, t)$, then $\|\langle t \rangle \varphi\|_{\mathcal{M}} \in N(w, ?t)$. (neg-introsp- N°)

Given a set $X \subseteq W$, the intuitive meaning of $X \in N(w, t)$ is that “at the world w , t is a justification for proposition X .”

It is easy to show that for all worlds $w \in W_N$:

$$(\mathcal{M}, w) \models \langle t \rangle \varphi \text{ iff } \|\neg\varphi\|_{\mathcal{M}} \notin N(w, t).$$

DEFINITION 4.4. Let CS be a constant specification for JL .

1. Given a neighborhood model $\mathcal{M} = (W, W_N, N, V)$ for JL_{CS} , a formula φ is globally true at $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if for all $w \in W_N$, we have $(\mathcal{M}, w) \models \varphi$.

2. φ is valid in JL_{CS} , written $\models_{\text{JL}_{\text{CS}}} \varphi$, if $\mathcal{M} \models \varphi$ for all neighborhood models $\mathcal{M}$ for JL_{CS} .
3. Given a set of formulas T and a formula φ of JL_{CS} , the (local) consequence relation is defined as follows: $T \models_{\text{JL}_{\text{CS}}} \varphi$ iff for all neighborhood models $\mathcal{M} = (W, W_N, N, V)$ for JL_{CS} , for all $w \in W_N$, if $(\mathcal{M}, w) \models \psi$ for all $\psi \in T$, then $(\mathcal{M}, w) \models \varphi$.

In the next sections, we also need to consider sub-classes of neighborhood models which satisfy certain conditions, as it is described in the following definition.

DEFINITION 4.5. Let $\mathcal{M} = (W, W_N, N, V)$ be a neighborhood model.

- $\mathcal{M}$ contains its core if for each $N(w, t)$ we have $\bigcap N(w, t) \in N(w, t)$.
- $\mathcal{M}$ contains normal worlds if each $N(w, t)$ contains W_N .
- $\mathcal{M}$ is closed under supersets if for each $N(w, t)$ and each $X \subseteq W$, if $X \in N(w, t)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w, t)$.
- $\mathcal{M}$ is closed under finite intersections if for each $N(w, t)$ and each $X_1, \dots, X_n \subseteq W$, if $X_1, \dots, X_n \in N(w, t)$, then $X_1 \cap \dots \cap X_n \in N(w, t)$.

The regularity rule

In the neighborhood semantics of modal logic, we have the following regularity rule:

$$\frac{\models \varphi \leftrightarrow \psi}{\models \Box \varphi \leftrightarrow \Box \psi} \text{ (RE)}$$

However, the fact that $\models_{\text{JL}_{\text{CS}}} \varphi \leftrightarrow \psi$ and the fact that φ is justified by reason t need not imply that ψ is justified by the same reason t . In other words, justifications are hyperintentional (see [4, 5, 13] for a more detailed exposition). We show that justification versions of the rule (RE) are not admissible in the basic justification logic J_{CS} (the reader may extend this result to other justification logics).

LEMMA 4.1. *The regularity rule*

$$\frac{\varphi \leftrightarrow \psi}{[t]\varphi \leftrightarrow [t]\psi} \text{ (JRE)}$$

is not validity preserving in J_{CS} , i.e. there exists $\varphi, \psi \in \text{Fm}$ and $t \in \text{Tm}$ such that $\models_{\text{J}_{\text{CS}}} \varphi \leftrightarrow \psi$ but $\not\models_{\text{J}_{\text{CS}}} [t]\varphi \leftrightarrow [t]\psi$.

PROOF. Let CS be an arbitrary constant specification for J. Consider the neighborhood model $\mathcal{M} = (W, W_N, N, V)$ as follows:

- $W = \{w, v\}$,
- $W_N = \{w\}$,
- $N(w, x) = N(w, c) = \{\{w\}\}$, for all $x \in \text{Var}$ and for all $c \in \text{Const}$,
 $N(w, t) = \wp(W)$, for all $t \in \text{Tm} \setminus (\text{Var} \cup \text{Const})$,
- $V(w) = \{p\}$, $V(v) = \{p \wedge p\}$.

We now show that $\mathcal{M}$ is a neighborhood model for JCS . For (constant-specification- N), suppose that $[t]\varphi \in \text{CS}$, for some $t \in \text{Const}$. Then $\|\varphi\|_{\mathcal{M}} = \{w\} \in N(w, t)$, as desired. Conditions (application- N) and (sum- N) hold because $N(w, t \cdot s) = N(w, t + s) = \wp(W)$ for all $t, s \in \text{Tm}$.

Let $x \in \text{Var}$ be a justification variable. We observe that $(\mathcal{M}, w) \not\models [x]p \leftrightarrow [x](p \wedge p)$ and thus $\not\models_{\text{JCS}} [x]p \leftrightarrow [x](p \wedge p)$, but $\models_{\text{JCS}} p \leftrightarrow p \wedge p$. $\dashv$

The axioms $jB^\diamond$ and $j5^\diamond$

Next, we show that the axiom $jB^\diamond$ is strictly weaker than jB and the axiom $j5^\diamond$ is strictly weaker than $j5$. The fact that $jB^\diamond$ and $j5^\diamond$ are weaker than jB and $j5$, respectively, is related to the hyperintensionality and the non-admissibility of the rule (JRE).

LEMMA 4.2. *Axiom jB is not valid in $\text{JB}_{\text{CS}}^\diamond$, i.e. there exists $\varphi \in \text{Fm}$ and $t \in \text{Tm}$ such that $\not\models_{\text{JB}_{\text{CS}}^\diamond} \neg\varphi \rightarrow [\bar{?}t]\neg[t]\varphi$.*

PROOF. Let CS be an arbitrary constant specification for $\text{JB}^\diamond$. Let $p \in \text{Prop}$ be a fixed atomic proposition. Consider the neighborhood model $\mathcal{M} = (W, W_N, N, V)$ as follows:

- $W = \{w, v\}$,
- $W_N = \{w\}$,
- $N(w, x) = N(w, c) = \{\{w\}\}$, for all $x \in \text{Var}$ and all $c \in \text{Const}$,
 $N(w, t \cdot s) = N(w, t + s) = \wp(W)$, for all $t, s \in \text{Tm}$,
 $N(w, \bar{?}t) = \{\{v\}, W\}$, for all $t \in \text{Tm}$,
- $V(w) = \emptyset$, $V(v) = \{p\} \cup \{\langle t \rangle \varphi \mid t \in \text{Tm}, \varphi \in \text{Fm}\}$.

It is easy to show that $\mathcal{M}$ is a neighborhood model for $\text{JB}_{\text{CS}}^\diamond$. We only check the condition (weak-neg-introsp- $N^\diamond$) (the proofs of other conditions (constant-specification- N), (application- N), and (sum- N) are similar to that of Lemma 4.1).

For (weak-neg-introsp- $N^\diamond$), suppose that $w \in \|\varphi\|_{\mathcal{M}}$. Then, $\|\langle t \rangle \varphi\|_{\mathcal{M}}$ is either $\{v\}$ or W , in either case $\|\langle t \rangle \varphi\|_{\mathcal{M}} \in N(w, \bar{?}t)$.

From the definition of $\mathcal{M}$, it follows that $(\mathcal{M}, w) \models \neg p$. Let $x \in \text{Var}$ be a fixed justification variable. We observe that $\|p\|_{\mathcal{M}} = \{v\} \notin N(w, x)$,

and thus $(\mathcal{M}, w) \models \neg[x]p$. In addition, since v is a non-normal world and $\neg[x]p \notin V(v)$, we get $(\mathcal{M}, v) \not\models \neg[x]p$. Hence, $\|\neg[x]p\|_{\mathcal{M}} = \{w\} \notin N(w, ?x)$, and hence $(\mathcal{M}, w) \not\models [?x]\neg[x]p$. Therefore, $(\mathcal{M}, w) \not\models \neg p \rightarrow [?x]\neg[x]p$. $\dashv$

LEMMA 4.3. *Axiom $j5$ is not valid in $J5_{CS}^\diamond$, i.e. there exists $\varphi \in Fm$ and $t \in Tm$ such that $\not\models_{J5_{CS}^\diamond} \neg[t]\varphi \rightarrow [?t]\neg[t]\varphi$.*

PROOF. Let CS be an arbitrary constant specification for $J5^\diamond$. Let $p \in Prop$ be a fixed atomic proposition. Consider the neighborhood model $\mathcal{M} = (W, W_N, N, V)$ as follows:

- $W = \{w, v\}$,
- $W_N = \{w\}$,
- $N(w, x) = N(w, c) = \{\{w\}\}$, for all $x \in Var$ and all $c \in Const$,
 $N(w, t \cdot s) = N(w, t + s) = \emptyset(W)$, for all $t, s \in Tm$,
 $N(w, ?t) = \{W\}$, for all $t \in Tm$,
- $V(w) = \{p\}$, $V(v) = \{p \wedge p\} \cup \{\langle t \rangle \varphi \mid t \in Tm, \varphi \in Fm\}$.

It is easy to show that $\mathcal{M}$ is a neighborhood model for $J5_{CS}^\diamond$. We only check the condition (neg-introsp- $N^\diamond$) (the proofs of other conditions are similar to that of Lemma 4.1).

For (neg-introsp- $N^\diamond$), suppose that $\|\neg\varphi\|_{\mathcal{M}} \notin N(w, t)$. Then, we get $(\mathcal{M}, w) \models \langle t \rangle \varphi$. On the other hand, from $\langle t \rangle \varphi \in V(v)$ we get $(\mathcal{M}, v) \models \langle t \rangle \varphi$. Thus, $\|\langle t \rangle \varphi\|_{\mathcal{M}} = W \in N(w, ?t)$.

Let $x \in Var$. We observe that $\|p \wedge p\|_{\mathcal{M}} = W \notin N(w, x)$, and thus $(\mathcal{M}, w) \models \neg[x](p \wedge p)$. In addition, since v is a non-normal world and $\neg[x](p \wedge p) \notin V(v)$, we get $(\mathcal{M}, v) \not\models \neg[x](p \wedge p)$. Hence, $\|\neg[x](p \wedge p)\|_{\mathcal{M}} = \{w\} \notin N(w, ?x)$, and hence $(\mathcal{M}, w) \not\models [?x]\neg[x](p \wedge p)$. Therefore, we have $(\mathcal{M}, w) \not\models \neg[x](p \wedge p) \rightarrow [?x]\neg[x](p \wedge p)$. $\dashv$

Completeness of JL with respect to neighborhood models

For the completeness proof, we employ the canonical model construction.

DEFINITION 4.6. The canonical neighborhood model

$$\mathcal{M} = (W, W_N, N, V)$$

for JL_{CS} is defined as follows:

1. $W := \wp(Fm)$,

2. $W_N := \text{MCS}_{\text{JLCS}}$,
3. $N(\Gamma, t) := \{|\varphi|_{\text{JLCS}} \mid [t]\varphi \in \Gamma\}$, for $\Gamma \in W_N$,
4. $V(\Gamma) := \Gamma$, for $\Gamma \in W \setminus W_N$,
 $V(\Gamma) := \text{Prop} \cap \Gamma$, for $\Gamma \in W_N$,

where $|\varphi|_{\text{JLCS}} := \{\Gamma \in W \mid \varphi \in \Gamma\}$ is the *proof set* of φ , and MCS_{JLCS} is the set of all maximal JLCS -consistent sets of formulas.

From the consistency of JLCS (Lemma 3.1) it follows that the canonical neighborhood model $\mathcal{M}$ is a quasi neighborhood model, and hence the truth of a formula at a pointed model can be defined as in Definition 4.2.

LEMMA 4.4 (Truth Lemma). *Let $\mathcal{M} = (W, W_N, N, V)$ be the canonical neighborhood model for JLCS . Then for any $\varphi \in \text{Fm}$*

$$\|\varphi\|_{\mathcal{M}} = |\varphi|_{\text{JLCS}}.$$

PROOF. By induction on φ . For $\varphi = p \in \text{Prop}$ and $\Gamma \in W$ we have

$$(\mathcal{M}, \Gamma) \models p \quad \text{iff} \quad p \in V(\Gamma) \quad \text{iff} \quad p \in \Gamma.$$

Thus $\|p\|_{\mathcal{M}} = |p|_{\text{JLCS}}$. The cases $\varphi = \perp$ and $\varphi = \varphi_1 \rightarrow \varphi_2$ are standard. For $\varphi = [t]\psi$, by the induction hypothesis we have $\|\psi\|_{\mathcal{M}} = |\psi|_{\text{JLCS}}$. For $\Gamma \in W \setminus W_N$, we have

$$(\mathcal{M}, \Gamma) \models [t]\psi \quad \text{iff} \quad [t]\psi \in V(\Gamma) = \Gamma.$$

For $\Gamma \in W_N$, we have

$$(\mathcal{M}, \Gamma) \models [t]\psi \text{ iff } \|\psi\|_{\mathcal{M}} \in N(\Gamma, t) \text{ iff } |\psi|_{\text{JLCS}} \in N(\Gamma, t) \text{ iff } [t]\psi \in \Gamma.$$

Thus $\|[t]\psi\|_{\mathcal{M}} = |[t]\psi|_{\text{JLCS}}$. ←

LEMMA 4.5. *The canonical neighborhood model $\mathcal{M} = (W, W_N, N, V)$ for JLCS is a neighborhood model for JLCS .*

PROOF. Given a justification logic JLCS , it suffices to show that the neighborhood function N of the canonical model $\mathcal{M}$ for JLCS satisfies the conditions of Definition 4.3. Let $\Gamma \in W_N = \text{MCS}_{\text{JLCS}}$.

(constant-specification- N): Suppose that $[t]\varphi \in \text{CS}$. Then $[t]\varphi \in \Gamma$. Thus $|\varphi|_{\text{JLCS}} = \|\varphi\|_{\mathcal{M}} \in N(\Gamma, t)$.

- (application- N): Suppose that $\|\varphi \rightarrow \psi\|_{\mathcal{M}} \in N(\Gamma, t)$ and $\|\varphi\|_{\mathcal{M}} \in N(\Gamma, s)$. Since by the truth lemma $\|\varphi \rightarrow \psi\|_{\mathcal{M}} = |\varphi \rightarrow \psi|_{\text{JLCS}}$ and $\|\varphi\|_{\mathcal{M}} = |\varphi|_{\text{JLCS}}$, we get $[t](\varphi \rightarrow \psi), [s]\varphi \in \Gamma$, and hence $[t \cdot s]\psi \in \Gamma$. Therefore $|\psi|_{\text{JLCS}} = \|\psi\|_{\mathcal{M}} \in N(\Gamma, t \cdot s)$.
- (sum- N): Suppose that $X \in N(\Gamma, t) \cup N(\Gamma, s)$. Then, either $X \in N(\Gamma, t)$ or $X \in N(\Gamma, s)$. In the former case, $X = |\varphi|_{\text{JLCS}}$, for some $\varphi \in \text{Fm}$ such that $[t]\varphi \in \Gamma$. Thus, $[t + s]\varphi \in \Gamma$. Therefore $X = |\varphi|_{\text{JLCS}} \in N(\Gamma, t + s)$. The proof for the case $X \in N(\Gamma, s)$ is similar.
- (reflexivity- N): Suppose that $jT \in \text{JL}$ and $X \in N(\Gamma, t)$. We want to show that $\Gamma \in X$. From $X \in N(\Gamma, t)$ it follows that $X = |\varphi|_{\text{JLCS}}$, for some $\varphi \in \text{Fm}$, such that $[t]\varphi \in \Gamma$. Since $[t]\varphi \rightarrow \varphi \in \Gamma$, we get $\varphi \in \Gamma$. Therefore, $\Gamma \in |\varphi|_{\text{JLCS}} = X$.
- (consistency- N): Suppose that $jD \in \text{JL}$. Since $\neg[t]\perp \in \Gamma$, we get $[t]\perp \notin \Gamma$. Thus $|\perp|_{\text{JLCS}} = \|\perp\|_{\mathcal{M}} \notin N(\Gamma, t)$.
- (noc- N): Suppose that $jD^\circ \in \text{JL}$ and $\|\varphi\|_{\mathcal{M}} \in N(\Gamma, t)$. By the truth lemma, $|\varphi|_{\text{JLCS}} \in N(\Gamma, t)$, and hence $[t]\varphi \in \Gamma$. Thus, $[t]\neg\varphi \notin \Gamma$. Therefore $|\neg\varphi|_{\text{JLCS}} = \|\neg\varphi\|_{\mathcal{M}} \notin N(\Gamma, t)$.
- (pos-introsp- N): Suppose that $j4 \in \text{JL}$ and $\|\varphi\|_{\mathcal{M}} \in N(\Gamma, t)$. By the truth lemma, $|\varphi|_{\text{JLCS}} \in N(\Gamma, t)$, and hence $[t]\varphi \in \Gamma$. Since $[t]\varphi \rightarrow [!t][t]\varphi \in \Gamma$, we get $[!t][t]\varphi \in \Gamma$. Hence, $[t]\varphi|_{\text{JLCS}} = \|[t]\varphi\|_{\mathcal{M}} \in N(\Gamma, !t)$.
- (weak-neg-introsp- N): Suppose that $jB \in \text{JL}$ and $\Gamma \notin \|\varphi\|_{\mathcal{M}}$. By the truth lemma, $\Gamma \notin |\varphi|_{\text{JLCS}}$, and hence $\neg\varphi \in \Gamma$. Since $\neg\varphi \rightarrow [\bar{?}t]\neg[t]\varphi \in \Gamma$, we get $[\bar{?}t]\neg[t]\varphi \in \Gamma$. Hence, $|\neg[t]\varphi|_{\text{JLCS}} = \|\neg[t]\varphi\|_{\mathcal{M}} \in N(\Gamma, \bar{?}t)$.
- (weak-neg-introsp- N°): Suppose that $jB^\circ \in \text{JL}$ and $\Gamma \in \|\varphi\|_{\mathcal{M}}$. By the truth lemma, $\Gamma \in |\varphi|_{\text{JLCS}}$, and hence $\varphi \in \Gamma$. Since $\varphi \rightarrow [\bar{?}t]\langle t \rangle \varphi \in \Gamma$, we get $[\bar{?}t]\langle t \rangle \varphi \in \Gamma$. Hence, $|\langle t \rangle \varphi|_{\text{JLCS}} = \|\langle t \rangle \varphi\|_{\mathcal{M}} \in N(\Gamma, \bar{?}t)$.
- (neg-introsp- N): Suppose that $j5 \in \text{JL}$ and $\|\varphi\|_{\mathcal{M}} \notin N(\Gamma, t)$. By the truth lemma, $|\varphi|_{\text{JLCS}} \notin N(\Gamma, t)$, and hence $\neg[t]\varphi \in \Gamma$. Since $\neg[t]\varphi \rightarrow [?t]\neg[t]\varphi \in \Gamma$, we get $[?t]\neg[t]\varphi \in \Gamma$. Hence, $|\neg[t]\varphi|_{\text{JLCS}} = \|\neg[t]\varphi\|_{\mathcal{M}} \in N(\Gamma, ?t)$.
- (neg-introsp- N°): Suppose that $j5^\circ \in \text{JL}$ and $\|\neg\varphi\|_{\mathcal{M}} \notin N(\Gamma, t)$. Then $[t]\neg\varphi \notin \Gamma$, and hence $\neg[t]\neg\varphi \in \Gamma$, namely $\langle t \rangle \varphi \in \Gamma$. Since $\langle t \rangle \varphi \rightarrow [?t]\langle t \rangle \varphi \in \Gamma$, we get $[?t]\langle t \rangle \varphi \in \Gamma$. Hence, $|\langle t \rangle \varphi|_{\text{JLCS}} = \|\langle t \rangle \varphi\|_{\mathcal{M}} \in N(\Gamma, ?t)$. $\dashv$

THEOREM 4.1 (Completeness). *For each formula φ and set of formulas T ,*

$$T \models_{\text{JL}_{\text{CS}}} \varphi \quad \text{iff} \quad T \vdash_{\text{JL}_{\text{CS}}} \varphi.$$

PROOF. Soundness is straightforward. For completeness, suppose that $T \not\models_{\text{JL}_{\text{CS}}} \varphi$. Thus, $T \cup \{\neg\varphi\}$ is JL_{CS} -consistent. By Lindenbaum's Lemma, there is $\Gamma \in \text{MCS}_{\text{JL}_{\text{CS}}}$ such that $T \cup \{\neg\varphi\} \subseteq \Gamma$. Construct the canonical neighborhood model $\mathcal{M}$ for JL_{CS} . By the Truth Lemma 4.4, $\Gamma \in \|\neg\varphi\|_{\mathcal{M}}$ and $\Gamma \in \|\psi\|_{\mathcal{M}}$ for all $\psi \in T$. Hence, $T \not\models_{\text{JL}_{\text{CS}}} \varphi$. $\dashv$

5. Relationship with other semantics

In this section, we study the relationship between some of the possible world semantics of justification logics (i.e., Fitting, modular, subset, relational, and neighborhood models). All the results in this section are stated for the logic of proofs LP, but they can be extended to other justification logics as well. We first give a precise definition for a kind of equivalence between possible world models.

DEFINITION 5.1. Given a possible world model $\mathcal{M} = (W, \dots)$, let $|\mathcal{M}| := W$. Let $\mathcal{M}$ and $\mathcal{M}'$ be two possible world models, and $S \subseteq |\mathcal{M}| \cap |\mathcal{M}'|$ is a non-empty subset. We say that $\mathcal{M}$ and $\mathcal{M}'$ are *world-equivalent with respect to S* provided that for all $w \in S$ and for all $\varphi \in \text{Fm}$:

$$(\mathcal{M}, w) \models \varphi \quad \text{iff} \quad (\mathcal{M}', w) \models \varphi.$$

In practice, we only consider models $\mathcal{M}$ and $\mathcal{M}'$ where either $|\mathcal{M}| = |\mathcal{M}'|$ or $|\mathcal{M}| \subseteq |\mathcal{M}'|$, and in either case we demonstrate world-equivalence with respect to $|\mathcal{M}| \cap |\mathcal{M}'|$, i.e. we let S to be $|\mathcal{M}|$. Some of the models we study contains normal worlds in their formulation (subset, relational, and neighborhood models), and others do not. We usually assume that if either $|\mathcal{M}|$ or $|\mathcal{M}'|$ contains normal worlds, then S includes all normal worlds as well.

The notion of world-equivalence enables us to reduce completeness with respect to one class of models to completeness with respect to another class of models. For example, suppose that we have proved completeness of a logic L with respect to a class of models $\mathcal{C}$, i.e.

$$T \models_{\mathcal{C}} \varphi \quad \text{only if} \quad T \vdash_{\text{L}} \varphi.$$

In order to show the completeness of logic L with respect to another class of models $\mathcal{C}'$, it suffices to show this claim: for each model $\mathcal{M} = (W, \dots) \in \mathcal{C}$ there is a model $\mathcal{M}' = (W, \dots) \in \mathcal{C}'$ that is world-equivalent to $\mathcal{M}$ with respect to W (it is also required that W includes all normal worlds, if either $\mathcal{M}$ or $\mathcal{M}'$ contains normal worlds). Having proved this, the completeness of L with respect to $\mathcal{C}'$ can be shown as follows: from $T \not\vdash_L \varphi$ it follows that there is a model $\mathcal{M} = (W, \dots) \in \mathcal{C}$ and a (normal) world $w \in W$ such that $(\mathcal{M}, w) \not\models \varphi$, and hence, by the above claim, there is a model $\mathcal{M}' = (W, \dots) \in \mathcal{C}'$ such that $(\mathcal{M}', w) \not\models \varphi$. Here, for simplicity we assumed that $|\mathcal{M}| = |\mathcal{M}'| = W$.

5.1. Relationship with Fitting models

In this section, we first recall Fitting models for the logic of proofs (cf. [14]), and then we show that Fitting models are world-equivalent to neighborhood models.

DEFINITION 5.2 (Fitting model). A Fitting model for LP_{CS} is a tuple $\mathcal{M} = (S, R, \mathcal{E}, V)$ where S is a non-empty set of worlds, $R \subseteq S \times S$ is a reflexive and transitive relation, $V : S \rightarrow \wp(\text{Prop})$ is a valuation, and $\mathcal{E} : S \times \text{Tm} \rightarrow \wp(\text{Fm})$ is an *evidence function* satisfying the following conditions. For all terms $s, t \in \text{Tm}$, all formulas $\varphi, \psi \in \text{Fm}$, and all worlds $v, w \in S$:

1. If $[t]\varphi \in CS$, then $\varphi \in \mathcal{E}(w, t)$. (constant-specification- $\mathcal{E}$)
2. If $\varphi \rightarrow \psi \in \mathcal{E}(w, t)$ and $\varphi \in \mathcal{E}(w, s)$, then $\psi \in \mathcal{E}(w, t \cdot s)$. (application- $\mathcal{E}$)
3. $\mathcal{E}(w, t) \cup \mathcal{E}(w, s) \subseteq \mathcal{E}(w, s + t)$. (sum- $\mathcal{E}$)
4. If $\varphi \in \mathcal{E}(w, t)$, then $[t]\varphi \in \mathcal{E}(w, !t)$. (pos-introsp- $\mathcal{E}$)
5. If wRv , then $\mathcal{E}(w, t) \subseteq \mathcal{E}(v, t)$. (monotonicity- $\mathcal{E}$)

The definition of truth for Boolean connectives is standard (see Definition 4.2). The truth of justification assertions is defined as follows:

$$(\mathcal{M}, w) \models [t]\varphi \text{ iff } \varphi \in \mathcal{E}(w, t) \text{ and } R(w) \subseteq \|\varphi\|_{\mathcal{M}}, \quad (1)$$

where $R(w) := \{v \in S \mid wRv\}$ and $\|\varphi\|_{\mathcal{M}} = \{w \in S \mid (\mathcal{M}, w) \models \varphi\}$.

Given a set of formulas T and a formula φ of LP_{CS} , $T \models_{LP_{CS}}^F \varphi$ iff for all Fitting models $\mathcal{M} = (S, R, \mathcal{E}, V)$ for LP_{CS} , for all $w \in S$, if $(\mathcal{M}, w) \models \psi$ for all $\psi \in T$, then $(\mathcal{M}, w) \models \varphi$.

LEMMA 5.1. *Given a neighborhood model $\mathcal{M}^N = (W, W_N, N, V)$ for LP_{CS} , there is a Fitting model $\mathcal{M}^F$ for LP_{CS} which is world-equivalent to $\mathcal{M}^N$ with respect to W_N .*

PROOF. Suppose that $\mathcal{M}^N = (W, W_N, N, V)$ is a neighborhood model for LP_{CS} . Let $\mathcal{M}^F = (W_N, R, \mathcal{E}, V^F)$ be defined as follows. For all $t \in \text{Tm}$, all $\varphi \in \text{Fm}$, and all $w, v \in W_N$:

- wRv iff for all $t \in \text{Tm}$ ($N(w, t) \subseteq N(v, t)$ & $v \in \bigcap N(w, t)$),
- $\varphi \in \mathcal{E}(w, t)$ iff $\|\varphi\|_{\mathcal{M}^N} \in N(w, t)$,
- $V^F := V \upharpoonright_{W_N}$ (= the restriction of V to W_N).

It is easy to show that $\mathcal{M}^F$ is a Fitting model for LP_{CS} . The fact that conditions (constant-specification- N), (application- N), (sum- N), and (pos-introsp- N) hold for $\mathcal{M}^N$ immediately yield that conditions (constant-specification- $\mathcal{E}$), (application- $\mathcal{E}$), (sum- $\mathcal{E}$), and (pos-introsp- $\mathcal{E}$) hold for $\mathcal{M}^F$.

Let us show that $\mathcal{M}^F$ satisfies condition (monotonicity- $\mathcal{E}$). Let wRv and $\varphi \in \mathcal{E}(w, t)$. By the definition of $\mathcal{E}$, we have $\|\varphi\|_{\mathcal{M}^N} \in N(w, t)$. From this and wRv , we conclude that $\|\varphi\|_{\mathcal{M}^N} \in N(v, t)$, and thus $\varphi \in \mathcal{E}(v, t)$.

The reflexivity of R follows easily from the condition (reflexivity- N) (i.e. $w \in \bigcap N(w, t)$).

We now show that R is transitive. Suppose that wRv and vRu . We have to show that wRu . Let t be an arbitrary term. From wRv and vRu , it follows that $N(w, t) \subseteq N(v, t)$ and $N(v, t) \subseteq N(u, t)$, and hence $N(w, t) \subseteq N(u, t)$. It is sufficient to show that $u \in \bigcap N(w, t)$. Let $X \in N(w, t)$ be arbitrary. From $N(w, t) \subseteq N(v, t)$ and $u \in \bigcap N(v, t)$, it follows that $u \in X$, as required.

Next, we show that $\mathcal{M}^F$ is world-equivalent to $\mathcal{M}^N$ with respect to W_N . We show that for all $\varphi \in \text{Fm}$ and for all $w \in W_N$,

$$(\mathcal{M}^N, w) \models \varphi \text{ iff } (\mathcal{M}^F, w) \models \varphi.$$

This can be proved by induction on φ , and we only show the case that φ is a justification assertion, say $\varphi = [t]\psi$.

First, suppose that $(\mathcal{M}^F, w) \models [t]\psi$. Thus $\psi \in \mathcal{E}(w, t)$ and $R(w) \subseteq \|\psi\|_{\mathcal{M}^F}$. By the definition of $\mathcal{E}$, we have $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$, and hence $(\mathcal{M}^N, w) \models [t]\psi$.

On the other hand, suppose that $(\mathcal{M}^N, w) \models [t]\psi$. Thus $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$, and hence $\psi \in \mathcal{E}(w, t)$. Now we show that $R(w) \subseteq \|\psi\|_{\mathcal{M}^F}$. Let $v \in W_N$ be an arbitrary world such that $v \in R(w)$. Then, by the definition of R , we have $v \in \bigcap N(w, t)$, and thus $v \in \|\psi\|_{\mathcal{M}^N}$. Hence, $(\mathcal{M}^N, v) \models \psi$. By the induction hypothesis, $(\mathcal{M}^F, v) \models \psi$, and thus $v \in \|\psi\|_{\mathcal{M}^F}$, as desired. $\dashv$

Completeness of LP_{CS} with respect to neighborhood models (Theorem 4.1) and world-equivalent of neighborhood models to some Fitting models (Lemma 5.1) give a new proof for the completeness of LP_{CS} with respect to Fitting models.

THEOREM 5.1 (Completeness). *For each formula φ and set of formulas T ,*

$$T \models_{LP_{CS}}^F \varphi \quad \text{iff} \quad T \vdash_{LP_{CS}} \varphi.$$

PROOF. Soundness is straightforward. Completeness follows from Theorem 4.1 and Lemma 5.1. $\dashv$

Next, we aim to show that Fitting models are world-equivalent to neighborhood models.

DEFINITION 5.3. A Fitting model $\mathcal{M} = (S, R, \mathcal{E}, V)$ is called *congruential* provided that there exists $u \in S$ with the property that for all formulas $\varphi, \psi \in \text{Fm}$ such that $\|\varphi\|_{\mathcal{M}} = \|\psi\|_{\mathcal{M}}$, we have that $\varphi \in \mathcal{E}(u, t)$ iff $\psi \in \mathcal{E}(u, t)$, for all $t \in \text{Tm}$. A Fitting model is called *non-congruential* if it is not congruential.

LEMMA 5.2. *Given a non-congruential Fitting model $\mathcal{M} = (S, R, \mathcal{E}, V)$ for LP_{CS} , there is a congruential Fitting model $\mathcal{M}'$ for LP_{CS} which is world-equivalent to $\mathcal{M}$ with respect to S .*

PROOF. The idea is to add a new world, say u , to the set of worlds of the non-congruential Fitting model $\mathcal{M} = (S, R, \mathcal{E}, V)$ in such a way that the new Fitting model is congruential. Define $\mathcal{M}' = (S', R', \mathcal{E}', V')$ as follows:

- $S' := S \cup \{u\}$, where $u \notin S$.
- $R' := R \cup \{(u, u)\}$.
- $V' \upharpoonright_S := V$ and $V'(u) = \emptyset$.
- $\mathcal{E}'(w, t) := \mathcal{E}(w, t)$, for all $w \in S$ and all $t \in \text{Tm}$; and $\mathcal{E}'(u, t) = \bigcup_{i \geq 0} \mathcal{E}_i(u, t)$, where $\mathcal{E}_i(u, t)$ is defined inductively as follows:

Base case:

- $\mathcal{E}_0(u, t) = \{\psi \mid [t]\psi \in \text{CS}\}$, provided that $[t]\varphi \in \text{CS}$ for some φ ; and $\mathcal{E}_0(u, t) = \emptyset$, for other terms t (i.e. either $t \in \text{Const}$ but there is no φ such that $[t]\varphi \in \text{CS}$, or $t \notin \text{Const}$).

Induction step: Let $i \geq 0$:

- $\mathcal{E}_{2i+1}(u, t) = \mathcal{E}_{2i}(u, t)$, for $t \in \text{Var} \cup \text{Const}$.
- $\mathcal{E}_{2i+1}(u, t \cdot s) = \mathcal{E}_{2i}(u, t \cdot s) \cup \{\psi \mid \varphi \rightarrow \psi \in \mathcal{E}_{2i}(u, t) \text{ and } \varphi \in \mathcal{E}_{2i}(u, s), \text{ for some } \varphi\}$.
- $\mathcal{E}_{2i+1}(u, s + t) = \mathcal{E}_{2i}(u, s + t) \cup \{\varphi \mid \varphi \in \mathcal{E}_{2i}(u, s) \text{ or } \varphi \in \mathcal{E}_{2i}(u, t)\}$.
- $\mathcal{E}_{2i+1}(u, !t) = \mathcal{E}_{2i}(u, !t) \cup \{[t]\varphi \mid \varphi \in \mathcal{E}_{2i}(u, t)\}$.
- $\mathcal{E}_{2i+2}(u, t) = \mathcal{E}_{2i+1}(u, t) \cup \{\psi \mid \|\psi\|_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}} \text{ and } \varphi \in \mathcal{E}_{2i+1}(u, t), \text{ for some } \varphi\}$.

It is not difficult to show that $\mathcal{E}_i(u, t) \subseteq \mathcal{E}_{i+1}(u, t)$, for all $t \in \text{Tm}$ and all $i \geq 0$. We now show that $\mathcal{M}'$ is a model for LP_{CS} . Note that since the new world u is not related to any worlds from S and since $\mathcal{M}$ satisfies (monotonicity- $\mathcal{E}$), the condition (monotonicity- $\mathcal{E}$) holds trivially for the $\mathcal{M}'$. The proof of other conditions follows from the definition of $\mathcal{E}'$. As an illustration, we show that $\mathcal{M}'$ satisfies (constant-specification- $\mathcal{E}$) and (pos-introsp- $\mathcal{E}$).

For (constant-specification- $\mathcal{E}$), suppose that $[t]\varphi \in \text{CS}$. Since $\mathcal{M}$ satisfies (constant-specification- $\mathcal{E}$), for all $w \in S$ we have $\varphi \in \mathcal{E}(w, t)$, and hence $\varphi \in \mathcal{E}'(w, t)$. In addition, $\varphi \in \mathcal{E}_0(u, t) \subseteq \mathcal{E}'(u, t)$. Thus, $\mathcal{M}'$ satisfies (constant-specification- $\mathcal{E}$).

For (pos-introsp- $\mathcal{E}$), we have to show that for all terms $t \in \text{Tm}$, all formulas $\varphi \in \text{Fm}$, and all worlds $w \in S'$: if $\varphi \in \mathcal{E}'(w, t)$, then $[t]\varphi \in \mathcal{E}'(w, !t)$. Note that this condition holds for worlds from S , because $\mathcal{M}$ satisfies (pos-introsp- $\mathcal{E}$). For the new world u , suppose that $\varphi \in \mathcal{E}'(u, t)$. Then, from the definition of $\mathcal{E}'$, it follows that $\varphi \in \mathcal{E}_i(u, t) \subseteq \mathcal{E}_{2i}(u, t)$, for some $i \geq 0$, and hence $[t]\varphi \in \mathcal{E}_{2i+1}(u, !t) \subseteq \mathcal{E}'(u, !t)$.

We next show that $\mathcal{M}'$ is world-equivalent to $\mathcal{M}$ with respect to S . It suffices to show that for all $\varphi \in \text{Fm}$

$$\|\varphi\|_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}'} \cap S.$$

This can be proved by induction on φ , and we only show the case that φ is a justification assertion, say $\varphi = [t]\psi$. Note that for $w \in S$ we have $R(w) = R'(w)$ and $\mathcal{E}(w, t) = \mathcal{E}'(w, t)$.

Given $w \in S$, $(\mathcal{M}, w) \models [t]\psi$ iff $\psi \in \mathcal{E}(w, t)$ and $R(w) \subseteq \|\psi\|_{\mathcal{M}}$ iff $\psi \in \mathcal{E}'(w, t)$ and $R'(w) \subseteq \|\psi\|_{\mathcal{M}'} \cap S$ iff $\psi \in \mathcal{E}'(w, t)$ and $R'(w) \subseteq \|\psi\|_{\mathcal{M}'}$ iff $(\mathcal{M}', w) \models [t]\psi$.

We now show that $\mathcal{M}'$ is congruential. Given arbitrary formulas $\varphi, \psi \in \text{Fm}$ and arbitrary $t \in \text{Tm}$ such that $\|\varphi\|_{\mathcal{M}'} = \|\psi\|_{\mathcal{M}'}$ and $\varphi \in \mathcal{E}'(u, t)$, we have to show that $\psi \in \mathcal{E}'(u, t)$. From $\|\varphi\|_{\mathcal{M}'} = \|\psi\|_{\mathcal{M}'}$ it follows that $\|\varphi\|_{\mathcal{M}'} \cap S = \|\psi\|_{\mathcal{M}'} \cap S$, and hence $\|\varphi\|_{\mathcal{M}} = \|\psi\|_{\mathcal{M}}$. On the other hand,

$\varphi \in \mathcal{E}'(u, t)$ implies that there is $i \geq 0$ such that $\varphi \in \mathcal{E}_i(u, t)$, and thus clearly $\varphi \in \mathcal{E}_{2i+1}(u, t)$. From the inductive construction of $\mathcal{E}'$ it follows that $\psi \in \mathcal{E}_{2i+2}(u, t)$, and therefore $\psi \in \mathcal{E}'(u, t)$, as required. The proof of the other direction, i.e. $\psi \in \mathcal{E}'(u, t)$ implies $\varphi \in \mathcal{E}'(u, t)$, is similar. $\dashv$

LEMMA 5.3. *Given a Fitting model $\mathcal{M}^F = (S, R, \mathcal{E}, V)$ for LP_{CS} , there is a neighborhood model $\mathcal{M}^N$ for LP_{CS} which is world-equivalent to $\mathcal{M}^F$ with respect to S .*

PROOF. Suppose that $\mathcal{M}^F = (S, R, \mathcal{E}, V)$ is a Fitting model for LP_{CS} . If $\mathcal{M}^F$ is non-congruential, by Lemma 5.2, we find a congruential Fitting model that is world-equivalent to $\mathcal{M}^F$ with respect to S . Thus, without loss of generality we assume that $\mathcal{M}^F$ is congruential. Define $\mathcal{M}^N = (S, W_N, N, V^N)$ as follows:

- Let W_N denote the set of all $w \in S$ such that for all $\varphi, \psi \in \text{Fm}$ and for all $t \in \text{Tm}$, if $\|\varphi\|_{\mathcal{M}^F} = \|\psi\|_{\mathcal{M}^F}$, then we have that $\varphi \in \mathcal{E}(w, t)$ iff $\psi \in \mathcal{E}(w, t)$. Since $\mathcal{M}^F$ is congruential, W_N is non-empty.
- $N(w, t) := \{\|\varphi\|_{\mathcal{M}^F} \mid \varphi \in \mathcal{E}(w, t) \text{ and } R(w) \subseteq \|\varphi\|_{\mathcal{M}^F}\}$, for all $w \in W_N$ and all $t \in \text{Tm}$.
- $V^N(w) := V(w)$, for all $w \in W_N$; and $V^N(w) := \{\varphi \mid (\mathcal{M}^F, w) \models \varphi\}$, for all $w \in S \setminus W_N$.

Note that the requirement of congruentiality of $\mathcal{M}^F$ implies that the neighborhood function N is well-defined.

In order to prove that $\mathcal{M}^N$ and $\mathcal{M}^F$ are world-equivalent with respect to S , we show that for all $\varphi \in \text{Fm}$:

$$\|\varphi\|_{\mathcal{M}^N} = \|\varphi\|_{\mathcal{M}^F}. \quad (2)$$

This can be shown by induction on φ , and we only show the case that φ is a justification assertion, say $\varphi = [t]\psi$. The proof is as follows:

Case $w \in S \setminus W_N$: $(\mathcal{M}^N, w) \models [t]\psi$ iff $[t]\psi \in V^N(w)$ iff $(\mathcal{M}^F, w) \models [t]\psi$.

Case $w \in W_N$: $(\mathcal{M}^F, w) \models [t]\psi$ iff $\psi \in \mathcal{E}(w, t)$ and $R(w) \subseteq \|\psi\|_{\mathcal{M}^F}$ iff $\|\psi\|_{\mathcal{M}^F} \in N(w, t)$ iff (by IH) $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$ iff $(\mathcal{M}^N, w) \models [t]\psi$.

It is easy to show that $\mathcal{M}^N$ is a neighborhood model. As an illustration, we show that $\mathcal{M}^N$ satisfies (application- N).

Suppose that $\|\varphi \rightarrow \psi\|_{\mathcal{M}^N} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}^N} \in N(w, s)$. By (2) we have $\|\varphi \rightarrow \psi\|_{\mathcal{M}^F} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}^F} \in N(w, s)$. Thus, we get

$\varphi \rightarrow \psi \in \mathcal{E}(w, t)$, $\varphi \in \mathcal{E}(w, s)$, $R(w) \subseteq \|\varphi\|_{\mathcal{M}^F}$ and $R(w) \subseteq \|\varphi \rightarrow \psi\|_{\mathcal{M}^F}$. By (application- $\mathcal{E}$), we obtain $\psi \in \mathcal{E}(w, t \cdot s)$. Moreover, since $\|\varphi\|_{\mathcal{M}^F} \cap \|\varphi \rightarrow \psi\|_{\mathcal{M}^F} \subseteq \|\psi\|_{\mathcal{M}^F}$, we get $R(w) \subseteq \|\psi\|_{\mathcal{M}^F}$. Hence $\|\psi\|_{\mathcal{M}^F} \in N(w, t \cdot s)$, and therefore by (2) we get $\|\psi\|_{\mathcal{M}^N} \in N(w, t \cdot s)$, as required. $\dashv$

Remark 5.1. The proof of Lemma 5.3 gives a world-equivalent neighborhood model that has the same set of worlds as the original Fitting model. This comes at a price. Namely, many of these worlds will be non-normal worlds in the neighborhood model. It would also be possible to construct a neighborhood model where all worlds of the Fitting model are normal worlds. To achieve this, however, one would have to add many additional non-normal worlds (like in the canonical model, where we have a world for each set of formulas).

In the literature of justification logic, evidence functions are usually interpreted as functions that determine only *admissible* pieces of evidence that may or may not justify a given proposition. In this case, $\varphi \in \mathcal{E}(w, t)$ reads as “at the world w , t is an admissible reason (not necessarily a decisive evidence) why φ is known.” If evidence functions are interpreted in this way, then the truth condition for justification assertions in Definition 5.2 suffers from a problem (cf. [7]). If φ is known for an agent in the sense of Kripke’s semantics of the standard modal logic, i.e. $R(w) \subseteq \|\varphi\|_{\mathcal{M}}$, then any reason, say t , that is admissible for φ , i.e. $\varphi \in \mathcal{E}(w, t)$, is a reason of why φ is known for the agent. Thus, an admissible reason t for a known formula φ always provides a reason for that formula. In particular, the semantics does not allow us to have two different pieces of reasons t and s , both admissible for why φ is known for the agent, and yet just one of these pieces of reasons is the agent’s decisive reason.

Remark 5.2 (Modular models). Modular models are introduced in [3] (cf. also [24]). We briefly compare them with Fitting models here.

A modular model $\mathcal{M} = (S, R, \mathcal{E}, V)$ for LP_{CS} is a Fitting model for LP_{CS} in which the condition (monotonicity- $\mathcal{E}$) is dropped. In addition, modular models should satisfy the following condition, called *Justification Yields Belief* (JYB): for all terms $t \in \text{Tm}$, and for all worlds $w \in S$

$$\mathcal{E}(w, t) \subseteq \Box_w, \quad (\text{JYB})$$

where $\Box_w := \{\varphi \in \text{Fm} \mid R(w) \subseteq \|\varphi\|_{\mathcal{M}}\}$. Truth of justification assertions is defined as follows:

$$(\mathcal{M}, w) \models [t]\varphi \text{ iff } \varphi \in \mathcal{E}(w, t). \quad (3)$$

It is easy to show that Fitting models and modular models are world-equivalent (see [3, 24]). By the truth condition (1) in a Fitting model, a formula $[t]\varphi$ is true at a world w iff t is a justification for φ at w (expressed by $\varphi \in \mathcal{E}(w, t)$) and φ is believed at w (expressed by $R(w) \subseteq \|\varphi\|_{\mathcal{M}}$). Thus, in a Fitting model the formula $[t]\varphi$ can be read “ t is a reason for believing φ ”. A good feature of modular models is that they separate the justification part from the belief part. By the truth condition (3) in a modular model, a formula $[t]\varphi$ is true at a world w iff t is a justification for φ at w (expressed by $\varphi \in \mathcal{E}(w, t)$). Thus, in a modular model the formula $[t]\varphi$ can be read “ t is a justification for φ ”. Then, the condition Justification Yields Belief (JYB) says that if the agent has a justification for a sentence, then she believes it. Thus, by (JYB), the truth of $[t]\varphi$ implies that the agent believes that φ is true.

However, it seems that (JYB) also suffers from some problems. For instance, an individual might have justification for believing in the existence of God but has not yet come to hold that belief. For another example, imagine a person grieving the loss of someone she holds dear and refusing to accept the reality of her loved one’s death. Despite having justifications to acknowledge the passing, she does not believe it to be true.

One way to justify the condition (JYB) is to change the meaning of the evidence function $\mathcal{E}$ in modular models. As Artemov mentioned in [3, page 23], in modular models $\mathcal{E}(w, t)$ reflects ‘believing for a reason t .’ Using this reading, (JYB) is plausible, since believing in a proposition for a reason t trivially yields believing in that proposition. However, this reading of $\mathcal{E}$ would violate the aforementioned separation of justification from belief which is assumed as a foundational reason for considering modular models by Artemov [3, page 27].

Finally, as it is mentioned in [24], a Fitting model for LP_{CS} (in which the truth condition for justification assertions is defined by (1)) is a modular model that satisfies the condition (**monotonicity- $\mathcal{E}$**). Thus, in this sense, modular models are more general than Fitting models.

5.2. Relationship with subset models

In this section, we first recall subset models for the logic of proofs (cf. [27, 28]), and then we show that subset models are world-equivalent to some neighborhood models.

DEFINITION 5.4 (Subset model). A subset model $\mathcal{M} = (W, W_N, E, V)$ for LP_{CS} is defined by:

1. W and W_N are non-empty sets of worlds and normal worlds, respectively, such that $W_N \subseteq W$.
2. $V : W \rightarrow \wp(\text{Fm})$ is a valuation function that satisfies the following conditions (we denote $\varphi \in V(w)$ by $(\mathcal{M}, w) \models \varphi$). For all normal worlds $w \in W_N$:

$$\begin{aligned} (\mathcal{M}, w) &\not\models \perp, \\ (\mathcal{M}, w) &\models \varphi \rightarrow \psi \text{ iff } (\mathcal{M}, w) \not\models \varphi \text{ or } (\mathcal{M}, w) \models \psi, \\ (\mathcal{M}, w) &\models [t]\varphi \text{ iff } E(w, t) \subseteq \|\varphi\|_{\mathcal{M}}. \end{aligned}$$

3. $E : W \times \text{Tm} \rightarrow \wp(W)$ is an evidence function that meets the following conditions. For all $w \in W_N$, and all $s, t \in \text{Tm}$:
 - If $[t]\varphi \in \text{CS}$, then $E(w, t) \subseteq \|\varphi\|_{\mathcal{M}}$.
 - $E(w, t \cdot s) \subseteq \|\text{APP}_w(t, s)\|_{\mathcal{M}}$,
where
 $\text{APP}_w(t, s) := \{\psi \in \text{Fm} \mid \exists \varphi \in \text{Fm} \text{ s.t. } E(w, t) \subseteq \|\varphi \rightarrow \psi\|_{\mathcal{M}} \text{ and } E(w, s) \subseteq \|\varphi\|_{\mathcal{M}}\}$ and $\|\Gamma\|_{\mathcal{M}} := \{w \in W \mid (\mathcal{M}, w) \models \varphi \text{ for all } \varphi \in \Gamma\}$
 for any set of formulas Γ .
 - $E(w, t + s) \subseteq E(w, t) \cap E(w, s)$.
 - $w \in E(w, t)$.
 - $E(w, !t) \subseteq \{v \in W \mid \forall \varphi \in \text{Fm} ([t]\varphi \in V(w) \Rightarrow [t]\varphi \in V(v))\}$.

Given a set of formulas T and a formula φ of LP_{CS} , $T \models_{\text{LP}_{\text{CS}}}^S \varphi$ iff for all subset models $\mathcal{M} = (W, W_N, E, V)$ for LP_{CS} , for all $w \in W_N$, if $(\mathcal{M}, w) \models \psi$ for all $\psi \in T$, then $(\mathcal{M}, w) \models \varphi$.

LEMMA 5.4. *Given a neighborhood model $\mathcal{M}^N = (W, W_N, N, V)$ for LP_{CS} such that $\mathcal{M}^N$ contains its core and is closed under supersets, then there is a subset model $\mathcal{M}^S$ for LP_{CS} which is world-equivalent to $\mathcal{M}^N$ with respect to W .*

PROOF. Suppose that $\mathcal{M}^N = (W, W_N, N, V)$ is a neighborhood model for LP_{CS} such that each $N(w, t)$ contains its core and is closed under

supersets. Let $\mathcal{M}^S = (W, W_N, E, V^S)$ be defined as follows. For all $t \in \text{Tm}$, all $\varphi \in \text{Fm}$, and all $w \in W$:

- $E(w, t) := \cap N(w, t)$, if $w \in W_N$; and $E(w, t) := \emptyset$, if $w \in W \setminus W_N$,
- $\varphi \in V^S(w)$ iff $w \in \|\varphi\|_{\mathcal{M}^N}$.

In order to prove that $\mathcal{M}^N$ and $\mathcal{M}^S$ are world-equivalent with respect to W , we show that for all $\varphi \in \text{Fm}$:

$$\|\varphi\|_{\mathcal{M}^N} = \|\varphi\|_{\mathcal{M}^S}. \quad (4)$$

This can be shown by induction on φ , and we only show the case $\varphi = [t]\psi$. The proof is as follows:

Case $w \in W \setminus W_N$: $(\mathcal{M}^N, w) \models [t]\psi$ iff $[t]\psi \in V(w)$ iff $w \in \|[t]\psi\|_{\mathcal{M}^N}$ iff $[t]\psi \in V^S(w)$ iff $(\mathcal{M}^S, w) \models [t]\psi$.

Case $w \in W_N$: First, suppose that $(\mathcal{M}^N, w) \models [t]\psi$. Thus $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$. Since $E(w, t) = \cap N(w, t) \subseteq \|\psi\|_{\mathcal{M}^N} = \|\psi\|_{\mathcal{M}^S}$ we obtain $(\mathcal{M}^S, w) \models [t]\psi$, as desired.

On the other hand, suppose that $(\mathcal{M}^S, w) \models [t]\psi$. Thus, $E(w, t) = \cap N(w, t) \subseteq \|\psi\|_{\mathcal{M}^S}$. Since $N(w, t)$ contains its core, we have $\cap N(w, t) \in N(w, t)$. Since $N(w, t)$ is closed under supersets, $\|\psi\|_{\mathcal{M}^S} \in N(w, t)$. By the induction hypothesis, $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$, and hence $(\mathcal{M}^N, w) \models [t]\psi$, as desired.

It is easy to prove that $\mathcal{M}^S$ is a subset model. We only show that

$$E(w, t \cdot s) \subseteq \|\text{APP}_w(t, s)\|_{\mathcal{M}}.$$

Suppose that $v \in E(w, t \cdot s)$ and $\psi \in \text{APP}_w(t, s)$. We have to show that $v \in \|\psi\|_{\mathcal{M}^S}$. From $\psi \in \text{APP}_w(t, s)$ it follows that there exists $\varphi \in \text{Fm}$ such that $E(w, t) \subseteq \|\varphi \rightarrow \psi\|_{\mathcal{M}^S}$ and $E(w, s) \subseteq \|\varphi\|_{\mathcal{M}^S}$. Since both $N(w, s)$ and $N(w, t)$ contain their cores, we have $E(w, s) = \cap N(w, s) \in N(w, s)$ and $E(w, t) = \cap N(w, t) \in N(w, t)$. Since $N(w, s)$ and $N(w, t)$ are closed under supersets, $\|\varphi \rightarrow \psi\|_{\mathcal{M}^S} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}^S} \in N(w, s)$. By the induction hypothesis, $\|\varphi \rightarrow \psi\|_{\mathcal{M}^N} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}^N} \in N(w, s)$. By (application- N), we get $\|\psi\|_{\mathcal{M}^N} \in N(w, t \cdot s)$. From this and the fact that $v \in E(w, t \cdot s) = \cap N(w, t \cdot s)$ we get $v \in \|\psi\|_{\mathcal{M}^N}$. By the induction hypothesis, we get $v \in \|\psi\|_{\mathcal{M}^S}$, as required. $\dashv$

LEMMA 5.5. *Given a subset model $\mathcal{M}^S = (W, W_N, E, V)$ for LP_{CS} , there is a neighborhood model $\mathcal{M}^N = (W, W_N, N, V^N)$ for LP_{CS} which is world-equivalent to $\mathcal{M}^S$ with respect to W .*

PROOF. Suppose that $\mathcal{M}^S = (W, W_N, E, V)$ is a subset model for LP_{CS} . Let $\mathcal{M}^N = (W, W_N, N, V^N)$ be defined as follows:

- $N(w, t) := \{\|\varphi\|_{\mathcal{M}^S} \mid E(w, t) \subseteq \|\varphi\|_{\mathcal{M}^S}\}$, for all $t \in \text{Tm}$, and all $w \in W_N$,
- $p \in V^N(w)$ iff $p \in V(w)$, for all $p \in \text{Prop}$, and all $w \in W_N$,
 $\varphi \in V^N(w)$ iff $\varphi \in V(w)$, for all $\varphi \in \text{Fm}$, and all $w \in W \setminus W_N$.

In order to prove that $\mathcal{M}^N$ and $\mathcal{M}^S$ are world-equivalent with respect to W , we show that for all $\varphi \in \text{Fm}$:

$$\|\varphi\|_{\mathcal{M}^N} = \|\varphi\|_{\mathcal{M}^S}. \quad (5)$$

This can be shown by induction on φ , and we only show the case $\varphi = [t]\psi$. The proof is as follows:

Case $w \in W \setminus W_N$: $(\mathcal{M}^N, w) \models [t]\psi$ iff $[t]\psi \in V^N(w)$ iff $[t]\psi \in V(w)$ iff $(\mathcal{M}^S, w) \models [t]\psi$.

Case $w \in W_N$: $(\mathcal{M}^N, w) \models [t]\psi$ iff $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$ iff $\|\psi\|_{\mathcal{M}^S} \in N(w, t)$ iff $E(w, t) \subseteq \|\psi\|_{\mathcal{M}^S}$ iff $(\mathcal{M}^S, w) \models [t]\psi$.

It is easy to prove that $\mathcal{M}^N$ is a neighborhood model. As an illustration, we show that $\mathcal{M}^N$ satisfies (application- N).

Suppose $\|\varphi \rightarrow \psi\|_{\mathcal{M}^N} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}^N} \in N(w, s)$. By (5) we have $\|\varphi \rightarrow \psi\|_{\mathcal{M}^S} \in N(w, t)$ and $\|\varphi\|_{\mathcal{M}^S} \in N(w, s)$. Thus, we get $E(w, s) \subseteq \|\varphi\|_{\mathcal{M}^S}$ and $E(w, t) \subseteq \|\varphi \rightarrow \psi\|_{\mathcal{M}^S}$. Thus, $\psi \in \text{APP}_w(t, s)$. We now show that $E(w, t \cdot s) \subseteq \|\psi\|_{\mathcal{M}^S}$. Let $v \in E(w, t \cdot s)$ be an arbitrary world. Since $E(w, t \cdot s) \subseteq \|\text{APP}_w(t, s)\|_{\mathcal{M}^S}$ we have $v \in \|\text{APP}_w(t, s)\|_{\mathcal{M}^S}$. Therefore, $v \in \|\psi\|_{\mathcal{M}^S}$. Hence, $E(w, s \cdot t) \subseteq \|\psi\|_{\mathcal{M}^S}$. Thus, $\|\psi\|_{\mathcal{M}^S} \in N(w, t \cdot s)$, and hence $\|\psi\|_{\mathcal{M}^N} \in N(w, t \cdot s)$, as required. $\dashv$

Completeness of LP_{CS} with respect to its subset models is proved in [27], and so the proof is omitted here.

THEOREM 5.2 (Completeness). *For each formula φ and set of formulas T ,*

$$T \models_{\text{LP}_{\text{CS}}}^S \varphi \quad \text{iff} \quad T \vdash_{\text{LP}_{\text{CS}}} \varphi.$$

5.3. Relationship with relational models

Purely relational models are first introduced in [43] for the justification logic $\text{J}_{\text{TCS}(\text{J})}$. We modify these models and present relational models for LP_{CS} , for an arbitrary constant specification CS .

DEFINITION 5.5. A *quasi relational model for LP_{CS}* is a tuple $\mathcal{M} = (W, W_N, R^t, V)_{t \in \mathsf{Tm}}$ where

1. W and W_N are non-empty sets of worlds and normal worlds, respectively, such that $W_N \subseteq W$,
2. $R^t \subseteq W \times W$ are binary relations,
3. V is a valuation function such that $V : W_N \rightarrow \wp(\mathsf{Prop})$ and $V : W \setminus W_N \rightarrow \wp(\mathsf{Fm})$.

Intuitively, wR^tv reads as “at the world w , v is a possible world according to reason t .”

DEFINITION 5.6. Let $\mathcal{M} = (W, W_N, R^t, V)_{t \in \mathsf{Tm}}$ be a quasi relational model. For $w \in W \setminus W_N$, let $(\mathcal{M}, w) \models \varphi$ if $\varphi \in V(w)$. Truth of justification assertions at worlds $w \in W_N$ is defined as follows:

$$(\mathcal{M}, w) \models [t]\varphi \text{ iff } (\mathcal{M}, v) \models \varphi, \text{ for all worlds } v \in W \text{ such that } wR^tv.$$

Let $\|\varphi\|_{\mathcal{M}} = \{w \in W \mid (\mathcal{M}, w) \models \varphi\}$ be the *truth set* of φ , and let

$$R^t(w) = \{v \in W \mid wR^tv\}.$$

Now, the truth condition of justification assertions can be expressed as follows:

$$(\mathcal{M}, w) \models [t]\varphi \text{ iff } R^t(w) \subseteq \|\varphi\|_{\mathcal{M}}.$$

The truth condition of the dual of justification assertions would be as follows:

$$(\mathcal{M}, w) \models \langle t \rangle \varphi \text{ iff } (\mathcal{M}, v) \models \varphi, \text{ for some world } v \in W \text{ such that } wR^tv.$$

DEFINITION 5.7 (Relational model). A *relational model*

$$\mathcal{M} = (W, W_N, R^t, V)_{t \in \mathsf{Tm}}$$

for LP_{CS} is a quasi relational model such that the relations R^t satisfy the following conditions. For all terms $s, t \in \mathsf{Tm}$, and all $w \in W_N$:

1. $R^t(w) \subseteq \|\varphi\|_{\mathcal{M}}$ for $[t]\varphi \in \mathsf{CS}$. (constant-specification- R)
2. $R^{t \cdot s}(w) \subseteq \{v \in W \mid \forall \varphi, \psi \in \mathsf{Fm}(w \in \|[t](\varphi \rightarrow \psi)\|_{\mathcal{M}} \cap \|[s]\varphi\|_{\mathcal{M}} \Rightarrow v \in \|\psi\|_{\mathcal{M}})\}$. (application- R)
3. $R^{t+s}(w) \subseteq R^t(w) \cap R^s(w)$. (sum- R)
4. wR^tw . (reflexivity- R)

5. $R^{lt}(w) \subseteq \{v \in W \mid \forall \varphi \in \mathbf{Fm} \ (w \in \llbracket t \rrbracket \varphi \parallel_{\mathcal{M}} \Rightarrow v \in \llbracket t \rrbracket \varphi \parallel_{\mathcal{M}})\}$.
 (pos-introsp- R)

Given a set of formulas T and a formula φ of $\mathbf{LP}_{CS}$, $T \models_{\mathbf{LP}_{CS}}^R \varphi$ iff for all relational models $\mathcal{M} = (W, W_N, R^t, V)_{t \in \mathbf{Tm}}$ for $\mathbf{LP}_{CS}$, for all $w \in W_N$, if $(\mathcal{M}, w) \models \psi$ for all $\psi \in T$, then $(\mathcal{M}, w) \models \varphi$.

Since the purely relational models of [43] developed for the logic $\mathbf{J}_{\mathbf{TCS}(\mathbf{J})}$, the conditions (constant-specification- R) and (pos-introsp- R) cannot be found there. Moreover, we omit the requirement of irreflexivity of R^t which is specified in [43]. Instead, we require that each R^t satisfies (reflexivity- R), which correspond to the axiom jT of $\mathbf{LP}$.

LEMMA 5.6. *Given a subset model $\mathcal{M}^S = (W, W_N, E, V)$ for $\mathbf{LP}_{CS}$, there is a relational model $\mathcal{M}^R = (W, W_N, R^t, V)_{t \in \mathbf{Tm}}$ for $\mathbf{LP}_{CS}$ which is world-equivalent to $\mathcal{M}^S$ with respect to W , and vice versa.*

PROOF. Suppose that $\mathcal{M}^R = (W, W_N, R^t, V)_{t \in \mathbf{Tm}}$ is a relational model and $\mathcal{M}^S = (W, W_N, E, V)$ is a subset model such that the following holds

$$wR^tv \quad \text{iff} \quad v \in E(w, t). \quad (6)$$

It is obvious that $R^t(w) = E(w, t)$, and hence $\mathcal{M}^R$ and $\mathcal{M}^S$ are world-equivalent with respect to W . $\dashv$

In fact, (6) suggests that relational models and subset models are notational variants of each other. Next, we prove the completeness of $\mathbf{LP}_{CS}$ with respect to relational models.

THEOREM 5.3 (Completeness). *For each formula φ and set of formulas T ,*

$$T \models_{\mathbf{LP}_{CS}}^R \varphi \quad \text{iff} \quad T \vdash_{\mathbf{LP}_{CS}} \varphi.$$

PROOF. Soundness is straightforward. Completeness follows from Theorem 5.2 and Lemma 5.6. $\dashv$

6. Topological models for $\mathbf{LP}$

Artemov and Nogina [6] presented topological semantics for various justification logics based on modal logic $\mathbf{S4}$. The systems studied in [6] have no operations on justifications, and so only variables are used as

justification terms. In this section, we extend their results to the logic of proofs and present topological semantics for LP_{CS} .

In [6], the authors interpret justification variables as *tests*, and justification assertions $[t]\varphi$ are read as “test t confirms φ ”. Given a topological space $(S, \mathcal{T})$, they used a test function $M : \text{Tm} \times \text{Fm} \rightarrow \wp(S)$ with the following meaning: $M(t, \varphi) \subseteq S$ represents a “potentially accessible (reachable) region of S associated with t and φ ” or “the set of possible outcomes of a test t of a property φ ”. Instead of this test function, we use a function $\mathbb{M} : S \times \text{Tm} \rightarrow \wp(\text{Fm})$ that acts similar to evidence functions of Fitting models. Here, $\varphi \in \mathbb{M}(w, t)$ says that “test t confirms φ at point w ” or “the point w is a possible outcome of the test t of the property φ ”.

DEFINITION 6.1 (Topological model). A quasi topological model for LP_{CS} is a tuple $\mathcal{M} = (S, \mathcal{T}, \mathbb{M}, V)$ where S is a non-empty set of worlds (or points), $(S, \mathcal{T})$ is a topological space where $\mathcal{T} \subseteq \wp(S)$, $V : S \rightarrow \wp(\text{Prop})$ is a valuation and $\mathbb{M} : S \times \text{Tm} \rightarrow \wp(\text{Fm})$ is a *test function*.

The definition of truth for Boolean connectives is standard (see Definition 4.2). The truth of justification assertions is defined as follows:

$$(\mathcal{M}, w) \models [t]\varphi \text{ iff } \varphi \in \mathbb{M}(w, t) \text{ and there is } O \in \mathcal{T}_w \text{ such that } O \subseteq \|\varphi\|_{\mathcal{M}},$$

where $\mathcal{T}_w := \{O \in \mathcal{T} \mid w \in O\}$, and $\|\varphi\|_{\mathcal{M}} := \{w \in S \mid (\mathcal{M}, w) \models \varphi\}$.

A topological model $\mathcal{M} = (S, \mathcal{T}, \mathbb{M}, V)$ for LP_{CS} is a quasi topological model such that the test function $\mathbb{M}$ satisfies the following conditions. For all terms $s, t \in \text{Tm}$, all formulas $\varphi, \psi \in \text{Fm}$, and all worlds $v, w \in S$:

1. If $[t]\varphi \in \text{CS}$, then $\varphi \in \mathbb{M}(w, t)$. (constant-specification- $\mathbb{M}$)
2. If $\varphi \rightarrow \psi \in \mathbb{M}(w, t)$ and $\varphi \in \mathbb{M}(w, s)$, then $\psi \in \mathbb{M}(w, t \cdot s)$. (application- $\mathbb{M}$)
3. $\mathbb{M}(w, t) \cup \mathbb{M}(w, s) \subseteq \mathbb{M}(w, t + s)$. (sum- $\mathbb{M}$)
4. If $\varphi \in \mathbb{M}(w, t)$, then $[t]\varphi \in \mathbb{M}(w, !t)$. (pos-introsp- $\mathbb{M}$)
5. If there exists $O \in \mathcal{T}_w$ such that $O \subseteq \|\varphi\|_{\mathcal{M}}$ and $w \in \|[t]\varphi\|_{\mathcal{M}}$, then there exists $O' \in \mathcal{T}_w$ such that $O' \subseteq \|[t]\varphi\|_{\mathcal{M}}$. (monotonicity- $\mathbb{M}$)

Given a set of formulas T and a formula φ of LP_{CS} , $T \models_{\text{LP}_{\text{CS}}}^T \varphi$ iff for all topological models $\mathcal{M} = (S, \mathcal{T}, \mathbb{M}, V)$ for LP_{CS} , for all $w \in S$, if $(\mathcal{M}, w) \models \psi$ for all $\psi \in T$, then $(\mathcal{M}, w) \models \varphi$.

Given a test function $\mathbb{M} : S \times \text{Tm} \rightarrow \wp(\text{Fm})$, define $M_{\mathbb{M}} : \text{Tm} \times \text{Fm} \rightarrow \wp(S)$ as follows:

$$M_{\mathbb{M}}(t, \varphi) := \{w \in S \mid \varphi \in \mathbb{M}(w, t)\}.$$

The function $M_{\mathbb{M}}$ is similar to the original test function used in [6]. It is easy to show that

$$\|[t]\varphi\|_{\mathcal{M}} = \mathbb{I}(\|\varphi\|_{\mathcal{M}}) \cap M_{\mathbb{M}}(t, \varphi),$$

where $\mathbb{I}$ denotes the interior function, i.e. for each $X \subseteq S$, $\mathbb{I}(X)$ gives the interior of the set X in the topological space $(S, \mathcal{T})$. Thus, $[t]\varphi$ represents the interior of the proposition φ , which is confirmed by the test t .

We now show that LP_{CS} is sound and complete with respect to its topological models. To this end, we use the completeness of LP_{CS} with respect to Fitting models.

LEMMA 6.1. *Given a Fitting model $\mathcal{M}^F = (S, R, \mathcal{E}, V)$ for LP_{CS} , there is a topological model $\mathcal{M}^T = (S, \mathcal{T}, \mathbb{M}, V)$ for LP_{CS} which is world-equivalent to $\mathcal{M}^F$ with respect to S .*

PROOF. Suppose that $\mathcal{M}^F = (S, R, \mathcal{E}, V)$ is a Fitting model for LP_{CS} . Define the topological model $\mathcal{M}^T = (S, \mathcal{T}, \mathbb{M}, V)$ as follows. For all $t \in \text{Tm}$, and all $w \in S$:

- $\mathcal{T} := \{O \subseteq S \mid (\forall u \in O)(\text{ if } uRv, \text{ then } v \in O)\},$
- $\mathbb{M}(w, t) := \mathcal{E}(w, t).$

It is easy to show that $(S, \mathcal{T})$ is a topological space. In order to prove that $\mathcal{M}^F$ and $\mathcal{M}^T$ are world-equivalent with respect to S , we show that for all $\varphi \in \text{Fm}$:

$$\|\varphi\|_{\mathcal{M}^F} = \|\varphi\|_{\mathcal{M}^T}. \quad (7)$$

This can be shown by induction on φ , and we only show the case $\varphi = [t]\psi$. The proof is as follows. Suppose that $w \in S$.

Suppose that $(\mathcal{M}^T, w) \models [t]\psi$. Then, $\varphi \in \mathbb{M}(w, t)$ and there exists $O \in \mathcal{T}_w$ such that $O \subseteq \|\psi\|_{\mathcal{M}^T}$. Hence, $\psi \in \mathcal{E}(w, t)$. Let $v \in S$ be an arbitrary world such that wRv . From wRv and $w \in O \in \mathcal{T}$, we get $v \in O$. Thus, $(\mathcal{M}^T, v) \models \psi$, and hence by the induction hypothesis $(\mathcal{M}^F, v) \models \psi$. Therefore, $(\mathcal{M}^F, w) \models [t]\psi$.

Conversely, suppose that $(\mathcal{M}^F, w) \models [t]\psi$. Then, $\psi \in \mathcal{E}(w, t)$, and hence $\varphi \in \mathbb{M}(w, t)$. Let

$$O := \|[t]\psi\|_{\mathcal{M}^F}.$$

It is obvious that $w \in O$. On the other hand, if $v \in O$ be an arbitrary world, then by the reflexivity of R we get $v \in \|\psi\|_{\mathcal{M}^F}$. By the induction

hypothesis, $v \in \|\psi\|_{\mathcal{M}^T}$. Thus, $O \subseteq \|\psi\|_{\mathcal{M}^T}$. We now show that $O \in \mathcal{T}$. Suppose that $u \in O$ and uRv . Then, $(\mathcal{M}^F, u) \models [t]\psi$. By the transitivity of R and (monotonicity- $\mathcal{E}$), we get $(\mathcal{M}^F, v) \models [t]\psi$. Thus, $v \in O$, as required.

It is easy to show that $\mathcal{M}^T$ is a topological model. Conditions (constant-specification- $\mathbb{M}$), (application- $\mathbb{M}$), (sum- $\mathbb{M}$), and (pos-introsp- $\mathbb{M}$) follows immediately from the conditions (constant-specification- $\mathcal{E}$), (application- $\mathcal{E}$), (sum- $\mathcal{E}$), and (pos-introsp- $\mathcal{E}$) respectively. Next, we show that $\mathcal{M}^T$ satisfies (monotonicity- $\mathbb{M}$).

Suppose that there exists $O \in \mathcal{T}_w$ such that $O \subseteq \|\varphi\|_{\mathcal{M}^T}$ and $w \in \|[t]\varphi\|_{\mathcal{M}^T}$. We have to show that there is an open set $O' \in \mathcal{T}_w$ such that $O' \subseteq \|[t]\varphi\|_{\mathcal{M}^T}$. Let

$$O' := \|[t]\varphi\|_{\mathcal{M}^T}.$$

It is obvious that $w \in O'$ and $O' \subseteq \|[t]\varphi\|_{\mathcal{M}^T}$. It is enough to show that $O' \in \mathcal{T}$. Suppose that $u \in O'$ and uRv . Then, $(\mathcal{M}^T, u) \models [t]\varphi$, and hence by (7), we get $(\mathcal{M}^F, u) \models [t]\varphi$. By the transitivity of R and (monotonicity- $\mathcal{E}$), we get $(\mathcal{M}^F, v) \models [t]\varphi$. By (7), we get $(\mathcal{M}^T, v) \models [t]\varphi$. Thus, $v \in O'$, as required. $\dashv$

THEOREM 6.1 (Completeness). *For each formula φ and set of formulas T ,*

$$T \models_{LP_{CS}}^T \varphi \quad \text{iff} \quad T \vdash_{LP_{CS}} \varphi.$$

PROOF. Soundness is straightforward. Completeness follows from Theorem 5.1 and Lemma 6.1. $\dashv$

In the rest of this section, we aim to determine which class of neighborhood models are world-equivalent to topological models, and vice versa.

LEMMA 6.2. *Given a neighborhood model $\mathcal{M}^N = (W, W_N, N, V)$ for LP_{CS} such that $\mathcal{M}^N$ contains normal worlds and is closed under finite intersections and supersets, then there is a topological model $\mathcal{M}^T$ for LP_{CS} which is world-equivalent to $\mathcal{M}^N$ with respect to W_N .*

PROOF. Suppose that $\mathcal{M}^N = (W, W_N, N, V)$ is a neighborhood model for LP_{CS} such that each $N(w, t)$ satisfies the conditions stated in the statement of the lemma. Let

$$\mathcal{T}[N] := \{O \subseteq W_N \mid (\forall v \in O) O \in \cup_{s \in \text{tm}} N(v, s)\}.$$

It is easy to show that $\emptyset, W_N \in \mathcal{T}[N]$. Since each $N(w, t)$ is closed under finite intersections, $\mathcal{T}[N]$ is also closed under finite intersections. Since each $N(w, t)$ is closed under supersets, $\mathcal{T}[N]$ is also closed under arbitrary unions. Therefore, $(W_N, \mathcal{T}[N])$ is a topological space.

Let $\mathcal{M}^T = (W_N, \mathcal{T}[N], \mathbb{M}, V^T)$ be defined as follows. For all $t \in \text{Tm}$, and all $w \in W_N$:

- $\varphi \in \mathbb{M}(w, t)$ iff $\|\varphi\|_{\mathcal{M}^N} \in N(w, t)$,
- $V^T := V \upharpoonright_{W_N}$.

In order to prove that $\mathcal{M}^N$ and $\mathcal{M}^T$ are world-equivalent with respect to W_N , we show that for all $\varphi \in \text{Fm}$:

$$\|\varphi\|_{\mathcal{M}^N \cap W_N} = \|\varphi\|_{\mathcal{M}^T}. \quad (8)$$

This can be shown by induction on φ , and we only show the case $\varphi = [t]\psi$. The proof is as follows. Suppose that $w \in W_N$.

Suppose that $(\mathcal{M}^T, w) \models [t]\psi$. Then, we get $\psi \in \mathbb{M}(w, t)$, and hence $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$. Thus, $(\mathcal{M}^N, w) \models [t]\psi$.

Conversely, suppose that $(\mathcal{M}^N, w) \models [t]\psi$. Then, $\|\psi\|_{\mathcal{M}^N} \in N(w, t)$, and hence $\varphi \in \mathbb{M}(w, t)$. Let

$$O := \|[t]\psi\|_{\mathcal{M}^N \cap W_N} = \{v \in W_N \mid \|\psi\|_{\mathcal{M}^N} \in N(v, t)\}. \quad (9)$$

It is obvious that $w \in O$. On the other hand, if $v \in O$ is an arbitrary world, then by (reflexivity- N) we get $v \in \|\psi\|_{\mathcal{M}^N}$. By (8), $v \in \|\psi\|_{\mathcal{M}^T}$. Thus, $O \subseteq \|\psi\|_{\mathcal{M}^T}$. We now show that $O \in \mathcal{T}[N]$. Suppose that $v \in O$. Then, $\|\psi\|_{\mathcal{M}^N} \in N(v, t)$. By (pos-introsp- N), we have $\|[t]\psi\|_{\mathcal{M}^N} \in N(v, !t)$. On the other hand, $W_N \in N(v, !t)$. Since $N(v, !t)$ is closed under finite intersection, $O = \|[t]\psi\|_{\mathcal{M}^N \cap W_N} \in N(v, !t)$. Thus, $O \in \mathcal{T}[N]$, as desired. Therefore, $(\mathcal{M}^T, w) \models [t]\psi$.

It is easy to show that $\mathcal{M}^T$ is a topological model. As an illustration, we show that $\mathcal{M}^T$ satisfies (monotonicity- $\mathbb{M}$). Suppose that there exists $O \in \mathcal{T}[N]$ such that $w \in O$, $O \subseteq \|\varphi\|_{\mathcal{M}^T}$, and $w \in \|[t]\varphi\|_{\mathcal{M}^T}$. We have to show that there is an open set $O' \in \mathcal{T}_w$ such that $O' \subseteq \|[t]\varphi\|_{\mathcal{M}^T}$. Let

$$O' := \|[t]\varphi\|_{\mathcal{M}^T}.$$

It is obvious that $w \in O'$ and $O' \subseteq \|[t]\varphi\|_{\mathcal{M}^T}$. It is enough to show that $O' \in \mathcal{T}[N]$. Suppose that $v \in O'$. Then, $(\mathcal{M}^T, v) \models [t]\varphi$, and hence $\varphi \in \mathbb{M}(v, t)$. Thus, $\|\varphi\|_{\mathcal{M}^N} \in N(v, t)$. By (pos-introsp- N), we

have $\|[t]\varphi\|_{\mathcal{M}^N} \in N(v, !t)$. On the other hand, $W_N \in N(v, !t)$. Hence, $\|[t]\psi\|_{\mathcal{M}^N} \cap W_N \in N(v, !t)$. From (8) it follows that $O' = \|[t]\varphi\|_{\mathcal{M}^T} \in N(v, !t)$. Thus, $O' \in \cup_{s \in \mathbf{Tm}} N(v, s)$, and so $O' \in \mathcal{T}[N]$, as desired. $\dashv$

Finally, we aim to show that topological models are world-equivalent to neighborhood models.

DEFINITION 6.2. A topological model $\mathcal{M} = (S, \mathcal{T}, \mathbb{M}, V)$ is called *congruential* provided that there exists $u \in S$ with the property that for all formulas $\varphi, \psi \in \mathbf{Fm}$ such that $\|\varphi\|_{\mathcal{M}} = \|\psi\|_{\mathcal{M}}$, we have that $\varphi \in \mathbb{M}(u, t)$ iff $\psi \in \mathbb{M}(u, t)$, for all $t \in \mathbf{Tm}$. A topological model is called *non-congruential* if it is not congruential.

We next show that non-congruential topological models can be converted to world-equivalent congruential topological models.

LEMMA 6.3. *Given a non-congruential topological model $\mathcal{M} = (S, \mathcal{T}, \mathbb{M}, V)$ for LP_{CS} , there is a congruential topological model $\mathcal{M}'$ for LP_{CS} which is world-equivalent to $\mathcal{M}$ with respect to S .*

PROOF. The proof is similar to that of Lemma 5.2, so we omit the details here. Given a non-congruential topological model $\mathcal{M} = (S, \mathcal{T}, \mathbb{M}, V)$ for LP_{CS} , define $\mathcal{M} = (S', \mathcal{T}', \mathbb{M}', V')$ as follows:

- $S' := S \cup \{u\}$, where $u \notin S$.
- $\mathcal{T}' := \mathcal{T} \cup \{S'\}$.
- $V' \upharpoonright_S := V$ and $V'(u) = \emptyset$.

The test function $\mathbb{M}'$ for worlds from S is defined as $\mathbb{M}$, i.e. $\mathbb{M}'(w, t) := \mathbb{M}(w, t)$, for all $w \in S$ and all $t \in \mathbf{Tm}$. For the new world u we define test functions $\mathbb{M}_i$ inductively as follows.

Base case:

- $\mathbb{M}_0(u, t) = \{\psi \mid [t]\psi \in \mathbf{CS}\}$, provided that $[t]\varphi \in \mathbf{CS}$ for some φ ; and $\mathbb{M}_0(u, t) = \emptyset$, for other terms t (i.e. either $t \in \mathbf{Const}$ but there is no φ such that $[t]\varphi \in \mathbf{CS}$, or $t \notin \mathbf{Const}$).

Induction step: Let $i \geq 0$:

- $\mathbb{M}_{2i+1}(u, t) = \mathbb{M}_{2i}(u, t)$, for $t \in \mathbf{Var} \cup \mathbf{Const}$.
- $\mathbb{M}_{2i+1}(u, t \cdot s) = \mathbb{M}_{2i}(u, t \cdot s) \cup \{\psi \mid \varphi \rightarrow \psi \in \mathbb{M}_{2i}(u, t) \text{ and } \varphi \in \mathbb{M}_{2i}(u, s), \text{ for some } \varphi\}$.
- $\mathbb{M}_{2i+1}(u, s + t) = \mathbb{M}_{2i}(u, s + t) \cup \{\varphi \mid \varphi \in \mathbb{M}_{2i}(u, s) \text{ or } \varphi \in \mathbb{M}_{2i}(u, t)\}$.

- $\mathbb{M}_{2i+1}(u, !t) = \mathbb{M}_{2i}(u, !t) \cup \{[t]\varphi \mid \varphi \in \mathbb{M}_{2i}(u, t)\}.$
- $\mathbb{M}_{2i+2}(u, t) = \mathbb{M}_{2i+1}(u, t) \cup \{\psi \mid \|\psi\|_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}} \text{ and } \varphi \in \mathbb{M}_{2i+1}(u, t), \text{ for some } \varphi\}.$

Note that $\mathbb{M}_i(u, t) \subseteq \mathbb{M}_{i+1}(u, t)$, for all $t \in \text{Tm}$ and all $i \geq 0$. Let $\text{Conf}(t) := \cap_{w \in S} \mathbb{M}(w, t)$. The set $\text{Conf}(t)$ contains all formulas that confirms by the test t at all points of the model $\mathcal{M}$. Finally, let

$$\mathbb{M}'(u, t) := \text{Conf}(t) \cap \bigcup_{i \geq 0} \mathbb{M}_i(u, t).$$

The proofs of the following facts are easy:

1. If $\varphi \in \mathbb{M}'(u, t)$, then $\varphi \in \mathbb{M}(w, t)$, for all $w \in S$.
2. If $\varphi \rightarrow \psi \in \text{Conf}(t)$ and $\varphi \in \text{Conf}(s)$, then $\psi \in \text{Conf}(t \cdot s)$.
3. $\text{Conf}(t) \subseteq \text{Conf}(t + s) \cap \text{Conf}(s + t)$.
4. If $\varphi \in \text{Conf}(t)$, then $[t]\varphi \in \text{Conf}(!t)$.

It is obvious that $(S', \mathcal{T}')$ is a topological space, and hence $\mathcal{M}'$ is a quasi topological model for LP_{CS} . We now show that $\mathcal{M}'$ is world-equivalent to $\mathcal{M}$ with respect to S . It suffices to show that for all $\varphi \in \text{Fm}$

$$\|\varphi\|_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}'} \cap S. \quad (10)$$

This can be proved by induction on φ , and we only show the case that φ is a justification assertion, say $\varphi = [t]\psi$. First, we show that for all $w \in S$ and all $\varphi \in \text{Fm}$, we have

$$\exists O \in \mathcal{T}_w \text{ s.t. } O \subseteq \|\varphi\|_{\mathcal{M}} \text{ iff } \exists O' \in \mathcal{T}'_w \text{ s.t. } O' \subseteq \|\varphi\|_{\mathcal{M}}. \quad (11)$$

The left-to-right direction follows from the fact that $\mathcal{T} \subseteq \mathcal{T}'$. For the right-to-left direction, we have two cases: $O' \in \mathcal{T}$ or $O' = S'$. In the former case, the proof is obvious. In the latter case, note that if $S' \subseteq \|\varphi\|_{\mathcal{M}'}$ then $S \subseteq \|\varphi\|_{\mathcal{M}'}$ and since $S \in \mathcal{T}_w$, we get $\exists O \in \mathcal{T}_w$ such that $O \subseteq \|\varphi\|_{\mathcal{M}'}$.

Next, we show that (10) holds for $\varphi = [t]\psi$. Given $w \in S$, $(\mathcal{M}, w) \models [t]\psi$ iff $\psi \in \mathbb{M}(w, t)$ and $\exists O \in \mathcal{T}_w$ s.t. $O \subseteq \|\psi\|_{\mathcal{M}}$ iff (by the definition of $\mathbb{M}'$ and (11)) $\psi \in \mathbb{M}'(w, t)$ and $\exists O' \in \mathcal{T}'_w$ s.t. $O' \subseteq \|\psi\|_{\mathcal{M}}$ iff (by IH) $\psi \in \mathbb{M}'(w, t)$ and $\exists O' \in \mathcal{T}'_w$ s.t. $O' \subseteq \|\psi\|_{\mathcal{M}'}$ iff $(\mathcal{M}', w) \models [t]\psi$.

Next, we show that $\mathcal{M}'$ is a topological model for LP_{CS} . We only check the details for the conditions (**application-M**) and (**monotonicity-M**) (the proof of other conditions follows from the construction of $\mathbb{M}'$ and Facts 3–4).

For (application- $\mathbb{M}$), we have to show that for all terms $s, t \in \mathbf{Tm}$, all formulas $\varphi, \psi \in \mathbf{Fm}$, and all worlds $w \in S'$: if $\varphi \rightarrow \psi \in \mathbb{M}'(w, t)$ and $\varphi \in \mathbb{M}'(w, s)$, then $\psi \in \mathbb{M}'(w, t \cdot s)$. Since $\mathcal{M}$ satisfies (application- $\mathbb{M}$), this condition holds for worlds from S . For the new world u , suppose that $\varphi \rightarrow \psi \in \mathbb{M}'(u, t)$ and $\varphi \in \mathbb{M}'(u, s)$. Then, $\varphi \rightarrow \psi \in \text{Conf}(t)$ and $\varphi \in \text{Conf}(s)$. From Fact (2) it follows that $\psi \in \text{Conf}(t \cdot s)$. From the definition of $\mathbb{M}'$, it follows that $\varphi \rightarrow \psi \in \mathbb{M}_{2i}(u, t)$ and $\varphi \in \mathbb{M}_{2i}(u, s)$, for some $i \geq 0$, and hence $\psi \in \mathbb{M}_{2i+1}(u, t \cdot s) \subseteq \mathbb{M}'(u, t \cdot s)$.

We first prove (monotonicity- $\mathbb{M}$) for worlds from S . Given an arbitrary world $w \in S$, suppose that there exists $O \in \mathcal{T}'_w$ such that $O \subseteq \|\varphi\|_{\mathcal{M}'}$ and $w \in \|[t]\varphi\|_{\mathcal{M}'}$. We distinguish two cases: $O \in \mathcal{T}$ or $O = S'$. In the former case, since $O \subseteq S$, we get $O \subseteq \|\varphi\|_{\mathcal{M}'} \cap S = \|\varphi\|_{\mathcal{M}}$. On the other hand, by (10), we have $w \in \|[t]\varphi\|_{\mathcal{M}}$. Since $\mathcal{M}$ satisfies (monotonicity- $\mathbb{M}$), there exists $O' \in \mathcal{T}'_w$ such that $O' \subseteq \|[t]\varphi\|_{\mathcal{M}'}$. Hence, we have $O' \in \mathcal{T}'_w$, and by (10), $O' \subseteq \|[t]\varphi\|_{\mathcal{M}'}$, as desired. In the latter case, from $S' \subseteq \|\varphi\|_{\mathcal{M}'}$, it follows that $S \subseteq \|\varphi\|_{\mathcal{M}'}$. The rest of the proof is similar to the first case.

We now prove (monotonicity- $\mathbb{M}$) for u . Note that the only open set containing u is S' . So suppose that $S' \subseteq \|\varphi\|_{\mathcal{M}'}$ and $u \in \|[t]\varphi\|_{\mathcal{M}'}$. We shall show that $S' \subseteq \|[t]\varphi\|_{\mathcal{M}'}$. Since $u \in \|[t]\varphi\|_{\mathcal{M}'}$, it is sufficient to show that $S \subseteq \|[t]\varphi\|_{\mathcal{M}'}$. Assume that $w \in S$ is an arbitrary world. From $u \in \|[t]\varphi\|_{\mathcal{M}'}$, it follows that $\varphi \in \mathbb{M}'(u, t)$. Fact (1) immediately yields $\varphi \in \mathbb{M}'(w, t)$. On the other hand, $S' \in \mathcal{T}'_w$ and $S' \subseteq \|\varphi\|_{\mathcal{M}'}$. Therefore, $(\mathcal{M}', w) \models [t]\varphi$, and hence $w \in \|[t]\varphi\|_{\mathcal{M}'}$, as required.

The proof of the congruentiality of $\mathcal{M}'$ is similar to that given in the proof of Lemma 5.2. Thus, we omit the details here. $\dashv$

In the next lemma, we will modify the notion of *neighborhood system* from [33, p. 23] as a neighborhood function induced by a topology.

LEMMA 6.4. *Given a topological model $\mathcal{M}^T = (S, \mathcal{T}, \mathbb{M}, V)$, there is a neighborhood model $\mathcal{M}^N$ for LP_{CS} which is world-equivalent to $\mathcal{M}^T$ with respect to S .*

PROOF. Suppose that $\mathcal{M}^T = (S, \mathcal{T}, \mathbb{M}, V)$ is a topological model for LP_{CS} . Without loss of generality we assume that $\mathcal{M}^T$ is congruential. Define $\mathcal{M}^N = (S, W_N, N_{\mathcal{M}^T}, V^N)$ as follows.

- W_N is the set of all $w \in S$ such that for all $\varphi, \psi \in \mathbf{Fm}$ and for all $t \in \mathbf{Tm}$, if $\|\varphi\|_{\mathcal{M}^T} = \|\psi\|_{\mathcal{M}^T}$, then we have that $\varphi \in \mathbb{M}(w, t)$ iff $\psi \in \mathbb{M}(w, t)$. Since $\mathcal{M}^T$ is congruential, W_N is non-empty.

- The neighborhood function $N_{\mathcal{M}^T} : W_N \times \mathbf{Tm} \rightarrow \wp(\wp(W))$ is defined as follows. For all $w \in W_N$ and all $t \in \mathbf{Tm}$:

$$N_{\mathcal{M}^T}(w, t) := \{\|\varphi\|_{\mathcal{M}^T} \mid \varphi \in \mathbb{M}(w, t) \text{ and } \exists O \in \mathcal{T}_w \text{ s.t. } O \subseteq \|\varphi\|_{\mathcal{M}^T}\}.$$

The neighborhood function $N_{\mathcal{M}^T}$ can be viewed as a neighborhood system induced by a topology and a test function.

- $V^N(w) := V(w)$, for all $w \in W_N$; and $V^N(w) := \{\varphi \mid (\mathcal{M}^T, w) \models \varphi\}$, for all $w \in S \setminus W_N$.

In order to prove that $\mathcal{M}^N$ and $\mathcal{M}^T$ are world-equivalent with respect to S , we show that for all $\varphi \in \mathbf{Fm}$:

$$\|\varphi\|_{\mathcal{M}^N} = \|\varphi\|_{\mathcal{M}^T}. \quad (12)$$

This can be shown by induction on φ , and we only show the case $\varphi = [t]\psi$. The proof is as follows:

Case $w \in S \setminus W_N$: $(\mathcal{M}^N, w) \models [t]\psi$ iff $[t]\psi \in V^N(w)$ iff $(\mathcal{M}^T, w) \models [t]\psi$.

Case $w \in W_N$: $(\mathcal{M}^N, w) \models [t]\psi$ iff $\|\psi\|_{\mathcal{M}^N} \in N_{\mathcal{M}^T}(w, t)$ iff $\|\psi\|_{\mathcal{M}^T} \in N_{\mathcal{M}^T}(w, t)$ iff $\varphi \in \mathbb{M}(w, t)$ and $\exists O \in \mathcal{T}_w$ s.t. $O \subseteq \|\varphi\|_{\mathcal{M}^T}$ iff $(\mathcal{M}^T, w) \models [t]\psi$.

It is easy to show that $\mathcal{M}^N$ is a neighborhood model. As an illustration, we show that $\mathcal{M}^N$ satisfies (application- N).

Suppose that $\|\varphi \rightarrow \psi\|_{\mathcal{M}^N} \in N_{\mathcal{M}^T}(w, t)$ and $\|\varphi\|_{\mathcal{M}^N} \in N_{\mathcal{M}^T}(w, s)$. By (12) we have $\|\varphi \rightarrow \psi\|_{\mathcal{M}^T} \in N_{\mathcal{M}^T}(w, t)$ and $\|\varphi\|_{\mathcal{M}^T} \in N_{\mathcal{M}^T}(w, s)$. Thus, we get $\varphi \rightarrow \psi \in \mathbb{M}(w, t)$, $\varphi \in \mathbb{M}(w, s)$, $\exists O_1 \in \mathcal{T}_w$ s.t. $O_1 \subseteq \|\varphi \rightarrow \psi\|_{\mathcal{M}^T}$, and $\exists O_2 \in \mathcal{T}_w$ s.t. $O_2 \subseteq \|\varphi\|_{\mathcal{M}^T}$. By (application- $\mathbb{M}$), we obtain $\psi \in \mathbb{M}(w, t \cdot s)$. Moreover, since $\|\varphi\|_{\mathcal{M}^T} \cap \|\varphi \rightarrow \psi\|_{\mathcal{M}^T} \subseteq \|\psi\|_{\mathcal{M}^T}$, we get $O_1 \cap O_2 \subseteq \|\psi\|_{\mathcal{M}^T}$. Since $\mathcal{T}$ is a topology, we have $O_1 \cap O_2 \in \mathcal{T}_w$. Therefore $\|\psi\|_{\mathcal{M}^T} \in N_{\mathcal{M}^T}(w, t \cdot s)$, and $\|\psi\|_{\mathcal{M}^N} \in N_{\mathcal{M}^T}(w, t \cdot s)$ as required. $\dashv$

7. Discussion

In this article, we compare various possible world semantics of justification logics, including neighborhood models, Fitting models, modular models, subset models, relational models, and topological models. Our presentation of the neighborhood models, relational models, and topological models for the logic of proof LP is novel.

We develop a notion of equivalence, called world-equivalence, for possible world models, which helps to reduce a completeness result from one class of possible world models to a completeness result for another class of models. We observe that Fitting models and modular models are world-equivalent to each other, and our relational models and subset models are notational variants of each other.

Furthermore, we demonstrate the completeness theorem of justification logics with respect to the neighborhood models by a canonical model construction using maximal consistent sets. Whereas, the completeness of LP with respect to relational models is established through the completeness of LP with respect to subset models, and further the completeness of LP with respect to topological models is established through the completeness of LP with respect to Fitting models. We leave a direct proof of the completeness of LP with respect to topological models to possible future work.

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