Relevant Justification Logic

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Abstract

We introduce a relevant justification logic, RJ4, which is a combination of the relevant logic R and the justification logic J4. We describe the corresponding class of models, provide the axiomatization and prove that our logic is sound and complete.

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1. Introduction

Relevant logics are non-classical logics that avoid the paradoxes of material and strict implication and provide a more intuitive deductive inference. The central systems of relevant logic, according to Anderson and Belnap [1], are the system of relevant implication R, as well as the logic of entailment E.

Justification logic replaces the $\Box$-operator of modal logic by explicit justifications [2, 5]. That is justification logic features formulas of the form $t : A$ meaning *A is believed for reason t*; hence we can reason with and about explicit justifications for an agent's belief. The framework of justification logic has been used to formalize and study a variety of epistemic situations [3, 6, 10, 11, 13, 14, 17].

However, traditional justification logic is based on classical logic and can lead to some paradoxical situations. One of those situations will be our running example in this paper.

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Example 1. *Consider a person A visiting a foreign town, which she does not know well. In order to get to a certain restaurant, she asks two persons B and C for the way. Person B says that A can take path P to the restaurant whereas person C replies that P does not lead to the restaurant and A should take another way. Person A now has a reason s to believe P and a reason t to believe $\neg P$. We can formalize this in justification logic by saying that both*

$$s : P \qquad \textit{and} \qquad t : \neg P \tag{1}$$

hold. However, under certain natural assumptions, there exists a justification $r(s, t)$ such that

$$r(s, t) : (P \wedge \neg P)$$

holds. Now this implies that for any formula F, there is a justification u such that

$$u : F \tag{2}$$

holds. That means for any formula F, person A has a reason to believe F, which, of course, is an undesirable consequence.

It is the aim of this paper to introduce a justification logic, RJ4, in which situations of this kind cannot occur, in particular, that means a logic in which (2) does not follow from (1). We achieve this by combining the relevant logic R with the justification logic J4.

Meyer [18] proposed the logic NR, which is the relevant logic R equipped with an $S4$-style theory of necessity, in order to investigate whether the resulting theory coincides with the theory of entailment provided by Anderson and Belnap [1]. Adapting the semantics for the logic R [19], Routley and Meyer provided a complete semantics for the logic NR [20].

Our logic RJ4 is similar to NR but instead of the $\Box$-operator, we use explicit justifications and since we deal with beliefs, we do not include the truth principle $t : A \to A$ in the list of axioms. The choice of axioms for the relevant logic R can be varied in different ways, e.g., see [12]. We decided to use the first 12 axioms from [20].

Our relevant justification logic RJ4 is not just a simple combination of R and J4. The reason is that justification logic includes an application operation on terms, which is related to implication, i.e., we have the following axiom

$$t : (A \to B) \to (s : A \to (t \cdot s) : B).$$

Hence, if the meaning of implication changes, then also the meaning of the application operation has to change. This is hidden in the axiomatization, but it becomes evident in the semantics. There, property ($p7$) models the relation between justifications and relevant implication. It shows that there is a true interaction between those two parts and we cannot simply juxtapose the semantics for R and the one for J4 to obtain a semantics for RJ4.

Another motivation for this work, i.e., for combining relevant logic with justification logic, comes from the philosophical point of view. Namely, if an implication of the form

$$s : A \to t : B$$

holds, then, we argue, the antecedent $s : A$ should be relevant for the consequent $t : B$. Note that we do not claim that the justification s itself must be relevant for the justification t (it is a different topic), but rather that the fact that s is a justification for A should be relevant for the whole consequent, i.e., that t is a justification for B.

The contents of this paper are as follows. In Section 2 we present the syntax of our logic, in Section 3 we provide the axiomatization, while in Section 4 the semantics is explained. In Section 5 soundness and completeness theorems are proved and we conclude in Section 6.

2. Syntax

In this section we propose the syntax of the logic RJ4.

Let

$\mathsf{Con} = \{c_0, c_1, \ldots, c_n, \ldots\}$ be a countable set of constants,

$\mathsf{Var} = \{x_0, x_1, \ldots, x_n, \ldots\}$ be a countable set of variables, and

$\mathsf{Prop} = \{p_0, p_1, \ldots, p_n, \ldots\}$ a countable set of atomic propositions.

Definition 1 (Terms). *Terms are built from the sets* Con *and* Var *as follows:*

$$t ::= c \mid x \mid t \cdot t \mid t \tilde{\wedge} t \mid t + t \mid !t,$$

where $c \in \mathsf{Con}$ *and* $x \in \mathsf{Var}$. *The set of terms will be denoted by* Tm.

Note that, in comparison to the definition of terms in the justification logic J4, we have an additional operation, $\tilde{\wedge}$, on terms.

Definition 2 (Formulas). *Formulas are build from the sets* Prop *and* Tm *as follows:*

$$A ::= p \mid \neg A \mid A \rightarrow A \mid A \wedge A \mid A \vee A \mid A \circ A \mid t : A,$$

where $p \in \mathsf{Prop}$ *and* $t \in \mathsf{Tm}$. *The set of formulas is denoted by* For.

We define $A \leftrightarrow B$ as

$$A \leftrightarrow B =_{def} (A \rightarrow B) \wedge (B \rightarrow A).$$

For sets of formulas X and Y, we will use the following notation:

$$X \cdot Y := \{F \mid G \rightarrow F \in X \text{ and } G \in Y, \text{ for some formula } G\},$$
$$X \wedge Y := \{F \mid F = G \wedge H, \text{ for some } G \in X \text{ and } H \in Y\},$$
$$t : X := \{t : F \mid F \in X\}.$$

3. Axiomatization

There are two groups of axioms for RJ4. The first group are the axioms of the logic R[1]:

(A1) $A \rightarrow A$

[1] There are many equivalent ways to axiomatize the logic R. We decided to take the axiomatizaiton from [20].

(A2) $A \to ((A \to B) \to B)$

(A3) $(A \to B) \to ((B \to C) \to (A \to C))$

(A4) $(A \to (A \to B)) \to (A \to B)$

(A5) $A \wedge B \to A$

(A6) $A \wedge B \to B$

(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(A9) $\neg\neg A \to A$

(A10) $(A \to \neg B) \to (B \to \neg A)$

(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$

(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

The second group consists of the axioms of J4 plus an additional axiom (A15):

(A13) $t : (A \to B) \to (s : A \to (t \cdot s) : B)$

(A14) $t : A \to \; !t : t : A$

(A15) $t : A \wedge s : B \to (t \tilde{\wedge} s) : (A \wedge B)$

(A16) $t : A \to (t + s) : A$ and $t : A \to (s + t) : A$

In the axiom ($A12$) we introduced the binary connective, $\circ$, which is called *fusion*, or *intensional conjunction*. It is defined via $\neg$ and $\to$, i.e., introducing it is the conservative extension of our language. Fusion plays an important role in the relevant logic R and its connection with implication is even stronger, namely the formula

$$((A \circ B) \to C) \leftrightarrow (A \to (B \to C)) \tag{3}$$

is valid formula of R. For more details about fusion see, e.g., [16].

To introduce the rules of our logic, we need the following definition:

Definition 3. Constant specification *is a set*

$$\mathsf{CS} \subseteq \{(c, A) \mid c \textit{ is a constant and } A \textit{ is an axiom of } \mathsf{RJ4}\}.$$

Constant specification CS *is called* axiomatically appropriate *if for each axiom* A *there exists a constant* $c \in \mathsf{Con}$*, such that* $(c, A) \in \mathsf{CS}$*.*

Given a constant specification CS, the deductive system $\mathsf{RJ4}_{\mathsf{CS}}$ is given by the axioms of $\mathsf{RJ4}$ and the following rules:

$$(\mathsf{MP})\,\frac{F \qquad F \to G}{G} \qquad\qquad (\mathsf{ADJ})\,\frac{F \qquad G}{F \wedge G} \qquad\qquad (\mathsf{AN})\,\frac{(c, A) \in \mathsf{CS}}{c : A}$$

where the first rule is called *modus ponens*, the second *adjunction* and the last rule is called *axiom necessitation.*

As usual in justification logics, we can show the following analogue of the necessitation rule.

Lemma 1 (Constructive necessitation). *Let* CS *be an axiomatically appropriate constant specification. For each formula* A*,*

$$\mathsf{RJ4}_{\mathsf{CS}} \vdash A \quad \textit{implies} \quad \mathsf{RJ4}_{\mathsf{CS}} \vdash t : A \textit{ for some term } t.$$

4. Semantics

The semantics for $\mathsf{RJ4}$ is based on a combination of possible world models for R and basic modular models for $\mathsf{J4}$.

In order to motivate our semantics, let us look closer at the Example 1, i.e., let us formally prove that in the justification logic $\mathsf{J4}$ with an axiomatically appropriate constant specification[2], for any formula F, there exists a justification term $u(s, t)$, such that

$$u(s, t) : F$$

does follow from

$$s : P \qquad \text{and} \qquad t : \neg P.$$

[2] for more details about $\mathsf{J4}$ and axiomatically appropriate constant specification see, e.g., [5]

Example 2. *Consider a justification logic* J4 *with an axiomatically appropriate constant specification and suppose that* $s : P$ *and* $t : \neg P$ *hold. Then, we have the following derivation*[3]*:*

1) $\vdash_{\mathsf{J4_{CS}}} P \to (\neg P \to (P \wedge \neg P))$ *PR*

2) $\vdash_{\mathsf{J4_{CS}}} r : (P \to (\neg P \to (P \wedge \neg P)))$ *AN*

3) $\vdash_{\mathsf{J4_{CS}}} r : (P \to (\neg P \to (P \wedge \neg P))) \to (s : P \to (r \cdot s : (\neg P \to (P \wedge \neg P))))$ *J*

4) $\vdash_{\mathsf{J4_{CS}}} s : P \to (r \cdot s : (\neg P \to (P \wedge \neg P)))$ *2),3) MP*

5) $\vdash_{\mathsf{J4_{CS}}} r \cdot s : (\neg P \to (P \wedge \neg P)) \to (t : \neg P \to ((r \cdot s) \cdot t : (P \wedge \neg P)))$ *J*

6) $\vdash_{\mathsf{J4_{CS}}} s : P \to (t : \neg P \to ((r \cdot s) \cdot t) : (P \wedge \neg P))$ *4),5) PR*

7) $\vdash_{\mathsf{J4_{CS}}} (s : P \wedge t : \neg P) \to ((r \cdot s) \cdot t : (P \wedge \neg P))$ *6) PR*

8) $\vdash_{\mathsf{J4_{CS}}} (r \cdot s) \cdot t : (P \wedge \neg P)$ *follows from our assumption and 7) using MP*

9) $\vdash_{\mathsf{J4_{CS}}} (P \wedge \neg P) \to F$*, for any formula* F*,* *PR*

10) $\vdash_{\mathsf{J4_{CS}}} t' : ((P \wedge \neg P) \to F),$ *AN*

11) $\vdash_{\mathsf{J4_{CS}}} t' : ((P \wedge \neg P) \to F) \to ((r \cdot s) \cdot t : (P \wedge \neg P) \to t' \cdot ((r \cdot s) \cdot t) : F),$ *J*

12) $\vdash_{\mathsf{J4_{CS}}} (r \cdot s) \cdot t : (P \wedge \neg P) \to t' \cdot ((r \cdot s) \cdot t) : F,$ *10),11) MP*

13) $\vdash_{\mathsf{J4_{CS}}} t' \cdot ((r \cdot s) \cdot t) : F,$ *8),12) MP.*

Note that, if "$\to$" represents relevant implication instead of implication of the classical propositional logic, the derivation above is not possible. Namely, the step 9) does not hold.

[3] PR stands for propositional reasoning, AN for axiom necessitation of the logic J4, J for $JAxiom$ of the logic J4 and MP for modus ponens.

Remark 1. *If the constant specification is not axiomatically appropriate, then step* 10) *in the above example does not hold. Therefore, restricting the constant specification could be another approach to prevent the derivation of the formula* $t' \cdot ((r \cdot s) \cdot t) : F$. *However, it is a natural assumption to have an axiomatically appropriate constant specification, i.e., for each axiom there is a reason to believe it. Hence we do not use restricted constant specifications but employ a logic based on relevant implication.*

Our models will be models for relevant logic R, namely we use Routley-Meyer semantics, i.e., Kripke-style structrure equipped with ternary relation[4], where each world is essentially basic modular model of justification logic (with an additional constraint for $\tilde{\wedge}$), see [4, 15].

Definition 4 (Model). *Let* CS *be an arbitrary constant specification. An* $\mathsf{RJ4}_{\mathsf{CS}}$*-model is a tuple of the form* $M = (K, 0, R, *, \spadesuit, \nu)$ *where:*

1. K *is a set,*
2. $0 \in K$,
3. R *is a ternary relation on* K,
4. $*$ *is a function* $* : K \to K$,
5. $\spadesuit$ *is a function* $\spadesuit : \mathsf{Tm} \times K \to \mathcal{P}(\mathsf{For})$,
6. ν *is a function* $\nu : K \to \mathcal{P}(\mathsf{Prop})$,

that satisfies the following properties:

(p1) $a \leq a$,

(p2) $Raaa$,

(p3) $R^2abcd \Rightarrow R^2acbd$,

[4]There is no universally accepted intuition behind the ternary relation. For example, $Rxyz$ can be viewed as that the combination of the pieces of information x and y is a piece of information in z as well as that set-ups x and y are compatible according to z. For more details about the ternary relation and various models of R, see [8, 9, 7].

(p4) $a \leq b \wedge Rbcd \Rightarrow Racd$,

(p5) $Rabc \Leftrightarrow Rac^{*}b^{*}$,

(p6) $a^{**} = a$,

(p7) $Rabc \Rightarrow t_a^{\spadesuit} \cdot s_b^{\spadesuit} \subseteq (t \cdot s)_c^{\spadesuit}$,

(p8) $a \leq b \Rightarrow t_a^{\spadesuit} \subseteq t_b^{\spadesuit}$,

(p9) $s_a^{\spadesuit} \cup t_a^{\spadesuit} \subseteq (s + t)_a^{\spadesuit}$,

(p10) $A \in t_0^{\spadesuit}$ *if* $(t, A) \in \mathsf{CS}$,

(p11) $t : (t_a^{\spadesuit}) \subseteq (!t)_a^{\spadesuit}$,

(p12) $s_a^{\spadesuit} \wedge t_a^{\spadesuit} \subseteq (s \tilde{\wedge} t)_a^{\spadesuit}$,

(p13) $a \leq b \Rightarrow \nu(a) \subseteq \nu(b)$,

where

$$a \leq b := R0ab,$$

and

$$R^2abcd := \exists x(Rabx \wedge Rxcd).$$

We write $t_a^{\spadesuit}$ for $\spadesuit(t, a)$ and we call an ordered pair $(\nu, \spadesuit)$ *valuation*.

The property $(p7)$ deserves more attention since it is the only property that includes both the ternary relation R and justifications, i.e., gives us connection between Routley-Meyer semantics and justification semantics. Note that the axiom that corresponds to this property is the axiom (A13). It is the only axiom in the second group of axioms that has an implication in consequent. Therefore, we need to guarantee the validity of that implication and the property $(p7)$ gives us a connection between the ternary relation R and justifications on the worlds that are related by R.

The property $(p7)$ can also be regarded as a generalization of the following principle from basic modular models:

$$s^{\spadesuit} \cdot t^{\spadesuit} \subseteq (s \cdot t)^{\spadesuit}.$$

Our worlds are basic modular models, so we want that, for any $a \in K$,

$$s_a^{\spadesuit} \cdot t_a^{\spadesuit} \subseteq (s \cdot t)_a^{\spadesuit} \tag{4}$$

holds. Indeed, (4) follows from ($p7$) together with ($p2$). Hence our semantics is a true generalization of the traditional semantics for justification logic.

Note that $\mathsf{RJ4}_{\mathsf{CS}}$-models do not feature the *justification yields belief* principle of modular models [4, 15]. As in models for NR, we could add a binary relation S on K to $\mathsf{RJ4}_{\mathsf{CS}}$-model and require that justification yields belief in the sense of S. This construction would yield modular models for $\mathsf{RJ4}$.

Definition 5 (Satisfiability relation). *Given a model $\mathcal{M} = (K, 0, R, *, \spadesuit, \nu)$ and $a \in K$ we define a relation $\models$ as follows:*

$$\begin{array}{rcl}
\mathcal{M}, a \models p & \textit{iff} & p \in \nu(a), \textit{for } p \in Prop \\
\mathcal{M}, a \models A \wedge B & \textit{iff} & \mathcal{M}, a \models A \textit{ and } \mathcal{M}, a \models B \\
\mathcal{M}, a \models A \vee B & \textit{iff} & \mathcal{M}, a \models A \textit{ or } \mathcal{M}, a \models B \\
\mathcal{M}, a \models A \circ B & \textit{iff} & Rxya \textit{ and } \mathcal{M}, x \models A \textit{ and } \mathcal{M}, y \models B, \textit{for some } x, y \in K \\
\mathcal{M}, a \models A \to B & \textit{iff} & Raxy \textit{ and } \mathcal{M}, x \models A \textit{ imply } \mathcal{M}, y \models B, \textit{for all } x, y \in K \\
\mathcal{M}, a \models \neg A & \textit{iff} & \mathcal{M}, a* \not\models A \\
\mathcal{M}, a \models t : A & \textit{iff} & A \in t_a^{\spadesuit}.
\end{array}$$

We say that a formula A is *true* at a in $\mathcal{M}$ if $\mathcal{M}, a \models A$. Formula A is *verified* in M, iff $\mathcal{M}, 0 \models A$. Finally, formula A is CS-*valid* iff A is verified in every $\mathsf{RJ4}_{\mathsf{CS}}$-model. We will often write $a \models A$ instead of $\mathcal{M}, a \models A$ when $\mathcal{M}$ is clear from a context. Also, we say that A *entails* B if for all $a \in K$, if $a \models A$ then $a \models B$.

We need a couple of auxiliary lemmas. Let $\mathcal{M} = (K, 0, R, *, \spadesuit, \nu)$ be an arbitrary model and $a, b \in K$ and $A, B \in For$.

Lemma 2 (Hereditary Lemma). *If $a \leq b$ and $a \models A$, then $b \models A$.*

Proof. In order to prove this Lemma we need a few auxiliary claims, namely:

(i) $Rabc \Rightarrow Rbac$;

(ii) $a \leq b \Rightarrow b^* \leq a^*$;

(iii) $Rabc$ and $c \leq d \Rightarrow Rabd$.

Proof of (i). $Rabc \Rightarrow R^2 0abc \Rightarrow R^2 0bac \Rightarrow Rbac$.
First "$\Rightarrow$" holds since both $R0aa$ $(p1)$ and $Rabc$ (assumption) hold. The second "$\Rightarrow$" is $(p3)$ and the third follows from $(p4)$, since $(p4)$ can be written as $R0ab \wedge Rbcd \Rightarrow Racd$, i.e., $R^2 0acd \Rightarrow Racd$.
Proof of (ii). Directly from definition of "$\leq$" and $(p5)$, the following derivation holds:

$$a \leq b \Rightarrow R0ab \Rightarrow R0b^*a^* \Rightarrow b^* \leq a^*.$$

Proof of (iii). Using $(p5), (i), (p4)$ together with $c \leq d$ and $(ii), (i)$, $(p5)$ respectively, the following holds:

$$Rabc \Rightarrow Rac^*b^* \Rightarrow Rc^*ab^* \Rightarrow Rd^*ab^* \Rightarrow Rad^*b^* \Rightarrow Rabd.$$

Proof of Lemma 2. By induction on a length of a formula A.

1) If $A = p \in Prop$, then the condition $(p13)$, $a \leq b \Rightarrow \nu(a) \subseteq \nu(b)$, ensures the claim.

2) The cases when $A = B \wedge C$ or $A = B \vee C$ are trivial.

3) Let $A = \neg B$ and $a \models A$. That means that $a^* \not\models B$. Since $a \leq b$, from (ii) we have $b^* \leq a^*$. Therefore $b^* \not\models B$, so $b \models A$.

4) Now, let $A = B \rightarrow C$ and $a \models A$. For all x, y with $Raxy$ we have that if $x \models B$ then $y \models C$. Suppose that $Rbcd$ and $c \models B$. The question is whether $d \models C$. Since $a \leq b$ and $Rbcd$, from $(p4)$ we obtain that $Racd$ and therefore as a direct consequence of our premise, which holds for every x, y, we have that $d \models C$.

5) Let $A = B \circ C$. From $a \models A$ we get that there exist x, y, such that $Rxya$ and $x \models B$ and $y \models C$. Furthermore, from (iii), since $a \leq b$ and $Rxya$ we know that $Rxyb$. Thus $b \models B \circ C$, i.e., $b \models A$.

6) Finally, let $A = t : B$ and $a \models A$. That means that $B \in t^{\spadesuit}_a$ and since $a \leq b$, we have $t^{\spadesuit}_a \subseteq t^{\spadesuit}_b$, so $B \in t^{\spadesuit}_b$, i.e. $b \models A$ as well. $\square$

In the following, since the majority of proofs are the same as in [19], we will give the proofs only for those cases that are new.

Lemma 3 (Entailment). *A entails B if and only if $A \rightarrow B$ is verified.*

5. Soundness and Completeness

5.1. Soundness

In order to prove soundness we need to prove that every instance of an axiom holds in arbitrary model and that inference rules preserve validity. We will consider only the axioms $(A13) - (A16)$.

Theorem 1 (Soundness). *Let CS be any constant specification. For each formula A we have*

$$\textit{If} \quad \mathsf{RJ4}_{\mathsf{CS}} \vdash A \quad \textit{then} \quad A \textit{ is } \mathsf{CS}\textit{-valid.}$$

Proof. Since our axioms are of the form $X \rightarrow Y$, using Lemma 3, it is enough to prove that for arbirtary $a \in K$, if $a \models X$ then $a \models Y$.

(A13) Suppose that $a \models t : (B \rightarrow C)$, i.e. $B \rightarrow C \in t^{\spadesuit}_a$. We need to show that $a \models s : B \rightarrow (t \cdot s) : C$. Suppose that $Rabc$ and $b \models s : B$, i.e. $B \in s^{\spadesuit}_b$. Since $B \rightarrow C \in t^{\spadesuit}_a$, we obtain that $C \in t^{\spadesuit}_a \cdot s^{\spadesuit}_b$ and, because $Rabc$, $t^{\spadesuit}_a \cdot s^{\spadesuit}_b \subseteq (t \cdot s)^{\spadesuit}_c$, $C \in (t \cdot s)^{\spadesuit}_c$, i.e. $c \models (t \cdot s) : C$.

(A14) Let $a \models t : B$, i.e. $B \in t^{\spadesuit}_a$. Then $t : B \in t : (t^{\spadesuit}_a) \subseteq (!t)^{\spadesuit}_a$. Therefore $a \models !t : t : B$.

(A15) Now, suppose that $a \models t : A \wedge s : B$. That means $A \in t^{\spadesuit}_a$ and $B \in s^{\spadesuit}_a$. Hence $A \wedge B \in t^{\spadesuit}_a \wedge s^{\spadesuit}_a \subseteq (t \tilde{\wedge} s)^{\spadesuit}_a$, so $a \models t \tilde{\wedge} s : (A \wedge B)$.

(A16) Finally, suppose $a \models t : A$, i.e., $A \in t^{\spadesuit}_a \subseteq t^{\spadesuit}_a \cup s^{\spadesuit}_a \subseteq (t + s)^{\spadesuit}_a$. Therefore $a \models (t + s) : A$. The other case is analoguous.

We need to prove that inference rules preserve the validity:

(MP) If a formula A is obtained from $B \to A$ and B, we have that $0 \models B \to A$ and $0 \models B$, hence $0 \models A$, since $R000$.

(ADJ) The case when a formula is obtained from adjunction is trivial.

(AN) If A is obtained by (AN), then $A = c : B$, where $(c, B) \in CS$. Therefore, $B \in c_0^{\spadesuit}$, so $0 \models c : B$. □

5.2. Completeness

In order to prove the completeness theorem, we will use a procedure based on [19]. That means that first we develop a calculus of intensional $\mathsf{RJ4}_{\mathsf{CS}}$-theories, then a calculus of intensional T-theories, for a regular intensional theory T and at the end a calculus of prime intensional theories.

With all this machinery, we are able to define the canonical model, which gives us the completeness theorem. Below we state all definitions and lemmas that we need. The majority of the proofs are identical as in [19], so they will be omitted. We state only the original proofs.

Definition 6. *Let $T \subseteq \mathsf{For}$ and CS any constant specification. We say that T is*

a) *an* intensional $\mathsf{RJ4}_{\mathsf{CS}}$-theory *iff T is closed under adjunction and if $A \in T$ and $\mathsf{RJ4}_{\mathsf{CS}} \vdash A \to B$, then $B \in T$;*

b) prime *iff it is an intensional $\mathsf{RJ4}_{\mathsf{CS}}$-theory and if $A \vee B \in T$, then $A \in T$ or $B \in T$;*

c) regular *iff it contains all theorems of $\mathsf{RJ4}_{\mathsf{CS}}$;*

d) consistent *iff it does not contain the negation of some theorem of $\mathsf{RJ4}_{\mathsf{CS}}$.*

Lemma 4. *Let $(\nu, \spadesuit)$ be a valuation in a structure $(K, 0, R, *)$ and let $a \in K$. The set of all formulas F such that $a \models F$, denoted by $T((\nu, \spadesuit), a)$, is a prime theory. If $0 \leq a$, then $T((\nu, \spadesuit), a)$ is regular.*

Definition 7 (Calculus of intensional theories). *Let* CS *be an arbitrary constant specification. The calculus of intensional theories is the structure* $\mathcal{H} = (\mathcal{H}, \subseteq, \circ, 0)$*, where*

1) $\mathcal{H}$ *is the collection of all intensional* $\mathsf{RJ4}_{\mathsf{CS}}$*-theories;*

2) $\subseteq$ *is set inclusion;*

3) $\circ$ *is a binary operation on* $\mathcal{H}$ *defined with*

$$S \circ T = \{C \mid \mathsf{RJ4}_{\mathsf{CS}} \vdash A \circ B \to C,\ \textit{for some } A \in S \textit{ and some } B \in T\};$$

4) 0 *is the set of all theorems of* $\mathsf{RJ4}_{\mathsf{CS}}$*.*

Lemma 5. *The calculus* $\mathcal{H}$ *is a partially ordered commutative monoid, that means,* $\circ$ *is associative and commutative operation and* 0 *is an identity with respect to* $\circ$*.*
Also, the following holds for all $a, b, c \in \mathcal{H}$:
if $a \subseteq b$ *then* $a \circ c \subseteq b \circ c$*;*
$a \circ a \subseteq a$ *(square decreasing).*

Definition 8 (Intensional T-theory). *An intensional* T*-theory is any set of formulas,* a*, which is an intensional* $\mathsf{RJ4}_{\mathsf{CS}}$*-theory and whenever* $A \in a$ *and* $A \to B \in T$*, then* $B \in a$*.*

Now we define a calculus of intensional T-theories.

Definition 9 (Calculus of intensional T-theories). *The calculus of intensional* T*-theories is the structure* $\mathcal{H}_{\mathbf{T}} = (\mathcal{H}_T, \subseteq, \circ, 0_T)$*, where* T *is a regular theory,* $\mathcal{H}_T$ *is a set of all intensional* T*-theories,* $0_T = T$ *and* $\circ$ *and* $\subseteq$ *are defined as above.*

Lemma 6. *The calculus* $\mathcal{H}_{\mathbf{T}}$ *is a sub-semigroup of* $\mathcal{H}$*.*

Definition 10 (Positive relevant structure (PRS)). *The structure* $(K, 0, R)$*, where* K *is a set,* $0 \in K$ *and* R *is a ternary relation on* K*, which satisfies properties* $(p1) - (p4)$ *will be called positive relevant structure (PRS).*

Let $\mathbf{M} = (M, \leq, \circ, 0)$ be commutative, partially ordered, square decreasing monoid satisfying that $a \leq b$ implies $a \circ c \leq b \circ c$. We say that PRS $(M, 0, R)$ is associated with $\mathbf{M}$ if M is equal to underlying set of $\mathbf{M}$, 0 is equal to identity of $\mathbf{M}$ and R is defined such that $Rabc$ iff $a \circ b \leq c$ in M.

Lemma 7. *If* $\mathbf{M}$ *is commutative, partially ordered, square decreasing monoid, then PRS* $(M, 0, R)$ *associated with* $\mathbf{M}$ *satisfies properties* $(p1) - (p4)$.
Furthermore, for all $a, b, c, d \in M$, $a \leq b$ *in* M *iff* $a \leq b$ *in* $\mathbf{M}$ *and* R^2abcd *iff* $a \circ b \circ c \leq d$ *in* $\mathbf{M}$.
The calculus $\mathcal{H}$ *is associated with PRS* $(\mathcal{H}, 0, R)$ *and the calculus* $\mathcal{H}_{\mathbf{T}}$ *is associated with PRS* $(\mathcal{H}_T, 0_T, R_T)$.

Let T be prime, regular, intensional $\mathsf{RJ4_{CS}}$-theory. Let $(\mathcal{H}_T, 0_T, R_T)$ be the PRS associated with $\mathcal{H}_{\mathbf{T}}$ and let $\mathcal{H}'_T$ be the subset of $\mathcal{H}_T$ which consists of all prime intensional theories in $\mathcal{H}_T$. Let $0'_T = T$ and R'_T restriction of R_T to $\mathcal{H}'_T$.

Lemma 8. $(\mathcal{H}'_T, 0'_T, R'_T)$ *is a PRS, i.e., satisfies* $(p1) - (p4)$.

For a prime intensional theory a, we define $a^* = \{A \mid \neg A \notin a\}$.

Lemma 9. *Let* $(\mathcal{H}'_T, 0'_T, R'_T)$ *and* $*$ *be defined as above. Then* $(\mathcal{H}'_T, 0'_T, R'_T, *)$ *is a relevant structure (RS), i.e.,* $*$ *is an operation on* $\mathcal{H}'_T$ *and properties* $(p1)-(p6)$ *are satisfied.*

Definition 11 (Canonical model). *Let* CS *be any constant specification.* $\mathsf{RJ4_{CS}}$-*model* $(\mathcal{H}'_T, 0'_T, R'_T, *, \nu, \spadesuit)$, *where* $(\mathcal{H}'_T, 0'_T, R'_T, *)$ *is RS from Lemma 9 and a valuation* $(\nu, \spadesuit)$ *defined with:*

a) $p \in \nu(a)$ *iff* $p \in a$;

b) $\spadesuit(t, a) = \{A \mid t : A \in a\}$,

will be called canonical T-model.

Lemma 10. *The canonical* T*-model is an* $\mathsf{RJ4_{CS}}$*-model, i.e., it satisfies properties* $(p1) - (p13)$.

Proof. It follows from Lemma 9 that $(p1) - (p6)$ are satisfied. Let us show the others.

(p7) Suppose that $R'_T abc$, i.e., $a \circ b \subseteq c$ and suppose that $A \in t_a^{\spadesuit} \cdot s_b^{\spadesuit}$. Hence, there exist $B \in s_b^{\spadesuit}$ such that $B \to A \in t_a^{\spadesuit}$. By definition of $\spadesuit$, we have that $s : B \in b$ and $t : (B \to A) \in a$. Therefore,

$$(t : (B \to A)) \circ (s : B) \in a \circ b \subseteq c.$$

Also, note that, because of 3, the Axiom $(A13)$ is equivalent to

$$(t : B \to A) \circ (s : B) \to (t \cdot s) : A, \tag{5}$$

so, since c is an intensional $\mathsf{RJ4_{CS}}$-theory, and antecedent of 5 belongs to c, we obtain that $(t \cdot s) : A \in c$, i.e., $A \in (t \cdot s)_c^{\spadesuit}$.

(p8) Let $a \subseteq b$ and $A \in t_a^{\spadesuit}$. Then, $t : A \in a \subseteq b$, so, $A \in t_b^{\spadesuit}$.

(p9) Let $A \in s_a^{\spadesuit} \cup t_a^{\spadesuit}$. First, suppose that $A \in s_a^{\spadesuit}$, i.e., $s : A \in a$. Directly from the Axiom $(A16)$ and intensionality of a we obtain the result. If $A \in t_a^{\spadesuit}$, the proof is analogous.

(p10) If $(t, A) \in \mathsf{CS}$, then, because of an axiom necessitation rule, we know that $\mathsf{RJ4_{CS}} \vdash t : A$ and, since T is regular, $t : A \in T = 0'_T$ which implies that $A \in t_0^{\spadesuit}$.

(p11) Suppose that $A \in t : (t_a^{\spadesuit})$, i.e., exists $B \in t_a^{\spadesuit}$, such that $A = t : B$. Since $B \in t_a^{\spadesuit}$, we know that $t : B \in a$ and hence, by Axiom $(A14)$, $!t : t : B \in a$ as well. That means $A = t : B \in (!t)_a^{\spadesuit}$.

(p12) Let $A \in s_a^{\spadesuit} \wedge t_a^{\spadesuit}$. There exist $B \in s_a^{\spadesuit}$ and $C \in t_a^{\spadesuit}$, such that $A = B \wedge C$. By definition of $\spadesuit$, we have that $s : B \in a$ and $t : C \in a$ and, because of adjunction, $s : B \wedge t : C \in a$. Again, intensionality gives us that $(s \tilde{\wedge} t) : (B \wedge C) \in a$, i.e., $A = B \wedge C \in (s \tilde{\wedge} t)_a^{\spadesuit}$.

(p13) If $a \subseteq b$ and $p \in \nu(a)$, we have that $p \in a$ and therefore $p \in b$ which concludes the proof. □

Lemma 11. *Let $A \in \mathsf{For}$. For all $a \in \mathcal{H}'_T$, $a \models A$ iff $A \in a$.*

Proof. We will prove only the case when $A = t : F$, for some $F \in \mathsf{For}$.
First suppose that $a \models t : F$. That means $F \in t_a^{\spadesuit}$, hence, by definition of $\spadesuit$, we obtain that $t : F \in a$.
For the other direction, suppose that $t : F \in a$. Again, directly by definition we obtain that $F \in t_a^{\spadesuit}$ and therefore $a \models t : F$. □

For the proof of the following lemma, Zorn's Lemma is necessary.

Lemma 12. *Let* CS *be an arbitrary constant specification. For every non-theorem, A, there exists a prime, regular* $\mathsf{RJ4_{CS}}$*-theory which does not contain A.*

Theorem 2 (Completeness). *For an arbitrary constant specification* CS*, the system* $\mathsf{RJ4_{CS}}$ *is semantically complete, i.e.,*

$$\textit{if } A \textit{ is } \mathsf{CS}\textit{-valid, then } \mathsf{RJ4_{CS}} \vdash A.$$

6. Conclusion

In this paper, the logic $\mathsf{RJ4}$ is introduced and its models, which are combination of Kripke-style models for relevant logic R and basic modular models for justification logics, are developed. We propose an axiomatization and prove the soundness and completeness theorem.

As mentioned in the introduction, there is a close relationship between NR and our logic of relevant justifications. Let RLP be the system $\mathsf{RJ4}$ plus the axiom $t : A \to A$ based on the total constant specification, i.e., every constant justifies every axiom (including $t : A \to A$). A *realization* is a mapping from modal formulas to formulas of justification logic that replaces each $\Box$ with some expression t: (different occurrences of $\Box$ may be replaced with different terms).

For further work, we plan to prove the realization theorem, i.e.:

Conjecture 1 (Realization). *There is a realization r such that for each modal formula A*

$$\mathsf{NR} \vdash A \quad \textit{implies} \quad \mathsf{RLP} \vdash r(A).$$

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