Temporal Justification Logic*

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Abstract

Justification logics are modal-like logics with the additional capability of recording the reason, or justification, for modalities in syntactic structures, called justification terms. Justification logics can be seen as explicit counterparts to modal logics. The behavior and interaction of agents in distributed system is often modeled using logics of knowledge and time. In this paper, we sketch some preliminary ideas on how the modal knowledge part of such logics of knowledge and time could be replaced with an appropriate justification logic.

1   Introduction

Justification logics are epistemic logics that feature explicit reasons for an agent's knowledge and belief. Originally, Artemov [1] developed the first justification logic, the Logic of Proofs LP, to provide a classical provability semantics for intuitionistic logic. Later, Fitting [9] introduced epistemic models for justification logic. This general reading of justification led to a big variety of epistemic justification logics for many different applications [2, 3, 4, 6, 11, 12, 17, 18]. Instead of an implicit statement $\Box\varphi$, which stands for *the agent knows* φ, justification logics include explicit statements of the form $[t]\varphi$, which mean *t justifies the agent's knowledge of* φ.

A common approach to model distributed systems of interacting agents is using logics of knowledge and time, with the interplay between these two modalities leading to interesting properties and questions [8, 15, 16, 24, 25]. While knowledge in such systems has typically been modeled using the modal logic S5, it is a natural question to ask what happens when we model knowledge in such logics using a justification logic.

This paper offers a first study on combing temporal logic and justification logic. We introduce a system $\mathsf{LPLTL}_{\mathcal{CS}}$ that combines linear time temporal logic LTL with the justification logic LP. In Sections 2 and 3 we present the language and the axioms of $\mathsf{LPLTL}_{\mathcal{CS}}$, respectively. In Section 4 we introduce interpreted systems with Fitting models as semantics

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for temporal justification logic. In Section 5 we establish soundness and completeness of
LPLTL$_{\mathcal{CS}}$. In Section 6 we present an extension LPLTL$^\star_{\mathcal{CS}}$ of LPLTL$_{\mathcal{CS}}$ that enjoys the in-
ternalization property. In Section 7 we introduce some additional principles concerning
interactions of knowledge, justifications, and time. In Section 8 we conclude the paper and
discuss some open problems.

2 Language

In the following, let h be a fixed number of agents, Const a countable set of justification
constants, Var a countable set of justification variables, and Prop a countable set of atomic
propositions.

The set of justification terms Tm is defined inductively by

$$t ::= c \mid x \mid {!}t \mid t + t \mid t \cdot t \,,$$

where $c \in$ Const and $x \in$ Var.

The set of formulas Fml is inductively defined by

$$\varphi ::= P \mid \bot \mid \varphi \to \varphi \mid \bigcirc\varphi \mid \varphi\,\mathcal{U}\,\varphi \mid [t]_i\varphi \,,$$

where $1 \le i \le h$, $t \in$ Tm, and $P \in$ Prop.

We use the following usual abbreviations:

$$\neg\varphi := \varphi \to \bot \qquad\qquad\qquad \top := \neg\bot$$
$$\varphi \vee \psi := \neg\varphi \to \psi \qquad\qquad \varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$$
$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi) \qquad \Diamond\varphi := \top\,\mathcal{U}\,\varphi$$
$$\Box\varphi := \neg\Diamond\neg\varphi.$$

Associativity and precedence of connectives, as well as the corresponding omission of
brackets, are handled in the usual manner.

Subformulas are defined as usual. The set of subformulas Sub(χ) of a formula χ is
inductively given by:

$$\mathsf{Sub}(P) := \{P\} \qquad\qquad\qquad \mathsf{Sub}(\bot) := \{\bot\}$$
$$\mathsf{Sub}(\varphi \to \psi) := \{\varphi \to \psi\} \cup \mathsf{Sub}(\varphi) \cup \mathsf{Sub}(\psi) \qquad \mathsf{Sub}(\bigcirc\varphi) := \{\bigcirc\varphi\} \cup \mathsf{Sub}(\varphi)$$
$$\mathsf{Sub}(\varphi\,\mathcal{U}\,\psi) := \{\varphi\,\mathcal{U}\,\psi\} \cup \mathsf{Sub}(\varphi) \cup \mathsf{Sub}(\psi) \qquad \mathsf{Sub}([t]_i\varphi) := \{[t]_i\varphi\} \cup \mathsf{Sub}(\varphi).$$

3 Axioms

The axiom system for temporal justification logic consists of three parts, namely propositional
logic, temporal logic, and justification logic.

Propositional Logic

For propositional logic, we take
1. all propositional tautologies   (Prop)
as axioms and the rule modus ponens, as usual:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\,(\mathsf{MP})\,.$$

Temporal Logic

For the temporal part, we use a system of [10, 13, 14] with axioms

2.	$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$	($\bigcirc$-k)
3.	$\square(\varphi \to \psi) \to (\square\varphi \to \square\psi)$	($\square$-k)
4.	$\bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$	(fun)
5.	$\square(\varphi \to \bigcirc\varphi) \to (\varphi \to \square\varphi)$	(ind)
6.	$\varphi \mathcal{U} \psi \to \Diamond\psi$	($\mathcal{U}1$)
7.	$\varphi \mathcal{U} \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi))$	($\mathcal{U}2$)

and rules

$$\frac{\varphi}{\bigcirc\varphi} \ (\bigcirc\text{-nec}), \qquad\qquad \frac{\varphi}{\square\varphi} \ (\square\text{-nec}).$$

Justification Logic

Finally, for the justification logic part, we use a multi-agent version of the Logic of Proofs [1, 6, 11, 26] with axioms

8.	$[t]_i(\varphi \to \psi) \to ([s]_i\varphi \to [t \cdot s]_i\psi)$	(application)
9.	$[t]_i\varphi \to [t+s]_i\varphi \quad [s]_i\varphi \to [t+s]_i\varphi$	(sum)
10.	$[t]_i\varphi \to \varphi$	(reflexivity)
11.	$[t]_i\varphi \to [!t]_i\,[t]_i\varphi$	(positive introspection)

and rule

$$\frac{[c]_i\varphi \in \mathcal{CS}}{[c]_i\varphi} \ (\text{ax-nec}),$$

where the constant specification $\mathcal{CS}$ is a set of formulas $[c]_i\varphi$, where $c \in \mathsf{Const}$ is a justification constant and φ is an axiom of propositional logic, temporal logic, or justification logic.

For a given constant specification $\mathcal{CS}$, we use $\mathsf{LPLTL}_{\mathcal{CS}}$ to denote the Hilbert system given by the axioms and rules for propositional logic, temporal logic, and justification logic as presented above. As usual, we write $\mathsf{LPLTL}_{\mathcal{CS}} \vdash \varphi$ or simlpy $\vdash_{\mathcal{CS}} \varphi$ if a formula φ is derivable in $\mathsf{LPLTL}_{\mathcal{CS}}$. Often the constant specification is clear from the context and we will only write $\vdash \varphi$ instead of $\vdash_{\mathcal{CS}} \varphi$.

The axiomatization for linear time temporal logic given in [10, 13, 14] includes an axiom

$$\square\varphi \to (\varphi \wedge \bigcirc\square\varphi).$$

The following lemma shows that we do not need this axiom since in our formalization $\square$ is a defined operator.

▶ **Lemma 1.** *We have*

$$\vdash_{\mathcal{CS}} \square\varphi \to (\varphi \wedge \bigcirc\square\varphi)$$

and (MP) *is the only rule that is used in this derivation.*

Proof. $\square\varphi$ stands for $\neg(\top \mathcal{U} \neg\varphi)$. Hence from ($\mathcal{U}2$) we get

$$\vdash_{\mathcal{CS}} \neg\varphi \vee \bigcirc(\top \mathcal{U} \neg\varphi) \to \top \mathcal{U} \neg\varphi.$$

Taking the contrapositive yields

$$\vdash_{\mathcal{CS}} \neg(\top \mathcal{U} \neg\varphi) \to \neg(\neg\varphi \vee \bigcirc(\top \mathcal{U} \neg\varphi)).$$

By propositional reasoning and (fun) we get

$$\vdash_{\mathcal{CS}} \neg(\top \mathcal{U} \neg\varphi) \rightarrow (\varphi \wedge \bigcirc\neg(\top \mathcal{U} \neg\varphi)),$$

which is

$$\vdash_{\mathcal{CS}} \Box\varphi \rightarrow (\varphi \wedge \bigcirc\Box\varphi). \qquad\qquad\qquad\qquad\qquad\qquad\qquad \blacktriangleleft$$

▶ Remark. As usual, we find that the following rule is derivable, see [5, Lemma 6] for a detailed derivation,

$$\frac{\chi \rightarrow \neg\psi \wedge \bigcirc\chi}{\chi \rightarrow \neg(\varphi \mathcal{U} \psi)} .$$

From this, we get that the following rule is also derivable

$$\frac{\chi \rightarrow \neg\psi \wedge \bigcirc(\chi \vee (\neg\varphi \wedge \neg\psi))}{\chi \rightarrow \neg(\varphi \mathcal{U} \psi)} (\mathcal{U}\text{-R}) .$$

A proof is given in [15, Lemma 4.5].

4   Semantics

In this section we introduce interpreted systems based on Fitting-models as semantics for temporal justification logic.

▶ **Definition 2.** A *frame* is a tuple $(S, R_1, \ldots, R_h)$ where
1. S is a non-empty set of states;
2. each $R_i \subseteq S \times S$ is a reflexive and transitive relation.
A *run* r on a frame is a function from $\mathbb{N}$ to states, i.e., $r : \mathbb{N} \rightarrow S$. A *system* $\mathcal{R}$ is a non-empty set of runs.

▶ **Definition 3.** Given a frame $(S, R_1, \ldots, R_h)$, a *$\mathcal{CS}$-evidence function for agent i* is a function

$$\mathcal{E}_i : S \times \mathsf{Tm} \rightarrow \mathcal{P}(\mathsf{Fml})$$

satisfying the following conditions. For all terms $s, t \in \mathsf{Tm}$ and all $v, w \in S$,
1. $\mathcal{E}_i(v, t) \subseteq \mathcal{E}_i(w, t)$, whenever $R_i(v, w)$ (monotonicity),
2. if $[c]_i\varphi \in \mathcal{CS}$, then $\varphi \in \mathcal{E}_i(w, c)$ (constant specification)
3. if $\varphi \rightarrow \psi \in \mathcal{E}_i(w, t)$ and $\varphi \in \mathcal{E}_i(w, s)$,
        then $\psi \in \mathcal{E}_i(w, t \cdot s)$ (application)
4. $\mathcal{E}_i(w, s) \cup \mathcal{E}_i(w, t) \subseteq \mathcal{E}_i(w, s + t)$ (sum)
5. if $\varphi \in \mathcal{E}_i(w, t)$, then $[t]_i\varphi \in \mathcal{E}_i(w, !t)$ (positive introspection)

▶ **Definition 4.** An *interpreted system for $\mathcal{CS}$* is a tuple

$$\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1 \ldots, \mathcal{E}_h, \nu)$$

where
1. $(S, R_1, \ldots, R_h)$ is a frame;
2. $\mathcal{R}$ is a system on that frame;
3. $\mathcal{E}_i$ is a $\mathcal{CS}$-evidence function for agent i for $1 \leq i \leq h$;
4. $\nu : S \rightarrow \mathcal{P}(\mathsf{Prop})$ is a valuation.

▶ **Definition 5.** Given an interpreted system

$$\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu),$$

a run $r \in \mathcal{R}$, and $n \in \mathbb{N}$, we define validity of a formula φ in $\mathcal{I}$ at state $r(n)$ inductively by

$(\mathcal{I}, r, n) \vDash P$ iff $P \in \nu(r(n))$,

$(\mathcal{I}, r, n) \nvDash \bot$,

$(\mathcal{I}, r, n) \vDash \varphi \to \psi$ iff $(\mathcal{I}, r, n) \nvDash \varphi$ or $(\mathcal{I}, r, n) \vDash \psi$,

$(\mathcal{I}, r, n) \vDash \bigcirc \varphi$ iff $(\mathcal{I}, r, n+1) \vDash \varphi$,

$(\mathcal{I}, r, n) \vDash \varphi \,\mathcal{U}\, \psi$ iff there is some $m \geq 0$ such that $(\mathcal{I}, r, n+m) \vDash \psi$

and $(\mathcal{I}, r, n+k) \vDash \varphi$ for all $0 \leq k < m$,

$(\mathcal{I}, r, n) \vDash [t]_i \varphi$ iff $\varphi \in \mathcal{E}_i(r(n), t)$ and $(\mathcal{I}, r', n') \vDash \varphi$

for all $r' \in \mathcal{R}$ and $n' \in \mathbb{N}$ such that $R_i(r(n), r'(n'))$.

As usual, we write $\mathcal{I} \vDash \varphi$ if for all $r \in \mathcal{R}$ and all $n \in \mathbb{N}$, we have $(\mathcal{I}, r, n) \vDash \varphi$. Further, we write $\vDash_{\mathcal{CS}} \varphi$ if $\mathcal{I} \vDash \varphi$ for all interpreted systems $\mathcal{I}$ for $\mathcal{CS}$.

▶ **Remark.** From the definitions of $\Box$ and $\Diamond$ it follows that:

$(\mathcal{I}, r, n) \vDash \Diamond \varphi$ iff $(\mathcal{I}, r, n+k) \vDash \varphi$ for some $k \geq 0$,

$(\mathcal{I}, r, n) \vDash \Box \varphi$ iff $(\mathcal{I}, r, n+k) \vDash \varphi$ for all $k \geq 0$.

5 Soundness and Completeness

The soundness proof for $\mathsf{LPLTL}_{\mathcal{CS}}$ is a straightforward combination of the soundness proofs for temporal logic and justification logic.

▶ **Theorem 6.** *Let $\mathcal{CS}$ be an arbitrary constant specification. For each formula φ,*

$$\vdash_{\mathcal{CS}} \varphi \quad implies \quad \vDash_{\mathcal{CS}} \varphi.$$

Our completeness proof for $\mathsf{LPLTL}_{\mathcal{CS}}$ follows the one given in [15]. First, we define $\Gamma \vdash_{\mathcal{CS}} \varphi$ as

there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathcal{CS}} (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$.

Following our convention, we will often write $\Gamma \vdash \varphi$ instead of $\Gamma \vdash_{\mathcal{CS}} \varphi$.

▶ **Definition 7.** Let $\mathcal{CS}$ be a constant specification. A set Γ of formulas is called $\mathcal{CS}$-*consistent* if $\Gamma \nvdash_{\mathcal{CS}} \bot$. That means $\nvdash_{\mathcal{CS}} \bigwedge \Sigma \to \bot$, for each finite $\Sigma \subseteq \Gamma$.

For a formula χ, let $\mathsf{Sub}^+(\chi) := \mathsf{Sub}(\chi) \cup \{\neg \psi \mid \psi \in \mathsf{Sub}(\chi)\}$. Let MCS_χ denote the set of all maximally $\mathcal{CS}$-consistent subsets of $\mathsf{Sub}^+(\chi)$. We have the following facts for $\Gamma \in \mathsf{MCS}_\chi$:
- If $\Gamma \vdash_{\mathcal{CS}} \varphi$, then $\vdash_{\mathcal{CS}} \bigwedge \Gamma \to \varphi$.
- If $\varphi \in \mathsf{Sub}(\chi)$ and $\varphi \notin \Gamma$, then $\neg \varphi \in \Gamma$.
- If $\varphi \in \mathsf{Sub}^+(\chi)$ and $\Gamma \vdash_{\mathcal{CS}} \varphi$, then $\varphi \in \Gamma$.
- If $\psi \in \mathsf{Sub}^+(\chi)$, $\varphi \in \Gamma$ and $\vdash_{\mathcal{CS}} \varphi \to \psi$, then $\psi \in \Gamma$.

We define the relation $R_\bigcirc$ on MCS_χ as follows:

$$\Gamma R_\bigcirc \Delta \quad \text{iff} \quad \nvdash_{\mathcal{CS}} \bigwedge \Gamma \to \neg \bigcirc \bigwedge \Delta.$$

From this definition we immediately get the following lemmas.

▶ **Lemma 8.** *Let* $\Gamma, \Delta \in \mathsf{MCS}_\chi$, $\Gamma R_\bigcirc \Delta$, *and* $\varphi \in \mathsf{Sub}(\chi)$.

1. *If* $\Gamma \vdash_{\mathcal{CS}} \bigcirc \varphi$, *then* $\varphi \in \Delta$.
2. *If* $\Gamma \vdash_{\mathcal{CS}} \neg \bigcirc \varphi$, *then* $\neg \varphi \in \Delta$.

Proof. 1. Suppose toward a contradiction that $\varphi \notin \Delta$. Thus $\neg \varphi \in \Delta$. Since $\Gamma \vdash_{\mathcal{CS}} \bigcirc \varphi$, we have $\vdash_{\mathcal{CS}} \bigwedge \Gamma \to \bigcirc \varphi$. Hence $\vdash_{\mathcal{CS}} \bigwedge \Gamma \to \bigcirc \neg \neg \varphi$. Therefore $\vdash_{\mathcal{CS}} \bigwedge \Gamma \to \bigcirc \neg \bigwedge \Delta$. Thus $\vdash_{\mathcal{CS}} \bigwedge \Gamma \to \neg \bigcirc \bigwedge \Delta$, which would contradict $\Gamma R_\bigcirc \Delta$.
2. The proof is similar to part 1.   ◀

▶ **Lemma 9.** *Let* $\Gamma \in \mathsf{MCS}_\chi$ *and let* $S := \{\Delta \in \mathsf{MCS}_\chi \mid \Gamma R_\bigcirc \Delta\}$. *We have*

$$\vdash \bigwedge \Gamma \to \bigcirc \bigvee \{\bigwedge \Delta \mid \Delta \in S\}.$$

Proof. First observe that for all $\Gamma, \Delta \in \mathsf{MCS}_\chi$ we have

$$(\text{not } \Gamma R_\bigcirc \Delta) \quad \text{implies} \quad \vdash \bigwedge \Gamma \to \neg \bigcirc \bigwedge \Delta. \tag{1}$$

We also have

$$\vdash \bigvee \{\bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi\}.$$

By necessitation we get

$$\vdash \bigcirc \bigvee \{\bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi\}$$

and thus

$$\vdash \bigvee \{\bigcirc \bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi\}. \tag{2}$$

By (1) we infer

$$\vdash \bigwedge \Gamma \to \bigvee \{\bigcirc \bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi \text{ with } \Gamma R_\bigcirc \Delta\}$$

and thus

$$\vdash \bigwedge \Gamma \to \bigcirc \bigvee \{\bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi \text{ with } \Gamma R_\bigcirc \Delta\}. \quad\quad ◀$$

▶ **Lemma 10.** *The relation* $R_\bigcirc$ *is serial. That is for each* $\Gamma \in \mathsf{MCS}_\chi$, *there exists* $\Delta \in \mathsf{MCS}_\chi$ *with* $\Gamma R_\bigcirc \Delta$.

Proof. Suppose towards a contradiction that for $\Gamma \in \mathsf{MCS}_\chi$ we have (not $\Gamma R_\bigcirc \Delta$) for all $\Delta \in \mathsf{MCS}_\chi$. Then $\vdash \bigwedge \Gamma \to \neg \bigcirc \bigwedge \Delta$, for all $\Delta \in \mathsf{MCS}_\chi$. Thus

$$\vdash \bigwedge \Gamma \to \bigwedge \{\neg \bigcirc \bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi\},$$

and hence,

$$\vdash \bigwedge \Gamma \to \neg \bigvee \{\bigcirc \bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi\}. \tag{3}$$

On the other hand, from (2) we deduce

$$\vdash \bigwedge \Gamma \to \bigvee \{\bigcirc \bigwedge \Delta \mid \Delta \in \mathsf{MCS}_\chi\}. \tag{4}$$

Since Γ is consistent, (3) and (4) leads to a contradiction.   ◀

▶ **Definition 11.** A finite sequence $(\Gamma_0, \Gamma_1, \ldots, \Gamma_n)$ of elements of MCS_χ is called a $\varphi\,\mathcal{U}\,\psi$-*sequence starting with* Γ if
1. $\Gamma_0 = \Gamma$,
2. $\Gamma_j R_\bigcirc \Gamma_{j+1}$, for all $j < n$,
3. $\psi \in \Gamma_n$,
4. $\varphi \in \Gamma_j$, for all $j < n$.

▶ **Lemma 12.** *For every* $\Gamma \in \mathsf{MCS}_\chi$*, if* $\varphi\,\mathcal{U}\,\psi \in \Gamma$*, then there exists a* $\varphi\,\mathcal{U}\,\psi$*-sequence starting with* Γ*.*

Proof. Suppose $\varphi\,\mathcal{U}\,\psi \in \Gamma$ and there exists no $\varphi\,\mathcal{U}\,\psi$-sequence starting with Γ. We let T be the smallest set of elements of MCS_χ such that
1. $\Gamma \in T$;
2. for each $\Delta' \in \mathsf{MCS}_\chi$, if $\Delta \in T$, $\Delta R_\bigcirc \Delta'$, and $\varphi \in \Delta'$, then $\Delta' \in T$.
We find that $\vdash \bigwedge \Delta \to \neg\psi$ for all $\Delta \in T$. Let

$$\rho := \bigvee \{ \bigwedge \Delta \mid \Delta \in T \}.$$

We have $\vdash \rho \to \neg\psi$.

Moreover, for each $\Delta \in T$ and each $\Delta' \in \mathsf{MCS}_\chi$ with $\Delta R_\bigcirc \Delta'$, we have either $\Delta' \in T$ or $\vdash \bigwedge \Delta' \to \neg\varphi \wedge \neg\psi$. Thus by Lemma 9 we get

$$\vdash \rho \to \bigcirc(\rho \vee (\neg\varphi \wedge \neg\psi)).$$

Using ($\mathcal{U}$-R), we obtain $\vdash \rho \to \neg(\varphi\,\mathcal{U}\,\psi)$. Since $\Gamma \in T$, this implies

$$\vdash \bigwedge \Gamma \to \neg(\varphi\,\mathcal{U}\,\psi),$$

which contradicts the assumption $\varphi\,\mathcal{U}\,\psi \in \Gamma$. ◀

▶ **Definition 13.** An infinite sequence $(\Gamma_0, \Gamma_1, \ldots)$ of elements of MCS_χ is called *acceptable* if
1. $\Gamma_n R_\bigcirc \Gamma_{n+1}$ for all $n \geq 0$, and
2. for all n, if $\varphi\,\mathcal{U}\,\psi \in \Gamma_n$, then there exists $m \geq n$ such that $\psi \in \Gamma_m$ and $\varphi \in \Gamma_k$ for all k with $n \leq k < m$.

▶ **Lemma 14.** *Every finite sequence* $(\Gamma_0, \Gamma_1, \ldots, \Gamma_n)$ *of elements of* MCS_χ *with* $\Gamma_j R_\bigcirc \Gamma_{j+1}$*, for all* $j < n$*, can be extended to an acceptable sequence.*

Proof. In order to fulfill the requirements of Definition 13, we shall extend the sequence $(\Gamma_0, \Gamma_1, \ldots, \Gamma_n)$ by the following algorithm.

Suppose $\varphi\,\mathcal{U}\,\psi \in \Gamma_0$. Then either $\psi \in \Gamma_0$ or $\neg\psi \in \Gamma_0$. In the former case the requirement is fulfilled for the formula $\varphi\,\mathcal{U}\,\psi$ in Γ_0, and we go to the next step. In the latter case, using axiom ($\mathcal{U}$2), $\Gamma_0 \vdash_{\mathcal{CS}} \varphi \wedge \bigcirc(\varphi\,\mathcal{U}\,\psi)$. Since $\Gamma_0 R_\bigcirc \Gamma_1$, by Lemma 8, we get $\varphi\,\mathcal{U}\,\psi \in \Gamma_1$.

We can repeat this argument for Γ_i for $1 \leq i \leq n$. We find that the requirement for $\varphi\,\mathcal{U}\,\psi \in \Gamma_0$ is either fulfilled in $(\Gamma_0, \Gamma_1, \ldots, \Gamma_n)$ or $\varphi\,\mathcal{U}\,\psi \in \Gamma_n$ and $\varphi \in \Gamma_i$ for $1 \leq i \leq n$. In the latter case, by Lemma 12, there exists a sequence $(\Gamma_n, \Gamma_{n+1}, \ldots, \Gamma_{n+m})$ such that $\varphi \in \Gamma_i$ for $n \leq i < n + m$, $\psi \in \Gamma_{n+m}$, and $\Gamma_i R_\bigcirc \Gamma_{i+1}$ for $n \leq i < n + m$. This gives a finite extension of the original sequence that satisfies the requirement imposed by $\varphi\,\mathcal{U}\,\psi \in \Gamma_0$.

In the next step we repeat this argument for the remaining obligations at Γ_0. Eventually we obtain a finite sequence that satisfies all requirements imposed by formulas at Γ_0.

We may move on to Γ_1 and apply the same procedure. It is clear that by iterating it we obtain in the limit an acceptable sequence that extends $(\Gamma_0, \Gamma_1, \ldots, \Gamma_n)$. ◀

▶ **Corollary 15.** *For every* $\Gamma \in \mathsf{MCS}_\chi$, *there is an acceptable sequence that starts with* Γ.

▶ **Definition 16.** The χ-canonical interpreted system

$$\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1 \ldots, \mathcal{E}_h, \nu)$$

for $\mathcal{CS}$ is defined as follows:
1. $\mathcal{R}$ consists of all mappings $r : \mathbb{N} \to \mathsf{MCS}_\chi$ such that $(r(0), r(1), \ldots)$ is an acceptable sequence;
2. $S := \mathsf{MCS}_\chi = \{r(n) \mid r \in \mathcal{R}, n \in \mathbb{N}\}$;
3. $R_i(\Gamma, \Delta)$ iff $\{\varphi \mid \Gamma \vdash [t]_i\varphi \text{ for some } t\} \subseteq \{\varphi \mid \Delta \vdash \varphi\}$;
4. $\mathcal{E}_i(\Gamma, t) := \{\varphi \mid \Gamma \vdash [t]_i\varphi\}$;
5. $\nu(\Gamma) := \{P \in \mathsf{Prop} \mid P \in \Gamma\}$.

▶ Remark. The χ-canonical interpreted system $\mathcal{I}$ for $\mathcal{CS}$ is a finite structure in the sense that the set of states S is finite. This is a novelty for completeness proofs of justification logics. Even the completeness proofs for justification logics with common knowledge [2, 6] work with infinite canonical structures. Note that this remark concerns epistemic Fitting-models. Of course, symbolic M-models [19] could be considered as single-world Fitting-models.

The fact that states of $\mathcal{I}$ are maximally $\mathcal{CS}$-consistent subsets of $\mathsf{Sub}^+(\chi)$—instead of just maximally $\mathcal{CS}$-consistent sets—matters for the definitions of R_i and $\mathcal{E}_i$. The usual definitions would be

$$R_i(\Gamma, \Delta) \text{ iff } \{\varphi \mid [t]_i\varphi \in \Gamma \text{ for some } t\} \subseteq \{\varphi \mid \varphi \in \Delta\} \quad \text{and}$$
$$\mathcal{E}_i(\Gamma, t) := \{\varphi \mid [t]_i\varphi \in \Gamma\}.$$

This, however, would not work for our finite canonical structure. In particular the next lemma could not be established as, for instance, $[t]_i\varphi \in \Gamma$ does not imply $[!t]_i\,[t]_i\varphi \in \Gamma$ for $\Gamma \in \mathsf{MCS}_\chi$.

▶ **Lemma 17.** *The χ-canonical interpreted system*

$$\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1 \ldots, \mathcal{E}_h, \nu)$$

for $\mathcal{CS}$ is an interpreted system for $\mathcal{CS}$.

Proof. The proof is essentially the same as the corresponding proof for single agent Fitting-models in [9]. Let us only show here the monotonicity condition for $\mathcal{E}_i$.

Suppose $\Gamma, \Delta \in S$ and $R_i(\Gamma, \Delta)$. Suppose that $\varphi \in \mathcal{E}_i(\Gamma, t)$. Thus $\Gamma \vdash [t]_i\varphi$. Hence $\Gamma \vdash [!t]_i\,[t]_i\varphi$. Since $R_i(\Gamma, \Delta)$, we have $\Delta \vdash [t]_i\varphi$. Hence $\varphi \in \mathcal{E}_i(\Delta, t)$ as desired. ◄

▶ **Lemma 18** (Truth Lemma). *Let $\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1 \ldots, \mathcal{E}_h, \nu)$ be the χ-canonical interpreted system for $\mathcal{CS}$. For every formula $\psi \in \mathsf{Sub}^+(\chi)$, every run r in $\mathcal{R}$, and every $n \in \mathbb{N}$ we have:*

$$(\mathcal{I}, r, n) \models \psi \quad \textit{iff} \quad \psi \in r(n).$$

Proof. As usual, the proof is by induction on the structure of ψ. We show only the following cases:
- $\psi = [t]_i\varphi$. ($\Rightarrow$) If $(\mathcal{I}, r, n) \models [t]_i\varphi$, then $\varphi \in \mathcal{E}_i(r(n), t)$. Thus, by definition, $r(n) \vdash [t]_i\varphi$. Hence $[t]_i\varphi \in r(n)$, since $[t]_i\varphi \in \mathsf{Sub}^+(\chi)$.

  ($\Leftarrow$) If $[t]_i\varphi \in r(n)$, then $r(n) \vdash [t]_i\varphi$. Hence, by definition, $\varphi \in \mathcal{E}_i(r(n), t)$. Now suppose that $R_i(r(n), r'(n'))$. We find $r'(n') \vdash \varphi$. Since $\varphi \in \mathsf{Sub}^+(\chi)$, we have $\varphi \in r'(n')$ and by I.H. we get $(\mathcal{I}, r', n') \models \varphi$. Since r' and n' were arbitrary, we conclude $(\mathcal{I}, r, n) \models [t]_i\varphi$.

- $\psi = \bigcirc \varphi$. ($\Rightarrow$) Suppose that $(\mathcal{I}, r, n) \models \bigcirc \varphi$ and $\bigcirc \varphi \notin r(n)$. Then $(\mathcal{I}, r, n+1) \models \varphi$, and hence by the induction hypothesis $\varphi \in r(n+1)$. On the other hand, $\neg \bigcirc \varphi \in r(n)$. Since $r(n) R_\bigcirc r(n+1)$, by Lemma 8, we get $\neg \varphi \in r(n+1)$, which is a contradiction.
  ($\Leftarrow$) If $\bigcirc \varphi \in r(n)$, then $\varphi \in r(n+1)$. By the induction hypothesis, $(\mathcal{I}, r, n+1) \models \varphi$, and hence $(\mathcal{I}, r, n) \models \bigcirc \varphi$.
- $\psi = \psi_1 \mathcal{U} \psi_2$. ($\Rightarrow$) If $(\mathcal{I}, r, n) \models \psi_1 \mathcal{U} \psi_2$, then $(\mathcal{I}, r, m) \models \psi_2$ for some $m \geq n$, and $(\mathcal{I}, r, k) \models \psi_1$ for all k with $n \leq k < m$. By I.H. we get $\psi_2 \in r(m)$, and $\psi_1 \in r(k)$ for all k with $n \leq k < m$. We have to show $\psi_1 \mathcal{U} \psi_2 \in r(n)$, which follows by induction on m as follows:
  - Base case $m = n$. Since $\psi_2 \in r(n)$ and $\vdash \psi_2 \to (\psi_1 \mathcal{U} \psi_2)$, we obtain $\psi_1 \mathcal{U} \psi_2 \in r(n)$.
  - Suppose $m > n$. It follows from the induction hypothesis that $\psi_1 \mathcal{U} \psi_2 \in r(n+1)$. From this and $r(n) R_\bigcirc r(n+1)$ we get that

$$r(n) \cup \{\bigcirc(\psi_1 \mathcal{U} \psi_2)\} \text{ is consistent.} \tag{5}$$

    Assume now

$$\neg(\psi_1 \mathcal{U} \psi_2) \in r(n). \tag{6}$$

    Then $r(n) \vdash \neg(\psi_1 \mathcal{U} \psi_2)$ and by axiom ($\mathcal{U} 2$) we find $r(n) \vdash \neg(\psi_1 \wedge \bigcirc(\psi_1 \mathcal{U} \psi_2))$. From $\psi_1 \in r(n)$ we get $r(n) \vdash \psi_1$ and thus $r(n) \vdash \neg \bigcirc (\psi_1 \mathcal{U} \psi_2)$, which contradicts (5). Hence the assumption (6) must be false and we conclude $\psi_1 \mathcal{U} \psi_2 \in r(n)$.
  ($\Leftarrow$) If $\psi_1 \mathcal{U} \psi_2 \in r(n)$, then since $(r(n), r(n+1), \ldots)$ is an acceptable sequence there exists $m \geq n$ such that $\psi_2 \in r(m)$, and $\psi_1 \in r(k)$ for all k with $n \leq k < m$. By I.H. we obtain $(\mathcal{I}, r, m) \models \psi_2$, and $(\mathcal{I}, r, k) \models \psi_1$ for all k with $n \leq k < m$. Thus $(\mathcal{I}, r, n) \models \psi_1 \mathcal{U} \psi_2$. ◄

▶ **Theorem 19** (Completeness). *For each formula φ,*

$$\models_{\mathcal{CS}} \varphi \quad \text{implies} \quad \vdash_{\mathcal{CS}} \varphi.$$

Proof. Suppose that $\nvdash_{\mathcal{CS}} \varphi$. Thus, $\{\neg\varphi\}$ is a $\mathcal{CS}$-consistent set. Therefore, there exists $\Gamma \in \mathsf{MCS}_\varphi$ with $\neg\varphi \in \Gamma$. By Corollary 15, there is an acceptable sequence starting with Γ. Thus there is a run r in the φ-canonical interpreted system $\mathcal{I}$ for $\mathcal{CS}$ with $r(0) = \Gamma$. Since $\neg\varphi \in \Gamma$, by the Truth Lemma, $(\mathcal{I}, r, 0) \nvDash \varphi$. Therefore, $\nvDash_{\mathcal{CS}} \varphi$. ◄

6   Internalization

It is desirable that a justification logic internalizes its own notion of proof. This is formalized in the following definition.

▶ **Definition 20.** A justification logic L satisfies *internalization* if for each formula φ with $\mathsf{L} \vdash \varphi$ and for each agent i, there exists a term t with $\mathsf{L} \vdash [t]_i \varphi$.

Usually, internalization is shown by induction on the derivation of φ. However, for $\mathsf{LPLTL}_{\mathcal{CS}}$ this seems not possible because it includes rules ($\bigcirc$-nec) and ($\square$-nec). In this section, we introduce an extension $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ of $\mathsf{LPLTL}_{\mathcal{CS}}$ that satisfies internalization.

The language of $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ includes a new unary operator $\star$ on justification terms. We define

$$\star^0 c := c \quad \text{and} \quad \star^n c := \star \star^{n-1} c \quad (\text{for } n \geq 1).$$

The set of terms $\mathsf{Tm}^\star$ of $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ is given by

$$t ::= \star^n c \mid x \mid !t \mid t+t \mid t \cdot t,$$

where $c \in \mathsf{Const}$, $n \geq 0$, and $x \in \mathsf{Var}$. The set of formulas $\mathsf{Fml}^\star$ of $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ is defined like Fml but using $\mathsf{Tm}^\star$ instead of Tm.

The axioms of $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ are:

1. all axioms of LPLTL
2. $\Box\varphi \to \bigcirc\varphi$ (mix)
3. $\Box([t]_i\varphi \to \varphi)$ (boxed reflexivity)

The rules of $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ are:

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \; (\mathsf{MP}) \qquad \text{and} \qquad \frac{[c]_{i_0}\varphi \in \mathcal{CS}}{[\star^n c]_{i_n}\Box\,[\star^{n-1}c]_{i_{n-1}}\Box\ldots[\star c]_{i_1}\Box\,[c]_{i_0}\varphi} \; (\mathsf{ax\text{-}nec})^\star,$$

where $n \geq 0$; so $(\mathsf{ax\text{-}nec})^\star$ subsumes $(\mathsf{ax\text{-}nec})$. Note that a constant specification for $\mathsf{LPLTL}^\star$ may include formulas of the form $[c]_i(\Box\varphi \to \bigcirc\varphi)$ and $[c]_i\Box([t]_i\varphi \to \varphi)$.

▶ **Remark.** The principles (mix) and (boxed reflexivity) are derivable in $\mathsf{LPLTL}_{\mathcal{CS}}$. However, their proofs require applications of $(\bigcirc\text{-nec})$ and $(\Box\text{-nec})$, respectively. Since these rules are not included in $\mathsf{LPLTL}^\star_{\mathcal{CS}}$, we have to include (mix) and (boxed reflexivity) as axioms.

▶ **Remark.** The $\star$-operation is very powerful. Its meaning can be explained as follows. If $[c]_i\varphi$ is contained in $\mathcal{CS}$, then $[c]_i\varphi$ is provable and hence $\Box\,[c]_i\varphi$ is provable, too (see Lemma 22). The evidence $\star c$ justifies this fact, i.e., $[\star c]_i\Box\,[c]_i\varphi$ is provable. Looking closely at $(\mathsf{ax\text{-}nec})^\star$ we see that we get even more. Indeed, for any agent j we have that $[\star c]_j\Box\,[c]_i\varphi$ is provable.

We could use a less general version of $(\mathsf{ax\text{-}nec})^\star$ where the $\star$-operation is indexed. This would be similar to the evidence verification operation of [26], see also Question 2. In that case we would obtain $\left[\star^j_i c\right]_j\Box\,[c]_i\varphi$. However, for the purpose of internalization we do not need this indexes and hence we dispense with them.

▶ **Definition 21.** A constant specification $\mathcal{CS}$ is *axiomatically appropriate* if for each axiom φ of $\mathsf{LPLTL}^\star$ and each agent i, there is a constant c with $[c]_i\varphi \in \mathcal{CS}$.

First we show that $\mathsf{LPLTL}^\star_{\mathcal{CS}}$ extends $\mathsf{LPLTL}_{\mathcal{CS}}$.

▶ **Lemma 22.** *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for $\mathsf{LPLTL}^\star$. The rules $(\Box\text{-nec})$ and $(\bigcirc\text{-nec})$ are derivable in $\mathsf{LPLTL}^\star_{\mathcal{CS}}$.*

Proof. We first show that $(\Box\text{-nec})$ is derivable in $\mathsf{LPLTL}^\star_{\mathcal{CS}}$. Suppose φ is provable in $\mathsf{LPLTL}^\star_{\mathcal{CS}}$. By induction on the proof of φ, we show that $\Box\varphi$ is provable in $\mathsf{LPLTL}^\star_{\mathcal{CS}}$.

In case φ is an axiom, since $\mathcal{CS}$ is axiomatically appropriate, there is a constant c such that $[c]_i\varphi \in \mathcal{CS}$. Using $(\mathsf{ax\text{-}nec})^\star$, we get $[\star c]_i\Box\,[c]_i\varphi$, and then using axiom (reflexivity) we get $\Box\,[c]_i\varphi$. Finally, using axioms (boxed reflexivity) and $(\Box\text{-k})$ we obtain $\Box\varphi$.

In case φ is derived by modus ponens, the claim is immediate by $(\Box\text{-k})$.

In case φ is $[\star^n c]_{i_n}\Box\,[\star^{n-1}c]_{i_{n-1}}\Box\ldots[\star c]_{i_1}\Box\,[c]_{i_0}\varphi$ derived using $(\mathsf{ax\text{-}nec})^\star$, we can use $(\mathsf{ax\text{-}nec})^\star$ also to obtain $\left[\star^{n+1}c\right]_{i_{n+1}}\Box\left[\star^n c\right]_{i_n}\Box\left[\star^{n-1}c\right]_{i_{n-1}}\Box\ldots[\star c]_{i_1}\Box\,[c]_{i_0}\varphi$. Then using (reflexivity) we get $\Box\left[\star^n c\right]_{i_n}\Box\left[\star^{n-1}c\right]_{i_{n-1}}\Box\ldots[\star c]_{i_1}\Box\,[c]_{i_0}\varphi$, that is $\Box\varphi$.

Derivability of $(\bigcirc\text{-nec})$ follows from $(\Box\text{-nec})$ and axiom (mix). ◀

Let $\mathcal{CS}$ be a constant specification for $\mathsf{LPLTL}^\star$. We set

$$\mathcal{CS}^r := \{[c]_i\varphi \mid [c]_i\varphi \in \mathcal{CS} \text{ and } \varphi \text{ is an axiom of } \mathsf{LPLTL}\}.$$

Obviously, $\mathcal{CS}^r$ is a constant specification for LPLTL. We get the following corollary.

▶ **Corollary 23.** *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for LPLTL*. For each formula φ of Fml,*

$$\text{LPLTL}_{\mathcal{CS}^r} \vdash \varphi \quad \text{implies} \quad \text{LPLTL}^\star_{\mathcal{CS}} \vdash \varphi.$$

We will now establish the internalization property. We need the following lemma.

▶ **Lemma 24.** *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification. For each formula φ and each i,*

$$\text{LPLTL}^\star_\emptyset \vdash \varphi \quad \text{implies} \quad \text{LPLTL}^\star_{\mathcal{CS}} \vdash [t]_i\varphi \text{ for some term } t.$$

Proof. We proceed by induction on the derivation of φ.

In case φ is an axiom, since $\mathcal{CS}$ is axiomatically appropriate, there is a constant c with

$$\text{LPLTL}^\star_{\mathcal{CS}} \vdash [c]_i\varphi.$$

In case φ is derived by modus ponens from $\psi \to \varphi$ and ψ, then, by induction hypothesis, there are term s_1 and s_2 such that $[s_1]_i(\psi \to \varphi)$ and $[s_2]_i\psi$ are provable. Using (application) and modus ponens, we obtain $[s_1 \cdot s_2]_i\varphi$. ◀

▶ **Theorem 25.** *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification. LPLTL$^\star_{\mathcal{CS}}$ enjoys internalization.*

Proof. We have to show that for each formula φ and each i

$$\text{LPLTL}^\star_{\mathcal{CS}} \vdash \varphi \quad \text{implies} \quad \text{LPLTL}^\star_{\mathcal{CS}} \vdash [t]_i\varphi \text{ for some term } t.$$

We proceed by induction on the derivation of φ.

The cases where φ is an axiom or φ is derived by modus ponens are like the corresponding cases in the previous lemma.

In case φ is $[\star^n c]_{i_n}\square \dots [\star c]_{i_1}\square\,[c]_{i_0}\psi$ derived using (ax-nec)*, we can use (ax-nec)* also to obtain $\left[\star^{n+1}c\right]_i\square\varphi$. By Lemma 1 we find $\text{LPLTL}^\star_\emptyset \vdash \square\varphi \to \varphi$. Hence by Lemma 24 there is a term t such that $\text{LPLTL}^\star_{\mathcal{CS}} \vdash [t]_i(\square\varphi \to \varphi)$. We finally conclude

$$\text{LPLTL}^\star_{\mathcal{CS}} \vdash \left[t \cdot \star^{n+1}c\right]_i\varphi\,. \qquad\qquad ◀$$

It is straightforward to adapt our semantics for LPLTL$_{\mathcal{CS}}$ to the extended language of LPLTL$^\star_{\mathcal{CS}}$. Soundness and completeness of LPLTL$^\star_{\mathcal{CS}}$ can then be shown similar to the case of LPLTL$_{\mathcal{CS}}$. However, for the completeness proof of LPLTL$^\star_{\mathcal{CS}}$ we require $\mathcal{CS}$ to be axiomatically appropriate in order to have the necessitation rules available.

▶ **Definition 26.** Let $\mathcal{CS}$ be a constant specification for LPLTL*. A $\mathcal{CS}$-evidence function for agent i for LPLTL* is a function $\mathcal{E}_i : S \times \text{Tm}^\star \to \mathcal{P}(\text{Fml}^\star)$ satisfying conditions 1–5 of Definition 3 and the following additional condition:

▬ if $[c]_{i_0}\varphi \in \mathcal{CS}$, then for all $w \in S$, all $n \geq 1$, and all agents $i_{n-1}, \dots, i_1$:

$$\square\left[\star^{n-1}c\right]_{i_{n-1}}\dots\square[\star c]_{i_1}\square\,[c]_{i_0}\varphi \in \mathcal{E}_i(w, \star^n c).$$

An LPLTL$^\star_{\mathcal{CS}}$-interpreted system is an interpreted system where we use evidence functions for LPLTL*. We write $\models^\star_{\mathcal{CS}} \varphi$ to mean $\mathcal{I} \vDash \varphi$ for all LPLTL$^\star_{\mathcal{CS}}$-interpreted systems $\mathcal{I}$.

▶ **Theorem 27** (Soundness and completeness). *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for $\mathsf{LPLTL^\star}$. For each formula φ,*

$$\models^\star_{\mathcal{CS}} \varphi \quad \textit{iff} \quad \mathsf{LPLTL}^\star_{\mathcal{CS}} \vdash \varphi.$$

We conclude this section by showing the conservativity of $\mathsf{LPLTL}^\star$ over LPLTL. First we need a lemma.

▶ **Lemma 28.** *Let $\mathcal{CS}$ be a constant specification for LPLTL, and $\mathcal{I}$ be an interpreted system of LPLTL for $\mathcal{CS}$. Then we can extend $\mathcal{I}$ to an $\mathsf{LPLTL}^\star_{\mathcal{CS}}$-interpreted system $\mathcal{I}^\star$ such that for every run r, every $n \in \mathbb{N}$, and every formula $\varphi \in \mathsf{Fml}$:*

$$(\mathcal{I}, r, n) \models \varphi \quad \Longleftrightarrow \quad (\mathcal{I}^\star, r, n) \models \varphi.$$

Proof. Let $\mathcal{I} = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1, \ldots, \mathcal{E}_h, \nu)$ be an arbitrary interpreted system of LPLTL for $\mathcal{CS}$. By a least fixed point construction, we can easily extend the $\mathcal{CS}$-evidence functions $\mathcal{E}_i$, for $1 \le i \le h$, to $\mathcal{CS}$-evidence functions $\mathcal{E}_i^\star$ such that
1. $\mathcal{I}^\star = (\mathcal{R}, S, R_1, \ldots, R_h, \mathcal{E}_1^\star, \ldots, \mathcal{E}_h^\star, \nu)$ is an $\mathsf{LPLTL}^\star_{\mathcal{CS}}$-interpreted system and
2. for each formula $\varphi \in \mathsf{Fml}$, each run r and each $n \in \mathbb{N}$:

$$(\mathcal{I}, r, n) \models \varphi \quad \Longleftrightarrow \quad (\mathcal{I}^\star, r, n) \models \varphi.$$ ◀

▶ **Theorem 29** (Conservativity). *Let $\mathcal{CS}$ be a constant specification for LPLTL and $\varphi \in \mathsf{Fml}$. If $\mathsf{LPLTL}^\star_{\mathcal{CS}} \vdash \varphi$, then $\vdash_{\mathcal{CS}} \varphi$.*

Proof. Suppose that $\nvdash_{\mathcal{CS}} \varphi$. Then, by Theorem 19, we have $\nvDash_{\mathcal{CS}} \varphi$. Thus there exists an interpreted system $\mathcal{I}$ of LPLTL for $\mathcal{CS}$ and a state $r(n)$ such that $(\mathcal{I}, r, n) \nvDash \varphi$. Now, by Lemma 28, we find an $\mathsf{LPLTL}^\star_{\mathcal{CS}}$-interpreted system $\mathcal{I}^\star$ such that $(\mathcal{I}^\star, r, n) \nvDash \varphi$. Therefore, by Theorem 27, we have $\mathsf{LPLTL}^\star_{\mathcal{CS}} \nvdash \varphi$ as desired.   ◀

7   Additional Principles

In $\mathsf{LPLTL}_{\mathcal{CS}}$, epistemic and temporal properties do not interact. It is therefore a natural question to consider some of the following principles, which create a connection between time and knowledge. We assume the language for terms to be augmented in the obvious way.

$$[t]_i \Box \varphi \to \Box [\Downarrow t]_i \varphi \qquad\qquad (\Box\text{-access})$$
$$\Box [t]_i \varphi \to [\Uparrow t]_i \Box \varphi \qquad\qquad (\text{generalize})$$
$$[t]_i \bigcirc \varphi \to \bigcirc [\Rightarrow t]_i \varphi \qquad\qquad (\bigcirc\text{-access})$$
$$\bigcirc [t]_i \varphi \to [\Leftarrow t]_i \bigcirc \varphi \qquad\qquad (\bigcirc\text{-left})$$

Some first remarks about these principles:

($\Box$-**access**) This is very plausible, if you have evidence that something always is true, then at every point in time you should be able to access this information.

(**generalize**) Using evidence this seems more plausible than just using knowledge, as one requires the evidence to be the same at every point in time.

($\bigcirc$-**access**) This seems plausible: agents do not forget evidence once they have gathered it and can "take it with them".

($\bigcirc$-**left**) This one seems less plausible as it implies some form of premonition.

The principle (generalize) is very strong. In particular, it makes internalization possible even in the presence of necessitation rules. Indeed, let $\mathsf{LPLTL}^{\mathsf{G}}_{\mathcal{CS}}$ be the system $\mathsf{LPLTL}_{\mathcal{CS}}$ extended by the axioms (generalize) and (mix)—this is also reflected by constant specification—and the iterated constant necessitation rule

$$\frac{[c]_i\varphi \in \mathcal{CS}}{[\star^n c]_{i_n} \ldots [\star c]_{i_1} [c]_i \varphi}$$

for arbitrary agents $i_1, \ldots, i_n$.

▶ **Theorem 30** (Internalization). *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification. The system $\mathsf{LPLTL}^{\mathsf{G}}_{\mathcal{CS}}$ enjoys internalization.*

Proof. We proceed by induction on the derivation of φ. There are two interesting cases:

In case φ is $\square\psi$, derived using ($\square$-nec), then, by the induction hypothesis, there is a term s such that $[s]_i\psi$ is provable. Now, we can use ($\square$-nec) in order to obtain $\square\,[s]_i\psi$ and then (generalize) and modus ponens to get $[\Uparrow s]_i\square\psi$.

In case φ is $\bigcirc\psi$, derived using ($\bigcirc$-nec), then, as above, we obtain $[\Uparrow s]_i\square\psi$. Since $\mathcal{CS}$ is axiomatically appropriate, there is a constant c with $[c]_i(\square\psi \to \bigcirc\psi)$. Thus we finally conclude $[c\cdot \Uparrow s]_i \bigcirc \psi$. ◀

It is obvious how to formulate conditions on evidence functions that correspond to the additional principles of this section such that soundness results can be obtained, see [5]. However, it is not clear how to show the existence of such models and how to show completeness for these additional principles.

8 Conclusions

We introduced the temporal justification logic $\mathsf{LPLTL}_{\mathcal{CS}}$ and showed that it is sound and complete with respect to interpreted systems that are based on Fitting-models. To achieve this we had to adapt the usual canonical model construction of justification logic such that it yields a finite Fitting-model. Further, we established that a suitable form of axiom necessitation can replace the necessitation rules for $\square$ and $\bigcirc$ and thus make internalization possible. Finally, we briefly discussed some additional principles that concern the interaction of knowledge, justifications, and time.

We finish this paper with some questions that show possible directions for future work.

▶ **Question 1.** How does a temporal justification logic based on $\mathsf{JT45}$, i.e. the justification counterpart of the modal logic $\mathsf{S5}$, look like? The problem is that $\mathsf{JT45}$-models must satisfy the strong evidence condition, i.e. for all $\mathcal{I}, r, n$ and each formula $[t]_i\varphi$

$$\varphi \in \mathcal{E}_i(r(n), t) \quad \text{implies} \quad (\mathcal{I}, r, n) \vDash [t]_i\varphi, \tag{7}$$

see [3, 20, 22, 23]. In infinite canonical models, the strong evidence property is an easy consequence of the Truth Lemma. In our temporal setting, we have a finite canonical model and the Truth Lemma is restricted to $\mathsf{Sub}^+(\chi)$. Hence it does not entail (7) for all formulas $[t]_i\varphi$.

▶ **Question 2.** How can the typical examples, e.g., protocols related to message transmission, be formalized in $\mathsf{LPLTL}_{\mathcal{CS}}$?

Yavorskaya [26] introduces multi-agent justification logics with interaction operations on the justification terms, in particular, she studies two principles:

$$[t]_i \varphi \to \left[!_i^j t \right]_j [t]_i \varphi \qquad\qquad \text{(Evidence verification)}$$

$$[t]_i \varphi \to \left[\uparrow_i^j t \right]_j \varphi \qquad\qquad \text{(Evidence conversion)}$$

We believe that principles of this kind will be important in the context of this question. For example, one might consider a temporal justification logic with principles such as

$$[t]_i \varphi \to \bigcirc \left[\mathsf{sent}_j^i(t) \right]_j \varphi \quad \text{or}$$

$$[t]_i \varphi \to \Diamond \left[\mathsf{sent}_j^i(t) \right]_j \varphi \,.$$

▶ **Question 3.** What happens if we require operations on justification terms to take time?

We could formalize this idea, e.g., by replacing the axioms (application), (sum), and (positive introspection) with

$$[t]_i (\varphi \to \psi) \to ([s]_i \varphi \to \bigcirc [t \cdot s]_i \psi)$$

$$[t]_i \varphi \vee [s]_i \varphi \to \bigcirc [t + s]_i \varphi$$

$$[t]_i \varphi \to \bigcirc [!t]_i [t]_i \varphi \,.$$

This might also relate to the logical omniscience problem [4].

▶ **Question 4.** Can dynamic epistemic justification logics be translated into temporal justification logic akin to [25]?

There are several dynamic justification logics available, e.g., [7, 18, 21], which feature not only traditional public announcements but also specific forms of evidence based updates and evidence elimination. It would be interesting to see what the relationship between those dynamic logics and temporal justification logic ist.

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