Deduction Chains for Common Knowledge

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Abstract

Deduction chains represent a syntactic and in a certain sense constructive method for proving completeness of a formal system. Given a formula ϕ, the deduction chains of ϕ are built up by systematically decomposing ϕ into its subformulae. In the case where ϕ is a valid formula, the decomposition yields a (usually cut-free) proof of ϕ. If ϕ is not valid, the decomposition produces a countermodel for ϕ. In the current paper, we extend this technique to a semiformal system for the Logic of Common Knowledge. The presence of fixed point constructs in this logic leads to potentially infinite-length deduction chains of a non-valid formula, in which case fairness of decomposition requires special attention. An adequate order of decomposition also plays an important role in the reconstruction of the proof of a valid formula from the set of its deduction chains.

1 Introduction

Modal logic may be employed to reason about knowledge. A necessity for this arises for example when modeling systems of distributed agents, say computers connected over a network. In this setting, an agent knowing some proposition ϕ in state s is usually understood as ϕ holding in all states reachable from s in one step and thus each agent's knowledge may be modeled using a respective box operator. Furthermore, through arbitrary nesting of boxes epistemic situations of considerable complexity become expressible. However, it is well known that any formula of modal logic can only talk about a finite portion of a model and that this is not sufficient to express certain epistemic situations of particular interest. One such example often encountered in problems of coordination and agreement is common knowledge of a proposition ϕ, which can roughly be viewed as the infinitary conjunction "all agents know ϕ and all agents know that all agents know ϕ and …". In order to express common knowledge in

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the setting of modal logic, a fixed point extension is required, yielding the so called Logic of Common Knowledge which was introduced in [5] and studied extensively from a model-theoretic point of view in [3]. A more proof-theoretic study of this logic is given in [1] and [2].

In the current study we aim to deepen the proof-theoretic understanding of Logic of Common Knowledge by giving an alternative completeness proof for an infinitary proof system for this logic using the method of deduction chains. Deduction chains represent a syntactic and in a certain sense constructive method for proving completeness of a formal system. Given a formula ϕ, the deduction chains of ϕ are built up by systematically decomposing ϕ into its subformulae. In the case where ϕ is a valid formula, the decomposition yields a (usually cut-free) proof of ϕ. If ϕ is not valid, the decomposition produces a countermodel for ϕ. The method of deduction chains was first introduced by Schütte in [9,11] and has been used mainly in the proof-theory of systems of first and second order arithmetic. See [6,8] for applications of the method in this field. In [10] Schütte extends deduction chains to modal logic and we extend this approach again to accommodate fixed-point constructs. The main additional difficulty is that the presence of fixed-points requires a fully deterministic procedure for the decomposition of a given formula in order to guarantee fairness in the case of an infinite deduction chain.

We begin our account by giving an introduction to the syntax and semantics of Logic of Common Knowledge. In particular we will state the infinitary proof system $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{n}}$, the completeness of which will be the main goal. In Section 3 we introduce the concept of deduction chains for formulae of Logic of Common Knowledge and prove some crucial properties required for the subsequent argument, chiefly fairness and saturation. We then proceed to prove the so called *principal semantic lemma*, which represents one half of the deduction chain argument. The principal semantic lemma secures the construction of a countermodel in case of an infinite deduction chain. Section 5 takes care of the other half of the argument, the so called *principal syntactic lemma* which yields the construction of a proof from the set of all deduction chains of a formula, if all of these chains are finite. Completeness is then obtained as a corollary to the two principal lemmata. In the concluding section we give a short overview of the main completeness argument.

2 Syntax and semantics

The language $\mathcal{L}^{n}_{\mathsf{C}}$ for Logic of Common Knowledge comprises a set of *atomic propositions* $\mathsf{p}, \mathsf{q}, \ldots$, the *propositional connectives* $\wedge$ and $\vee$, the *epistemic operators* $\mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_n$ and the *common knowledge operator* C. Additionally, we assume there is an auxiliary symbol $\sim$ to form complements of atomic propositions and dual epistemic operators. The formulae $\alpha, \beta, \gamma, \dots$ (possibly with subscripts) of $\mathcal{L}_{\mathsf{C}}^{n}$ are defined inductively as follows.

(1) All atomic propositions p and their complements $\sim\mathsf{p}$ are $\mathcal{L}_{\mathsf{C}}^{n}$ formulae.
(2) If α and β are $\mathcal{L}_{\mathsf{C}}^{n}$ formulae, so are $(\alpha \vee \beta)$ and $(\alpha \wedge \beta)$.
(3) If α is an $\mathcal{L}_{\mathsf{C}}^{n}$ formula, so are $\mathsf{K}_i\alpha$ and $\sim\mathsf{K}_i\alpha$.
(4) If α is an $\mathcal{L}_{\mathsf{C}}^{n}$ formula, so are $\mathsf{C}\alpha$ and $\sim\mathsf{C}\alpha$.

Often we omit parentheses if there is no possible confusion. We can define the *negation* $\neg\alpha$ of general $\mathcal{L}_{\mathsf{C}}^{n}$ formulae α by making use of de Morgan's laws and the law of double negation.

(1) If α is the atomic proposition p, then $\neg\alpha$ is $\sim\alpha$; if α is the formula $\sim\mathsf{p}$, then $\neg\alpha$ is p.
(2) If α is the formula $(\beta \vee \gamma)$, then $\neg\alpha$ is $(\neg\beta \wedge \neg\gamma)$; if α is the formula $(\beta \wedge \gamma)$, then $\neg\alpha$ is $(\neg\beta \vee \neg\gamma)$.
(3) If α is the formula $\mathsf{K}_i\beta$, then $\neg\alpha$ is $\sim\mathsf{K}_i(\neg\alpha)$; if α is the formula $\sim\mathsf{K}_i\beta$, then $\neg\alpha$ is $\mathsf{K}_i(\neg\alpha)$;
(4) If α is the formula $\mathsf{C}\beta$, then $\neg\alpha$ is $\sim\mathsf{C}(\neg\alpha)$; if α is the formula $\sim\mathsf{C}\beta$, then $\neg\alpha$ is $\mathsf{C}(\neg\alpha)$;

We set

$$\mathsf{E}\alpha := \mathsf{K}_1\alpha \wedge \cdots \wedge \mathsf{K}_n\alpha.$$

The formula $\mathsf{K}_i\alpha$ can be interpreted as "agent i knows that α". Thus $\mathsf{E}\alpha$ means "everybody knows that α". We will also need iterations $\mathsf{E}^m\alpha$ for all natural numbers m, formally defined by

$$\mathsf{E}^0\alpha := \top,\ \mathsf{E}^1\alpha := \mathsf{E}\alpha \text{ and } \mathsf{E}^{m+1}\alpha := \mathsf{E}\mathsf{E}^m\alpha,$$

where $\top$ is taken to refer to some trivially valid formula as for example $\mathsf{p} \vee \sim\mathsf{p}$ where p is an atomic proposition.

The semantics for logics of common knowledge is given by *Kripke structures*

$$\mathcal{M} = (S, \mathcal{K}_1, \dots, \mathcal{K}_n, \pi)$$

where S is a non-empty set of *worlds*, $\mathcal{K}_1, \dots, \mathcal{K}_n$ are binary relations on S and π is a *valuation function* assigning to each atomic proposition a subset of S. We say w is a world of $\mathcal{M} = (S, \mathcal{K}_1, \dots, \mathcal{K}_n, \pi)$, expressed by $w \in \mathcal{M}$, if w is an element of S. The truth set $\|\alpha\|^{\mathcal{M}}$ of an $\mathcal{L}_{\mathsf{C}}^{n}$ formula α with respect to the Kripke structure $\mathcal{M} = (S, \mathcal{K}_1, \dots, \mathcal{K}_n, \pi)$ is defined by induction on the complexity of α:

$$
\begin{aligned}
\|\mathsf{p}\|^{\mathcal{M}} &:= \pi(\mathsf{p}) \\
\| \sim\mathsf{p}\|^{\mathcal{M}} &:= S \setminus \|\mathsf{p}\|^{\mathcal{M}}, \\
\|\alpha \vee \beta\|^{\mathcal{M}} &:= \|\alpha\|^{\mathcal{M}} \cup \|\beta\|^{\mathcal{M}}, \\
\|\alpha \wedge \beta\|^{\mathcal{M}} &:= \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}, \\
\|\mathsf{K}_i\alpha\|^{\mathcal{M}} &:= \{v \in S : w \in \|\alpha\|^{\mathcal{M}} \text{ for all } w \text{ with } (v,w) \in \mathcal{K}_i\}, \\
\| \sim\mathsf{K}_i\alpha\|^{\mathcal{M}} &:= S \setminus \|\mathsf{K}_i\neg\alpha\|^{\mathcal{M}}, \\
\|\mathsf{C}\alpha\|^{\mathcal{M}} &:= \bigcap\{\|\mathsf{E}^m\alpha\|^{\mathcal{M}} : m \geq 1\}, \\
\| \sim\mathsf{C}\alpha\|^{\mathcal{M}} &:= S \setminus \|\mathsf{C}\neg\alpha\|^{\mathcal{M}}.
\end{aligned}
$$

Using these truth sets, we can express that a formula α is *valid* in a world w of a Kripke structure $\mathcal{M}$. This is the case if $w \in \|\alpha\|^{\mathcal{M}}$. We will employ the following notation:

$$\mathcal{M}, w \models \alpha \;:\Longleftrightarrow\; w \in \|\alpha\|^{\mathcal{M}}.$$

Next, we are going to present the semiformal Tait-style calculus $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ for common knowledge. Tait-style calculi [12,14] are one-sided Gentzen calculi which derive finite sets of formulae. This kind of calculi is particularly well-suited for the study of cut-elimination and meta-mathematical investigations. $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ has been introduced by Alberucci and Jäger [1,2]. It incorporates an analogue of the ω rule which permits the derivation of the formula $\mathsf{C}\alpha$ from the infinitely many premises

$$\mathsf{E}^1\alpha, \mathsf{E}^2\alpha, \ldots, \mathsf{E}^m\alpha, \ldots$$

for all natural numbers $m \geq 1$. The system $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ is called *semiformal* since, as opposed to formal systems, it has basic inferences with infinitely many premises [11].

The system $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ derives finite sets of $\mathcal{L}^n_{\mathsf{C}}$ formulae which are denoted by $\Gamma, \Delta, \Sigma, \Pi, \ldots$ (possibly with subscripts). Usually we will write for example $\alpha, \beta, \Delta, \Gamma$ for the union $\{\alpha, \beta\} \cup \Delta \cup \Gamma$. Moreover, if Γ is the set $\{\alpha_1, \ldots, \alpha_m\}$, then we use the following abbreviations:

$$
\begin{aligned}
\bigvee \Gamma &:= \alpha_1 \vee \cdots \vee \alpha_m, \\
\neg\Gamma &:= \{\neg\alpha_1, \ldots, \neg\alpha_m\}, \\
\neg\mathsf{K}_i\Gamma &:= \{\neg\mathsf{K}_i\alpha_1, \ldots, \neg\mathsf{K}_i\alpha_m\}, \\
\neg\mathsf{C}\Gamma &:= \{\neg\mathsf{C}\alpha_1, \ldots, \neg\mathsf{C}\alpha_m\}.
\end{aligned}
$$

The axioms and rules of $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ consist of the usual propositional axioms and rules of Tait calculi, rules for the epistemic operators K_i with additional side formulae $\neg\mathsf{C}\Delta$ plus rules dealing with common knowledge. Note that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ includes neither an induction rule nor a cut rule.

Definition 2.1 *The infinitary Tait-style calculus* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ *over the language* $\mathcal{L}^n_{\mathsf{C}}$ *is defined by the following axioms and inference rules:*

$$\Gamma, \mathsf{p}, \neg\mathsf{p} \quad (ID)$$

$$\frac{\Gamma, \alpha, \beta}{\Gamma, \alpha \vee \beta} \ (\vee) \qquad \frac{\Gamma, \alpha \quad \Gamma, \beta}{\Gamma, \alpha \wedge \beta} \ (\wedge)$$

$$\frac{\neg\mathsf{C}\Delta, \neg\Gamma, \alpha}{\neg\mathsf{C}\Delta, \neg\mathsf{K}_i\Gamma, \mathsf{K}_i\alpha, \Sigma} \ (\mathsf{K}_i)$$

$$\frac{\Gamma, \neg\mathsf{E}\alpha}{\Gamma, \neg\mathsf{C}\alpha} \ (\neg\mathsf{C}) \qquad \frac{\Gamma, \mathsf{E}^k\alpha \textit{ for all } k \in \omega}{\Gamma, \mathsf{C}\alpha} \ (\mathsf{C}^\omega)$$

The infinitary system $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n}$ is formulated over the finitary language $\mathcal{L}^n_\mathsf{C}$ and derives finite sets of formulae. It is infinitary only because of the rule (C^ω) for introducing common knowledge. This rule has infinitely many premises and thus may give rise to infinite proof trees. For arbitrary ordinals α and finite sets Γ of $\mathcal{L}^n_\mathsf{C}$ formulae we define the derivability relation $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma$ as usual by induction on α.

(1) If Γ is an axiom of $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n}$, then we have $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma$ for all ordinals α.
(2) If $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha'_i}} \Gamma_i$ and $\alpha'_i < \alpha$ for all premises of a rule of $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n}$, then we have $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma$ for the conclusion Γ of this rule.

We will write $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash \Gamma$ if $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma$ for some ordinal α.

Now we have to mention some structural properties of $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n}$ which will be important in the sequel. The first two, weakening and inversion, are easily shown by induction on the length of the involved derivations.

Lemma 2.2 (Weakening) *If $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma$ and $\Gamma \subset \Gamma'$, then also $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma'$.*

Lemma 2.3 (Inversion)

(1) If $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \phi_1 \wedge \phi_2$, then $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \phi_1$ and $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \phi_2$.
(2) If $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \phi_1 \vee \phi_2$, then $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \phi_1, \phi_2$.
(3) If $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \mathsf{C}\phi$, then $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \mathsf{E}^k\phi$ for every $k \in \omega$.

Lemma 2.4 *If $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \neg\mathsf{E}^k\phi$ for some $k \in \omega$, then $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha+1}} \Gamma, \neg\mathsf{C}\phi$.*

PROOF. We proceed by induction on k. The base case of $k = 1$ holds directly by the rule $(\neg\mathsf{C})$. We thus assume $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \neg\mathsf{E}^{k+1}\phi$, which by iteration of Lemma 2.3 means

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \neg\mathsf{K}_1\mathsf{E}^k\phi, \ldots, \neg\mathsf{K}_n\mathsf{E}^k\phi \tag{1}$$

and show $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha+1}} \Gamma, \neg\mathsf{C}\phi$ by induction on length α of the proof. The case of $\alpha = 0$ is trivial, thus assume that the claim holds for all $\alpha' < \alpha$. We make a case distinction as to the last rule applied to derive (1).

Case 1) The last rule was (K_i) for some $1 \leq i \leq n$: Then there exists a formula $\mathsf{K}_i\xi \in \Gamma$ such that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha}} \neg\mathsf{C}\Delta_1, \neg\mathsf{K}_i\Delta_2, \mathsf{K}_i\xi, \Sigma$ and $\neg\mathsf{K}_j\mathsf{E}^k\phi \in \Sigma$ for all $j \neq i$. If we also have $\neg\mathsf{K}_i\mathsf{E}^k\phi \in \Sigma$, then the claim is trivial. Otherwise we must have $\neg\mathsf{K}_i\mathsf{E}^k\phi \in \neg\mathsf{K}_i\Delta_2$ and by the premise of (K_i)

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha'}} \neg\mathsf{C}\Delta_1, \neg\Delta_2, \xi,$$

where $\alpha' < \alpha$ and $\neg\mathsf{E}^k\phi \in \neg\Delta_2$. By the hypothesis of the outer induction $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha}} \neg\mathsf{C}\Delta_1, \neg\mathsf{C}\phi, \neg\Delta_2', \xi$, where $\neg\Delta_2' = \neg\Delta_2 \setminus \{\neg\mathsf{E}^k\phi\}$. Therefore, applying (K_i) yields $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha+1}} \neg\mathsf{C}\Delta_1, \neg\mathsf{C}\phi, \neg\mathsf{K}_i\Delta_2', \mathsf{K}_i\xi, \Sigma$, meaning $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha+1}} \Gamma, \neg\mathsf{C}\phi$.

Case 2) The last rule was not (K_i) for any $1 \leq i \leq n$: In this case the claim follows directly by applying the hypothesis of the inner induction to the premise of the respective rule. □

Transfinite induction on the length of derivations yields the correctness of $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ with respect to the semantics for logics of common knowledge. That is we have the following theorem.

Theorem 2.5 *For all finite sets Γ of $\mathcal{L}^n_{\mathsf{C}}$ formulae, all Kripke structures $\mathcal{M}$ and all worlds $w \in \mathcal{M}$ we have that*

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma \implies \mathcal{M}, w \models \bigvee \Gamma.$$

3 Deduction chains

In this section we are going to define the notion of deduction chain in the context of $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$. Schütte [9] originally introduced deduction chains for classical logic. Later, he showed in [10] how to extend this technique to the case of intuitionistic and modal logics. We adapt his method and apply it to show completeness of our infinitary fixed point logic.

In the sequel we will make use of the following notation for projections. If a is a tuple (x, y), then $a_1 := x$ and $a_2 := y$.

We start by defining labeled index trees. Such trees will provide the frame on which the countermodel of a non-valid formula ψ is based. The set of worlds will consist of all nodes of the labeled index trees of a deduction chain for ψ. The accessibility relation for agent i will be given the successor relation σ_i.

Definition 3.1 *A labeled index tree is a set I of pairs (k, α), where k is in $\{0, \ldots, n\}$ and α is a sequence of natural numbers such that I has the following properties*

(1) $(0, (0)) \in I$

(2) For every $m \in \omega$ we have that

$$(k, (\alpha, m+1)) \in I \text{ for some } k \in \{1, \ldots, n\}$$

implies

$$(l, (\alpha, m)) \in I \text{ for some } l \in \{1, \ldots, n\}.$$

(3) If there exists a $k \in \{1, \ldots, n\}$ with $(k, (\alpha, 0)) \in I$, then there exists an $l \in \{1, \ldots, n\}$ such that $(l, \alpha) \in I$

(4) If $(k, \alpha) \in I$ and $(l, \alpha) \in I$, then $k = l$.

Definition 3.2 *Let I be a labeled index tree and $a, b \in I$. We define the following binary relations on I:*

$$\begin{aligned}
a = b &:\Leftrightarrow a_2 = b_2 \\
a\sigma_i b &:\Leftrightarrow a = (j, \alpha) \text{ and } b = (i, (\alpha, l)) \\
&\qquad \text{for some sequence } \alpha,\ j \in \{1, \ldots, n\} \text{ and } l \in \omega \\
a \prec b &:\Leftrightarrow a_2 \text{ is a prefix of } b_2 \\
a \preccurlyeq b &:\Leftrightarrow a = b \text{ or } a \prec b \\
a \sqsubset b &:\Leftrightarrow (a \prec b) \text{ or } \\
&\qquad (a_2 = (\alpha, l) \text{ and } b_2 = (\alpha, k) \text{ and } l < k)
\end{aligned}$$

Definition 3.3 *A literal is a formula of the form p or $\sim \mathsf{p}$ where p is an atomic formula. A formula ϕ is reducible if it is not a literal.*

A deduction chain for a formula ϕ is built by decomposing ϕ. It is crucial for our argument that this decomposition satisfies certain fairness conditions. In particular, formulae of the form $\sim\mathsf{C}\alpha$ need special care. When we treat such a formula for the first time, we create a new formula $\neg\mathsf{E}^1\alpha$. When we deal with it for the second time, then we create $\neg\mathsf{E}^2\alpha$ and so on. Moreover, if there is another formula $\sim\mathsf{C}\beta$, we have to pay attention that we consider $\sim\mathsf{C}\alpha$ and $\sim\mathsf{C}\beta$ in alternation. In order to guarantee this, we need some bookkeeping which is achieved using so-called iteration histories.

Definition 3.4 *Let $\mathcal{L}^n_{\mathsf{C}}|_{\neg\mathsf{C}}$ denote the set of all formulae of the language $\mathcal{L}^n_{\mathsf{C}}$ which have the form $\sim\mathsf{C}\beta$ for some $\beta \in \mathcal{L}^n_{\mathsf{C}}$. An iteration history is a finite set $E \subset \mathcal{L}^n_{\mathsf{C}}|_{\neg\mathsf{C}} \times \omega \times \omega$ such that for any $e, f \in E$, we have $e = f$ if $e_1 = f_1$.*

Definition 3.5 *Given an iteration history E, we define*

$$dom_E := \{\alpha \in \mathcal{L}^n_{\mathsf{C}}|_{\neg\mathsf{C}}; \exists e \in E \text{ such that } e_1 = \alpha\}$$

Furthermore, for all $\alpha \in dom_E$ *and* $k \in \omega$ *we define the following functions:*

$$add_E(\sim\mathsf{C}\beta, k) = \begin{cases} E \cup \{(\sim\mathsf{C}\beta, k, 0)\} & \text{if } \sim\mathsf{C}\beta \notin dom_E \\ E & \text{otherwise} \end{cases}$$

$$lookup_E(\alpha) = (k, l) \text{ where } (\alpha, k, l) \in E$$

$$ord_E(\alpha) = (lookup_E(\alpha))_1$$

$$deg_E(\alpha) = (lookup_E(\alpha))_2$$

$$max_E = \begin{cases} \max\{ord_E(\beta); \beta \in dom_E\} & \text{if } dom_E \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$min_E = \begin{cases} \min\{ord_E(\beta); \beta \in dom_E\} & \text{if } dom_E \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.6 *A formula sequence* S *is an* $n+2$*-tuple* $(\Gamma, \Delta_1, \ldots, \Delta_n, E)$*, where* Γ *is a finite sequence of formulae of* $\mathcal{L}^n_{\mathsf{C}}$*,* Δ_i *are finite sequences of formulae of the form* $\neg\alpha$*, where* $\alpha \in \mathcal{L}^n_{\mathsf{C}}$ *and* E *is an iteration history. We will use* ϵ *to denote the empty sequence. The distinguished formula of* S *is the rightmost reducible formula appearing in* Γ*, if such a formula exists. For any finite sequence of formulae* Λ*, we denote by* $set(\Lambda)$ *the set of all formulae appearing in* Λ*. We define* $set(S) := set(\Gamma) \cup dom_E$*,*

$$set^+(S) := set(S) \cup \{\sim\mathsf{K}_i\beta; \neg\beta \in set(\Delta_1)\} \cup \ldots \cup \{\sim\mathsf{K}_n\beta; \neg\beta \in set(\Delta_n)\},$$

$max_S := max_E$*,* $min_S := min_E$ *and* $dom_S := dom_E$*. Further, for all formulae* $\beta \in dom_S$ *we set* $ord_S(\beta) := ord_E(\beta)$*. Let* FS *be the set of all formula sequences.*

A sequence tree is a labeled index tree of formula sequences. That is we annotate each node of the index tree with a formula sequence. In the construction of a countermodel for a non-valid formula, the sequence at a node will be the basis for defining the valuation function π at that node. In particular, π will be defined such that if a formula ψ belongs to the annotation of a node, then ψ will not hold at that node.

Definition 3.7 *Let* I *be a labeled index tree. A sequence tree over* I *is a function*

$$\mathbf{R} : I \longrightarrow FS$$

We adopt the notation $\mathbf{R}_a$ *for* $\mathbf{R}(a)$*, where* $a \in I$ *and define* $max(\mathbf{R})$ *as* $max\{max_{\mathbf{R}_a}; a \in I\}$*. Furthermore, given a formula* α *and an iteration history* E *we define the operation*

$$it(\mathbf{R}, \alpha, E) = \begin{cases} (E \setminus \{(\alpha, k, l)\}) \cup \{(\alpha, max(\mathbf{R}) + 1, l + 1)\} & \text{if } \alpha \in dom_E \\ E & \text{otherwise} \end{cases}$$

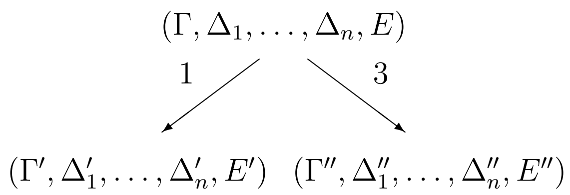


Fig. 1. A sequence tree

Definition 3.8 *Let* $\mathbf{R}$ *be a sequence tree over* I*. Further, let* J *be the set* $\{a \in I; dom_{\mathbf{R}_a} \neq \emptyset\}$*. We define the relation* $\sqsubset^*$ *for all* $a, b \in J$ *as follows:*

$$a \sqsubset^* b \quad :\Leftrightarrow \quad min_{\mathbf{R}_a} < min_{\mathbf{R}_b} \text{ or } [min_{\mathbf{R}_a} = min_{\mathbf{R}_b} \text{ and } a \sqsubset b]$$

The redex of a sequence tree is the formula that will be decomposed next. It is basically found as follows. The rightmost reducible formula of the main sequence of a node a of $\mathbf{R}$ is called distinguished formula of $\mathbf{R}$ at a (see Definition 3.6). The redex of $\mathbf{R}$ is defined as the topmost distinguished formula if such a formula exists; otherwise as the formula of the form $\sim\mathsf{C}\alpha$ (if such a formula exists) which has to be treated next according to information given by the iteration histories. If neither of these two conditions apply, then $\mathbf{R}$ has no redex.

Definition 3.9 *Let* $\mathbf{R}$ *be a sequence tree over* I *and* $a \in I$*. A formula* ϕ *is called redex of* $\mathbf{R}$ *at* a *if one of the following two conditions holds:*

(1) ϕ *is the distinguished formula of* $\mathbf{R}_a$ *and* a *is* $\sqsubset$*-minimal among all* $b \in I$*.*
(2) there are no distinguished formulae in $\mathbf{R}$*,* $\phi \in dom_{\mathbf{R}_a}$*,* $ord_{\mathbf{R}_a}(\phi) = min_{\mathbf{R}_a}$ *and* a *is* $\sqsubset^*$*-minimal in* $\mathbf{R}$*.*

Note that for a sequence tree $\mathbf{R}$ over I there is at most one $a \in I$ and one formula ϕ such that ϕ is the redex of $\mathbf{R}$ at a.

Definition 3.10 *Let* α *be a formula,* $S = (\Gamma, \Delta_1, \ldots, \Delta_n, E)$ *a formula sequence in a sequence tree* $\mathbf{R}$ *and* Γ' *the sequence* α, Γ*. Define the operation*

$$\alpha \circ S = \begin{cases} S \text{ if } \alpha \text{ is already in } \Gamma, \\ (\Gamma', \Delta_1, \ldots, \Delta_n, E) \text{ if } \alpha \text{ not in } \Gamma \text{ and not of the form } \sim\mathsf{C}\beta, \\ (\Gamma, \Delta_1, \ldots, \Delta_n, add_E(\alpha, max(\mathbf{R}) + 1)) \text{ if } \alpha \text{ not in } \Gamma \text{ and} \\ \qquad\qquad\qquad\qquad\qquad \text{of the form } \sim\mathsf{C}\beta \end{cases}$$

Given a finite sequence $\Lambda = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *of formulae and a formula sequence* S*, we write* $\Lambda \circ S$ *for* $\alpha_1 \circ (\alpha 2 \circ (\ldots \circ (\alpha_n \circ S)))$

Definition 3.11 *A sequence tree* $\mathbf{R}$ *over* I *is called reducible, if* $\mathbf{R}$ *has a redex.* $\mathbf{R}$ *is called axiomatic if there exists an* $a \in I$ *and an atomic proposition* p*, such that* $\mathbf{R}_a = (\Gamma, \Delta_1, \ldots, \Delta_n, E)$ *and both* p *and* $\sim\mathsf{p}$ *appear in* Γ*. Generally, we say that a formula* α *appears in* $\mathbf{R}$ *at some* $a \in I$ *if* $\alpha \in set(\mathbf{R}_a)$*.*

A deduction chain is a sequence $\Theta_0, \Theta_1, \Theta_2, \ldots$ of sequence trees. If Θ_i is axiomatic, then Θ_i is the last element of the deduction chain. Θ_i is also the last element of the deduction chain if it does not contain a redex. If Θ_i is not axiomatic and has a redex ψ at a, then ψ will be decomposed and a new sequence tree Θ_{i+1} is added to the deduction chain. Θ_{i+1} is obtained from Θ_i by removing ψ and adding

(1) ψ_1, ψ_2 at a if $\psi = \psi_1 \vee \psi_2$,

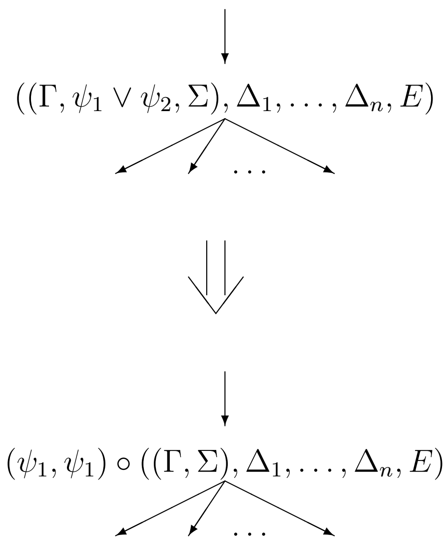


Fig. 2. Type 1 reduction

(2) ψ_1 or ψ_2 at a if $\psi = \psi_1 \wedge \psi_2$,

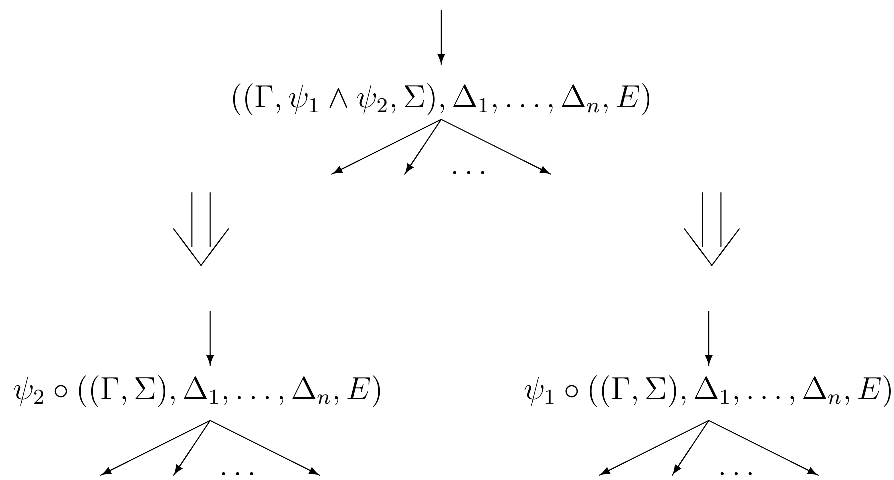


Fig. 3. Type 2 reduction

(3) $\neg\psi_1$ at every successor of a (and remembering $\neg\psi_1$ at a) if $\psi = \sim\mathsf{K}_i\psi_1$,

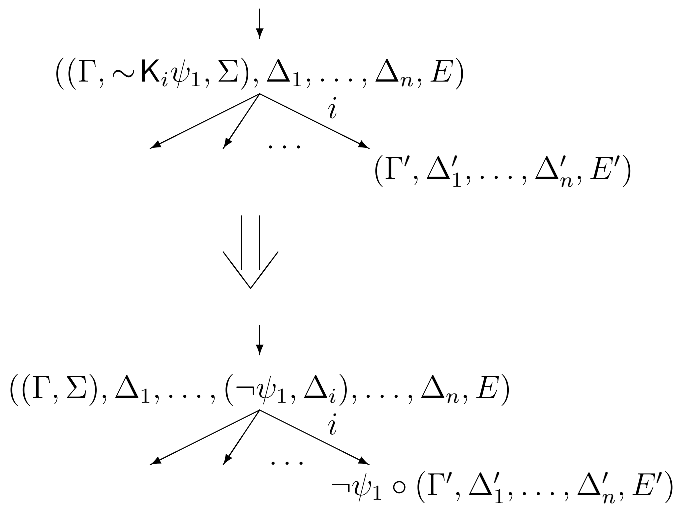


Fig. 4. Type 3 reduction

(4) a new successor of a initialized with ψ_1 (plus anything remembered at a) if $\psi = \mathsf{K}_i\psi_1$,

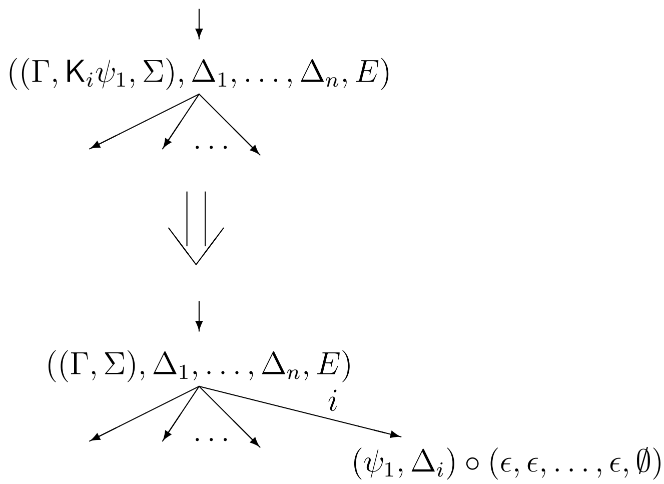


Fig. 5. Type 4 reduction

(5) $\mathsf{E}^k\psi_1$ at a for some k if $\psi = \mathsf{C}\psi_1$,

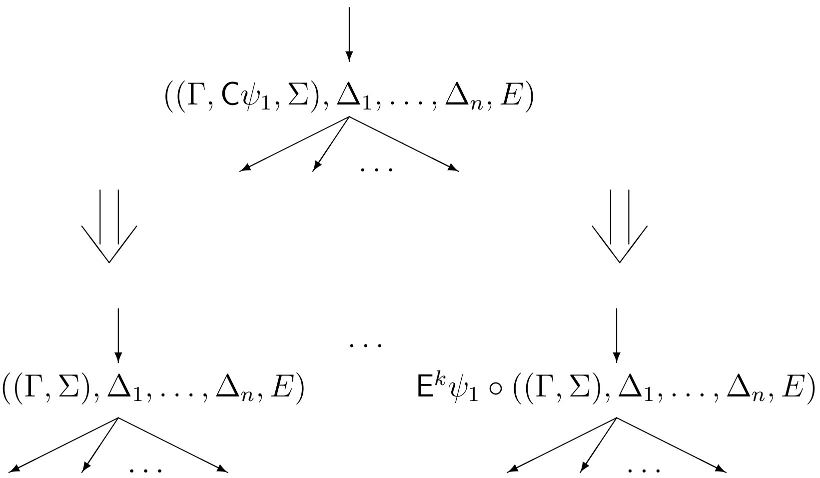
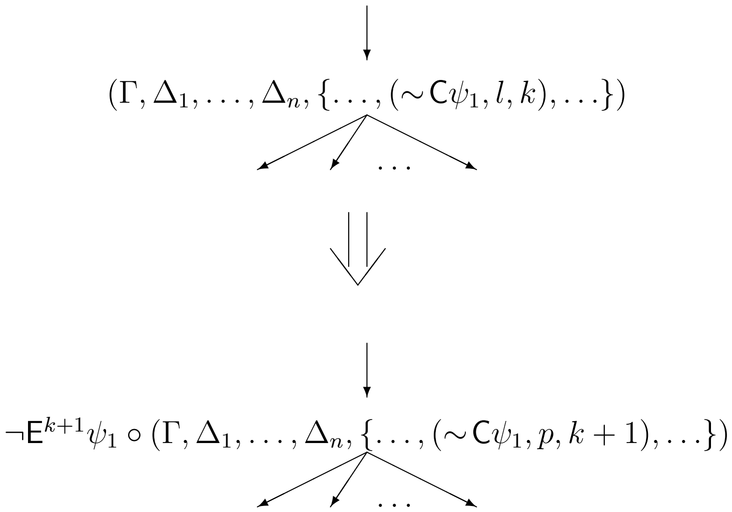


Fig. 6. Type 5 reduction

(6) $\neg\mathsf{E}^{k+1}\psi_1$ at a where k is the maximum number of iterations tried at a if $\psi = \sim\mathsf{C}\psi_1$.



where p is maximal for the whole sequence tree.

Fig. 7. Type 6 reduction

These six cases will be made precise in the next definition.

Definition 3.12 *Let* $\mathbf{R}$ *be a sequence tree. A deduction chain of* $\mathbf{R}$ *is a finite or infinite sequence*

$$\Theta_0, \Theta_1, \Theta_2, \ldots$$

of sequence trees with the following properties:

(1) $\Theta_0 = \mathbf{R}$
(2) If Θ_m *is axiomatic or not reducible, then* Θ_m *is the last element of the sequence.*

(3) *If* Θ_m *is not axiomatic and reducible, then* Θ_{m+1} *is derived from* Θ_m *in the following manner:*
Let Θ_m *be the sequence tree* $\mathbf{S}$ *over index tree* I *and let* ϕ *be the redex of* $\mathbf{S}$ *at* $a \in I$. *If* $\phi \notin \mathcal{L}^n_{\mathsf{C}}|_{\neg\mathsf{C}}$, *then* $\mathbf{S}_a = (\Gamma, \Delta_1, \ldots, \Delta_n, E)$ *and* $\Gamma = \Omega, \phi, \Omega'$, *where* Ω' *is a sequence of non-reducible formulae.*
Case 1: $\phi = \psi_1 \vee \psi_2$
Then Θ_{m+1} *is the sequence tree* $\mathbf{T}$ *over* I, *where*

$$\begin{aligned} \Gamma' &= \Omega, \Omega' \\ \mathbf{T}_a &= (\psi_1, \psi_2) \circ (\Gamma', \Delta_1, \ldots, \Delta_n, E) \\ \mathbf{T}_b &= \mathbf{S}_b \textit{ for all other } b \in I \end{aligned}$$

In this case we say that Θ_m *has type 1 successor* Θ_{m+1}.
Case 2: $\phi = \psi_1 \wedge \psi_2$
Then Θ_{m+1} *is the sequence tree* $\mathbf{T}$ *over* I, *where*

$$\begin{aligned} \Gamma' &= \Omega, \Omega' \\ \mathbf{T}_a &= \psi_1 \circ (\Gamma', \Delta_1, \ldots, \Delta_n, E) \textit{ or} \\ \mathbf{T}_a &= \psi_2 \circ (\Gamma', \Delta_1, \ldots, \Delta_n, E) \\ \mathbf{T}_b &= \mathbf{S}_b \textit{ for all other } b \in I \end{aligned}$$

In this case we say that Θ_m *has type 2 successor* Θ_{m+1}.
Case 3: $\phi = \sim \mathsf{K}_i \psi$
Then Θ_{m+1} *is the sequence tree* $\mathbf{T}$ *over* I, *where*

$$\begin{aligned} \Gamma' &= \Omega, \Omega' \\ \Delta'_i &= \neg\psi, \Delta_i \\ \mathbf{T}_a &= (\Gamma', \Delta_1, \ldots, \Delta'_i, \ldots, \Delta_n, E) \end{aligned}$$

and for all $b \in I$ *such that* $a\sigma_i b$

$$\mathbf{T}_b = \neg\psi \circ \mathbf{S}_b$$

and $\mathbf{T}_c := \mathbf{S}_c$ *for all other* $c \in I$.
In this case we say that Θ_m *has type 3 successor* Θ_{m+1}.
Case 4: $\phi = \mathsf{K}_i \psi$
Let $a = (l, \alpha)$ *and* k *be the smallest number such that* $(j, (\alpha, k)) \notin I$ *for any number* j. *Then* Θ_{m+1} *is the sequence tree* $\mathbf{T}$ *over* $I \cup \{b\}$, *where* $b = (i, (\alpha, k))$ *and*

$$\begin{aligned} \Gamma' &= \Omega, \Omega' \\ \mathbf{T}_a &= (\Gamma', \Delta_1, \ldots, \Delta_n, E) \\ \mathbf{T}_b &= (\psi, \Delta_i) \circ (\epsilon, \epsilon, \ldots, \epsilon, \emptyset) \\ \mathbf{T}_c &= \mathbf{S}_c \textit{ for all other } c \in I \end{aligned}$$

In this case we say that Θ_m *has type 4 successor* Θ_{m+1}.

Case 5: $\phi = \mathsf{C}\psi$

Then Θ_{m+1} *is the sequence tree* $\mathbf{T}$ *over* I*, where*

$$\Gamma' = \Omega, \Omega'$$
$$\mathbf{T}_a = \mathsf{E}^i\psi \circ (\Gamma', \Delta_1, \ldots, \Delta_n, E) \text{ for some } i \in \omega$$
$$\mathbf{T}_b = \mathbf{S}_b \text{ for all other } b \in I$$

In this case we say that Θ_m *has type 5 successor* Θ_{m+1}*.*

If $\phi \in \mathcal{L}^n_{\mathsf{C}}|_{\neg\mathsf{C}}$*, then we proceed as follows:*

Case 6: $\phi = \sim\mathsf{C}\psi$ *Then* Θ_{m+1} *is the sequence tree* $\mathbf{T}$ *over* I*, where*

$$\mathbf{T}_a = \neg\mathsf{E}^k\psi \circ (\Gamma, \Delta_1, \ldots, \Delta_n, it(\Theta_m, \sim\mathsf{C}\psi, E)) \quad \text{where } k = deg_E(\sim\mathsf{C}\psi) + 1$$
$$\mathbf{T}_b = \mathbf{S}_b \text{ for all other } b \in I$$

In this case we say that Θ_m *has type 6 successor* Θ_{m+1}*.*

Definition 3.13 *Let* ϕ *be an* $\mathcal{L}^n_{\mathsf{C}}$ *formula. A deduction chain of* ϕ *is a deduction chain of the sequence tree* $\mathbf{R}$ *which is given by the function mapping the index tree* $\{(0, (0)\}$ *to the formula sequence* $\phi \circ (\epsilon, \epsilon, \ldots, \epsilon, \emptyset)$*.*

4 Principal semantic lemma

The principal semantic lemma states that if there exists a deduction chain of a formula ψ which is infinite or ends in a non-axiomatic sequence tree, then there exists a countermodel for ψ. For this section we assume $\Theta_0, \Theta_1, \Theta_2, \ldots$ is such a deduction chain and we let $I_0, I_1, I_2, \ldots$ be the respective labeled index trees.

The Kripke structure K_Θ that will serve as countermodel is (roughly) constructed as $\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \ldots$ where $\pi(p) = \{a; \neg p \text{ appers at node } a\}$. Fairness in the construction of the deduction chain ensures that if $\phi \in a$, then $\mathsf{K}_\Theta, a \not\models \phi$. Finally we observe that ψ is an element of the root of K_Θ.

The following three lemmata follow directly from the definition of deduction chain.

Lemma 4.1 *If a literal* α *appears in* Θ_i *at* $a \in I_i$*, then* α *also appears in every* Θ_j *at* $a \in I_j$ *for* $j \geq i$*.*

Lemma 4.2 *For every* Θ_i *we have: There does not exist an* $a \in I_i$ *such that for some atomic formula* p *both* p *and* $\sim\mathsf{p}$ *appear in* Θ_i *at* a*.*

Lemma 4.3 *For each Θ_k there exists an $l \geq k$, such that Θ_l has no distinguished formulae.*

Lemma 4.4 *If $\mathbf{R} = \Theta_k$, $\sim\mathsf{C}\beta$ appears in $\mathbf{R}$ at a and $ord_{\mathbf{R}_a}(\sim\mathsf{C}\beta)$ is minimal in $\mathbf{R}$, then there exists an $l \geq k$, such that $\sim\mathsf{C}\beta$ is the redex of Θ_l at a.*

PROOF. By definition of deduction chains and the operations *it* and $\circ$ there can only be one formula and one $a \in I_k$, such that $ord_{\mathbf{R}_a}(\sim\mathsf{C}\beta)$ is minimal in Θ_k. By Lemma 4.3 there exists an $l \geq k$, such that Θ_l has no distinguished formulae. Then a is $\sqsubset^*$-minimal in Θ_l and so $\sim\mathsf{C}\beta$ is the redex of Θ_l at a. $\square$

Lemma 4.5 *For every Θ_k and $m \geq 0$ there exists an $l \geq k$, such that the (finite) set*

$$d_{\Theta_l}(m) := \{(\sim\mathsf{C}\beta, a); \sim\mathsf{C}\beta \text{ appears in } \Theta_l \text{ at } a \text{ and } ord_{(\Theta_l)_a}(\sim\mathsf{C}\beta) \leq m\}$$

is empty.

PROOF. The claim is trivial if Θ_k does not contain any formulae of the form $\sim\mathsf{C}\alpha$. We thus assume otherwise and prove the claim by induction on m.

<u>$m = 0$:</u> The set $d_{\Theta_k}(0)$ can only contain a pair $(\sim\mathsf{C}\beta, a)$, where we have $ord_{(\Theta_k)_a}(\sim\mathsf{C}\beta) = 0$. Since $ord_{(\Theta_k)_a}(\sim\mathsf{C}\beta)$ must be minimal in Θ_k by Lemma 4.4 there exists an $l \geq k$, such that $\sim\mathsf{C}\beta$ at a is redex of Θ_l. Then by the definition of deduction chains $d_{\Theta_{l+1}}(0) = \emptyset$.

<u>$m \to m+1$:</u> By the induction hypothesis there exists an $l' \geq k$, such that the set

$$d_{\Theta_{l'}}(m) := \{(\sim\mathsf{C}\beta, a); \Theta_{l'} \text{ contains } \sim\mathsf{C}\beta \text{ at } a \text{ and } ord_{(\Theta_{l'})_a}(\sim\mathsf{C}\beta) \leq m\}$$

is empty. Thus the set $d_{\Theta_{l'}}(m+1)$ contains only the pair $(\sim\mathsf{C}\gamma, a)$ such that $ord_{(\Theta_{l'})_a}(\sim\mathsf{C}\gamma) = m+1$. Since $ord_{(\Theta_{l'})_a}(\sim\mathsf{C}\gamma)$ is minimal in $\Theta_{l'}$ by Lemma 4.4 there exists an $l'' \geq l'$ such that $\sim\mathsf{C}\gamma$ at a is the redex of $\Theta_{l''}$. Therefore, again by the definition of deduction chains $d_{\Theta_{l''+1}}(m+1)$ must be empty.

Thus we have shown the claim for all $m \geq 0$. $\square$

Lemma 4.6 (Fairness) *If a reducible formula ϕ appears in Θ_k at $b \in I_k$, then there exists an $l \geq k$, such that ϕ is the redex of Θ_l at $b \in I_l$.*

PROOF. Due to the definition of redex, we must distinguish the following two cases:

Case 1) ϕ is not of the form $\sim\mathsf{C}\psi$: Then the claim follows by Lemma 4.3.
Case 2) ϕ is of the form $\sim\mathsf{C}\psi$: Then the claim follows by Lemma 4.5. □

Definition 4.7 *Define the Kripke structure $\mathsf{K}_\Theta = (S_\Theta, \mathcal{K}_1, \ldots, \mathcal{K}_n, \pi)$ as follows:*

(i) $S_\Theta := \bigcup I_i$
(ii) *for each $a \in S_\Theta$ define $B_a := \bigcup set(\mathbf{R}^i{}_a)$, where $\mathbf{R}^i := \Theta_i$*
(iii) *$\pi(\mathsf{p}) := \{a \in S_\Theta; \neg\mathsf{p} \in B_a\}$, for each atomic formula p*
(iv) *$\mathcal{K}_i := \sigma_i$, for each $i \in \{1, \ldots, n\}$*

We write $a \in \mathsf{K}_\Theta$ for $a \in S_\Theta$.

Lemma 4.8 (Saturation) *Let $a \in \mathsf{K}_\Theta$.*

(1) If $\phi \vee \psi \in B_a$, then $\phi \in B_a$ and $\psi \in B_a$
(2) If $\phi \wedge \psi \in B_a$, then $\phi \in B_a$ or $\psi \in B_a$
(3) If $\mathsf{K}_i\phi \in B_a$, then there exists a node $c \in \mathsf{K}_\Theta$, such that $a\mathcal{K}_i c$ and $\phi \in B_c$
(4) If $\sim\mathsf{K}_i\phi \in B_a$, then $\neg\phi \in B_c$ for all $c \in S_\Theta$ such that $a\mathcal{K}_i c$.
(5) If $\mathsf{E}^k\phi \in B_a$ for some $k \in \omega$, then there exists a $c \in S_\Theta$, reachable in k steps from a such that $\phi \in B_c$
(6) If $\neg\mathsf{E}^k\phi \in B_a$ for some $k \in \omega$, then $\neg\phi \in B_c$ for all $c \in S_\Theta$ reachable in k steps from a.
(7) If $\mathsf{C}\phi \in B_a$, then $\mathsf{E}^k\phi \in B_a$ for some $k \in \omega$
(8) If $\sim\mathsf{C}\phi \in B_a$, then $\neg\mathsf{E}^k\phi \in B_a$ for all $k \in \omega$

PROOF. All claims are consequences of Definition 3.12, Definition 4.7 and Lemma 4.6. □

Lemma 4.9 *For every formula $\phi \in \mathcal{L}^n_{\mathsf{C}}$ and every $a \in S_\Theta$*

(1) If $\phi \in B_a$, then $\mathsf{K}_\Theta, a \nvDash \phi$
(2) If $\neg\phi \in B_a$, then $\mathsf{K}_\Theta, a \vDash \phi$

PROOF. We prove the claims by induction on the structure of ϕ.

$\underline{\phi = \mathsf{p}:}$
(1): $\mathsf{p} \in B_a \overset{\text{Lemma 4.2}}{\Longrightarrow} \sim\mathsf{p} \notin B_a \implies a \notin \pi(\mathsf{p}) \implies \mathsf{K}_\Theta, a \nvDash \mathsf{p}$
(2): $\neg\mathsf{p} \in B_a \implies a \in \pi(\mathsf{p}) \implies \mathsf{K}_\Theta, a \vDash \mathsf{p}$
$\underline{\phi = \sim\mathsf{p}:}$ Dually to the previous case.
$\underline{\phi = \psi_1 \wedge \psi_2:}$
(1): $\psi_1 \wedge \psi_2 \in B_a \overset{\text{Lemma 4.8}}{\Longrightarrow} \psi_1 \in B_a$ or $\psi_2 \in B_a$
$\overset{\text{ind. hyp.}}{\Longrightarrow} \mathsf{K}_\Theta, a \nvDash \psi_1$ or $\mathsf{K}_\Theta, a \nvDash \psi_2 \implies \mathsf{K}_\Theta, a \nvDash \psi_1 \wedge \psi_2$

(2): $\neg(\psi_1 \wedge \psi_2) \in B_a \overset{\text{Lemma 4.8}}{\Longrightarrow} \neg\psi_1 \in B_a$ and $\neg\psi_2 \in B_a$
$\overset{\text{ind. hyp.}}{\Longrightarrow} \mathsf{K}_\Theta, a \models \psi_1$ and $\mathsf{K}_\Theta, a \models \psi_2 \implies \mathsf{K}_\Theta, a \models \psi_1 \wedge \psi_2$

$\underline{\phi = \psi_1 \vee \psi_2:}$ Dually to the previous case.

$\underline{\phi = \mathsf{K}_i\psi:}$

(1): If $\mathsf{K}_i\psi \in B_a$, then by Lemma 4.8 there exists a $c \in S_\Theta$ such that $a\mathcal{K}_ic$ and $\psi \in B_c$. Thus by induction hypothesis there exists a $c \in S_\Theta$ such that $a\mathcal{K}_ic$ and $\mathsf{K}_\Theta, c \not\models \psi$. Therefore $\mathsf{K}_\Theta, a \not\models \mathsf{K}_i\psi$

(2): If $\neg\mathsf{K}_i\psi \in B_a$, then by Lemma 4.8 $\sim\psi \in B_c$ for all $c \in S_\Theta$ such that $a\mathcal{K}_ic$. Thus by induction hypothesis $\mathsf{K}_\Theta, c \models \psi$ for all $c \in S_\Theta$ such that $a\mathcal{K}_ic$ and therefore $\mathsf{K}_\Theta, a \models \mathsf{K}_i\psi$.

$\underline{\phi = \sim\mathsf{K}_i\psi:}$

(1): If $\sim\mathsf{K}_i\psi \in B_a$, then by the previous case $\mathsf{K}_\Theta, a \models \mathsf{K}_i\psi$. Thus also $\mathsf{K}_\Theta, a \not\models \neg\mathsf{K}_i\psi$.

(2): $\neg\sim\mathsf{K}_i\psi$ is the formula $\mathsf{K}_i\psi$. Thus if $\neg\sim\mathsf{K}_i\psi \in B_a$, then by the previous case $\mathsf{K}_\Theta, a \not\models \psi$. Therefore $\mathsf{K}_\Theta, a \models \sim\mathsf{K}_i\psi$.

$\underline{\phi = \mathsf{C}\psi:}$

(1): If $\mathsf{C}\psi \in B_a$, then by Lemma 4.8 $\mathsf{E}^k\psi \in B_a$ for some $k \in \omega$. Then, again by Lemma 4.8 there exists a $c \in S_\Theta$ which is reachable from a in k steps and $\psi \in B_a$. Thus by induction hypothesis there exists a $c \in S_\Theta$ which is reachable from a in k steps and $\mathsf{K}_\Theta, c \not\models \psi$. Therefore $\mathsf{K}_\Theta, a \not\models \mathsf{E}^k\psi$ and thus also $\mathsf{K}_\Theta, a \not\models \mathsf{C}\psi$.

(2): If $\sim\mathsf{C}\psi \in B_a$, then by Lemma 4.8 $\neg\mathsf{E}^k\psi \in B_a$ for all $k \in \omega$. Thus by induction hypothesis $\mathsf{K}_\Theta, a \models \mathsf{E}^k\psi$ for all $k \in \omega$ and therefore $\mathsf{K}_\Theta, a \models \mathsf{C}\psi$.

$\underline{\phi = \sim\mathsf{C}\psi:}$

(1): If $\sim\mathsf{C}\psi \in B_a$, then by the previous case $\mathsf{K}_\Theta, a \models \mathsf{C}\psi$, thus trivially $\mathsf{K}_\Theta, a \not\models \neg\mathsf{C}\psi$.

(2): $\neg\sim\mathsf{C}\psi$ is the formula $\mathsf{C}\psi$. Thus by the previous case, if $\neg\sim\mathsf{C}\psi \in B_a$, then $\mathsf{K}_\Theta, a \not\models \mathsf{C}\psi$. Therefore, trivially $\mathsf{K}_\Theta, a \models \sim\mathsf{C}\psi$

This concludes the proof of (1) and (2) for all cases and thus the claim is shown. □

An immediate consequence of the previous lemma is the principle semantic lemma stated as follows.

Lemma 4.10 (Principle semantic lemma) *Let ϕ be a formula of $\mathcal{L}^n_{\mathsf{C}}$. If there exists a deduction chain of ϕ which does not end with an axiomatic sequence, then we can find a Kripke structure $\mathcal{M}$ and a world w such that $\mathcal{M}, w \not\models \phi$.*

5 Principal syntactic lemma

The principle syntactic lemma says that if all deduction chains for a formula ψ end in axiomatic sequence trees, then there exists a proof of ψ in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$. Hence, together with the principal semantic lemma we obtain either a proof or a countermodel for each formula ψ of $\mathcal{L}^n_{\mathsf{C}}$. This amounts to a (constructive) completeness result for $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$.

The principle syntactic lemma is proven along the following lines.

(1) Code each sequence tree $\mathbf{R}$ in the deduction tree (consisting of all deduction chains) of ψ as a set of formulae $C^{\mathbf{R}}$.
(2) Show that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{L}}$ for each leaf $\mathbf{L}$ of the deduction tree.
(3) Show by induction along the Kleene-Brouwer ordering of the deduction tree that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}$ if $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S_i}}$ for all successors $\mathbf{S_i}$ of $\mathbf{R}$.
(4) Finally, observe $C^{\mathbf{R}} = \psi$ for the root $\mathbf{R}$ of the deduction tree.

However, in order to prove step (3) of the above procedure, we need a series of lemmata. They state that (in certain cases) the rules of $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ may also be applied deep inside $\mathcal{L}^n_{\mathsf{C}}$ formulae. These lemmata are shown first.

Definition 5.1 *We extend the alphabet of the language $\mathcal{L}^n_{\mathsf{C}}$ by a propositional variable x. Let $\mathcal{L}^n_{\mathsf{C},\mathsf{x}}$ be the set of all formulae over this new alphabet. Let ϕ and ψ be formulae in $\mathcal{L}^n_{\mathsf{C},\mathsf{x}}$. $\phi[\psi]$ shall denote the formula which results from substituting all occurrences of x in ϕ with ψ. Furthermore, we define $\hat{\mathcal{L}}^n_{\mathsf{C}}$ to be the set of all formulae of $\mathcal{L}^n_{\mathsf{C}}$ which are of the form p, $\sim\mathsf{p}$, $\mathsf{K}_i\beta$, $\sim\mathsf{K}_i\beta$ or $\sim\mathsf{C}\beta$ for some β in $\mathcal{L}^n_{\mathsf{C}}$. Let dis$\hat{\mathcal{L}}^n_{\mathsf{C}}$ denote the set of disjunctions over elements of $\hat{\mathcal{L}}^n_{\mathsf{C}}$.*

Definition 5.2 *Let # denote the natural sum operation on ordinals. For all formulae $\alpha \in \mathcal{L}^n_{\mathsf{C}}$, we inductively define a complexity measure comp(α) as follows:*

1. *comp$(\alpha) = 1$ for all $\alpha \in \hat{\mathcal{L}}^n_{\mathsf{C}}$*
2. *comp$(\alpha \wedge \beta) = 1 \,\#\, \text{comp}(\alpha) \,\#\, \text{comp}(\beta)$*
3. *comp$(\alpha \vee \beta) = 1 \,\#\, \text{comp}(\alpha) \,\#\, \text{comp}(\beta)$*
4. *comp$(\mathsf{C}\alpha) = \omega^{\text{comp}(\alpha)}$*

Furthermore, given a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_l\} \subset \mathcal{L}^n_{\mathsf{C}}$, we define

$$\text{comp}(\Gamma) = \text{comp}(\gamma_1) \,\#\, \ldots \,\#\, \text{comp}(\gamma_l).$$

Remark 5.3 *By Definition 5.2 we have comp$(\mathsf{E}^k\xi) <$ comp$(\mathsf{C}\xi)$ for any formula ξ of $\mathcal{L}^n_{\mathsf{C}}$ and any $k \in \omega$. Furthermore, for any finite $\Gamma \subset \mathcal{L}^n_{\mathsf{C}}$ we have comp$(\Gamma) \geq |\Gamma|$. In particular, we have comp$(\Gamma) = |\Gamma|$ if $\Gamma \subset \hat{\mathcal{L}}^n_{\mathsf{C}}$.*

Definition 5.4 *We inductively define the subsets $\mathcal{A}_{\mathsf{x}}^k$ of $\mathcal{L}_{\mathsf{C},\mathsf{x}}^n$ as follows:*

$$\mathcal{A}_{\mathsf{x}}^0 := \{\phi \in \mathcal{L}_{\mathsf{C},\mathsf{x}}^n; \phi = \psi \vee \mathsf{x} \textit{ and } \psi \in \mathcal{L}_{\mathsf{C}}^n\}$$
$$\mathcal{A}_{\mathsf{x}}^{k+1} := \{\phi \in \mathcal{L}_{\mathsf{C},\mathsf{x}}^n; \phi = \psi \vee \mathsf{K}_i\delta[\mathsf{x}] \textit{ where } \psi \in dis\hat{\mathcal{L}}_{\mathsf{C}}^n \textit{ and } \delta[\mathsf{x}] \textit{ is in } \mathcal{A}_{\mathsf{x}}^k\}$$

Furthermore we define $\mathcal{A}_{\mathsf{x}}$ as $\bigcup \mathcal{A}_{\mathsf{x}}^k$ and for $\phi \in \mathcal{A}_{\mathsf{x}}$ depth(ϕ) as the least k, such that $\phi \in \mathcal{A}_{\mathsf{x}}^k$.

Lemma 5.5 *Let A be a formula in $\mathcal{A}_{\mathsf{x}}$ and Γ be a finite subset of $\hat{\mathcal{L}}_{\mathsf{C}}^n$. The following implications hold:*

1. *If $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\mathsf{E}^k\phi]$ for every $k \in \omega$, then $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\mathsf{C}\phi]$*
2. *If $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\phi]$ and $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\psi]$, then $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\phi \wedge \psi]$*
3. *If $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi]$ for some $k \in \omega$, then $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, A[\sim\mathsf{C}\phi]$*

PROOF. All three clauses are shown by induction on $d :=$ depth(A).
Clause 1: The base case of $d = 0$ follows directly by Lemma 2.3 and the rule (C^{ω}). We thus consider the induction step and assume that

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash_{\alpha_k} \Gamma, \psi \vee \mathsf{K}_i\delta[\mathsf{E}^k\phi]$$

for all $k \in \omega$ where depth$(\delta) = d$. Therefore, by iterated applications of Lemma 2.3 and the fact that $\psi \in dis\hat{\mathcal{L}}_{\mathsf{C}}^n$ we have

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash_{\alpha_k} \Gamma, \psi_1, \ldots, \psi_l, \mathsf{K}_i\delta[\mathsf{E}^k\phi] \tag{2}$$

for all $k \in \omega$ and suitable $\psi_1, \ldots, \psi_l$. We claim that

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, \psi_1, \ldots, \psi_l, \mathsf{K}_i\delta[\mathsf{C}\phi] \tag{3}$$

and distinguish two cases:

(i) For some $m \in \omega$ $\mathsf{K}_i\delta[\mathsf{E}^m\phi]$ was obtained by weakening in the derivation of (2), say after some $\beta_m \leq \alpha_m$.
(ii) For all $k \in \omega$ $\mathsf{K}_i\delta[\mathsf{E}^k\phi]$ was obtained by an application of the rule (K_i) in the derivation of (2), each one say after $\beta_k \leq \alpha_k$ respectively.

In case (i) we may instead conclude $\mathsf{K}_i\delta[\mathsf{C}\phi]$ after β_m and due to the fact that $\Gamma, \psi_1, \ldots, \psi_l \subset \hat{\mathcal{L}}_{\mathsf{C}}^n$ we may use the same inferences henceforth to conclude $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \Gamma, \psi_1, \ldots, \psi_l, \mathsf{K}_i\delta[\mathsf{C}\phi]$.
In case (ii) by the premise of the rule (K_i) we have for each $k \in \omega$

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}^{\omega} \vdash \neg\mathsf{C}\Delta_1^k, \neg\Delta_2^k, \delta[\mathsf{E}^k\phi] \tag{4}$$

where $\neg\mathsf{C}\Delta_1^k \subset \hat{\mathcal{L}}_{\mathsf{C}}^n$ and $\neg\Delta_2^k \subset \mathcal{L}_{\mathsf{C}}^n$ are suitable finite sets of formulae. Now define $\Gamma' := \Gamma, \psi_1, \ldots, \psi_l$, $\Gamma'|_{\neg\mathsf{C}} := \{\sim\mathsf{C}\xi \in \Gamma'\}$ and $\Gamma'|_{\neg\mathsf{K}_i} := \{\xi; \sim\mathsf{K}_i\xi \in \Gamma'\}$. By the fact that $\Gamma' \subset \hat{\mathcal{L}}_{\mathsf{C}}^n$ the following two statements hold for every $k \in \omega$:

$$\neg\mathsf{C}\Delta_1^k \subset \Gamma'|_{\neg\mathsf{C}} \tag{5}$$

$$\neg\Delta_2^k \subset \neg\Gamma'|_{\neg\mathsf{K}_i} \tag{6}$$

Clearly, we also have

$$\Gamma'|_{\neg\mathsf{C}} \subset \Gamma' \tag{7}$$

$$\neg\mathsf{K}_i\Gamma'|_{\neg\mathsf{K}_i} \subset \Gamma' \tag{8}$$

By Lemma 2.2, (4), (5) and (6) we get

$$\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \neg\Gamma'|_{\neg\mathsf{K}_i}, \delta[\mathsf{E}^k\phi] \tag{9}$$

for every $k \in \omega$. We show that

$$\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \neg\Gamma'|_{\neg\mathsf{K}_i}, \delta[\mathsf{C}\phi] \tag{10}$$

by induction on $\gamma := \mathrm{comp}(\neg\Gamma'|_{\neg\mathsf{K}_i})$. As the base case we have $\gamma = |\neg\Gamma'|_{\neg\mathsf{K}_i}|$ by Remark 5.3. But in this case $\neg\Gamma'|_{\neg\mathsf{K}_i}$ is either empty or a subset of $\hat{\mathcal{L}}_{\mathsf{C}}^n$. Therefore, the claim follows by induction hypothesis of the outer induction. Now assume that the claim holds for all $\gamma' < \gamma$. Then there exists a set $\Sigma \subset \mathcal{L}_{\mathsf{C}}^n$ and formulae ξ_1, ξ_2, ξ such that one of the following three cases holds

(a) $\Sigma, \xi_1 \wedge \xi_2 = \neg\Gamma'|_{\neg\mathsf{K}_i}$ and $\mathrm{comp}(\xi_1), \mathrm{comp}(\xi_2) < \mathrm{comp}(\xi_1 \wedge \xi_2)$
(b) $\Sigma, \xi_1 \vee \xi_2 = \neg\Gamma'|_{\neg\mathsf{K}_i}$ and $\mathrm{comp}(\xi_1), \mathrm{comp}(\xi_2) < \mathrm{comp}(\xi_1 \vee \xi_2)$
(c) $\Sigma, \mathsf{C}\xi = \neg\Gamma'|_{\neg\mathsf{K}_i}$ and by Remark 5.3 $\mathrm{comp}(\mathsf{E}^k\xi) < \mathrm{comp}(\mathsf{C}\xi)$ for all $k \in \omega$.

Case (a): By (9) and Lemma 2.3 we have

$$\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \Sigma, \xi_1, \delta[\mathsf{E}^k\phi] \text{ and}$$
$$\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \Sigma, \xi_2, \delta[\mathsf{E}^k\phi]$$

for all $k \in \omega$. Thus by the induction hypothesis of the inner induction

$$\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \Sigma, \xi_1, \delta[\mathsf{C}\phi] \text{ and}$$
$$\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \Sigma, \xi_2, \delta[\mathsf{C}\phi]$$

and again by the rule ($\wedge$) we obtain the claim.
Case (b) and case (c) are treated in analogous ways, using Lemma 2.3. From (10) using (K_i), we obtain $\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma'|_{\neg\mathsf{C}}, \neg\mathsf{K}_i\Gamma'|_{\neg\mathsf{K}_i}, \mathsf{K}_i\delta[\mathsf{C}\phi]$. With (7), (8) and Lemma 2.2 we conclude $\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma', \mathsf{K}_i\delta[\mathsf{C}\phi]$. Thus (3) holds in both cases (i) and (ii). Then, by an iterated application of ($\vee$) $\mathsf{T}^{\omega}_{\mathsf{K}_n^{\mathsf{C}}} \vdash \Gamma, A[\mathsf{C}\phi]$ follows and this clause is shown.

Clause 2: The base case of $d = 0$ follows by Lemma 2.3, the rule $(\wedge)$ and finally an application of the rule $(\vee)$. The induction step is analogous to clause 1 only that in this case we are dealing with just two premises instead of infinitely many.

Clause 3: The base case of $d = 0$ follows by Lemmata 2.3 and 2.4. We therefore consider the induction step and assume that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \mid\!\frac{}{\alpha}\, \Gamma, \psi \vee \mathsf{K}_i\delta[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi]$, where $\text{depth}(\delta) = d$. Therefore, by iterated applications of Lemma 2.3 and the fact that $\psi \in dis\hat{\mathcal{L}}^n_{\mathsf{C}}$ we have

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \mid\!\frac{}{\alpha}\, \Gamma, \psi_1, \ldots, \psi_l, \mathsf{K}_i\delta[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi] \tag{11}$$

For suitable $\psi_1, \ldots, \psi_l$. We claim that

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, \psi_1, \ldots, \psi_l, \mathsf{K}_i\delta[\sim\mathsf{C}\phi] \tag{12}$$

and distinguish two cases:

(i) $\mathsf{K}_i\delta[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi]$ was introduced by weakening in the derivation of (11), say after some $\beta < \alpha$.

(ii) $\mathsf{K}_i\delta[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi]$ was obtained by the rule (K_i) in the derivation of (11).

In case (i) we may instead introduce $\mathsf{K}_i\delta[\sim\mathsf{C}\phi]$ with weakening after β and due to the fact that $\Gamma, \psi_1, \ldots, \psi_l \subset \hat{\mathcal{L}}^n_{\mathsf{C}}$ we may use the same inferences henceforth to conclude the claim. In case (ii) we have $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \neg\mathsf{C}\Delta_1, \neg\Delta_2, \delta[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi]$ for suitable sets Δ_1 and Δ_2. Then by induction hypothesis and an identical argument to the corresponding case in clause 1 we obtain

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \neg\mathsf{C}\Delta_1, \neg\Delta_2, \delta[\sim\mathsf{C}\phi].$$

The rule (K_i) yields $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \neg\mathsf{C}\Delta_1, \neg\mathsf{K}_i\Delta_2, \mathsf{K}_i\delta[\sim\mathsf{C}\phi]$ Then by the fact that $\Gamma, \psi_1, \ldots, \psi_l \subset \hat{\mathcal{L}}^n_{\mathsf{C}}$, we may use the same inferences again to arrive at the claim. Thus (12) holds in both cases (i) and (ii). Therefore, by an iteration of the rule $(\vee)$ we arrive at $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, A[\sim\mathsf{C}\phi]$ and the clause is shown. □

Definition 5.6 *Let $\psi_1, \ldots, \psi_l$ be formulae of $\mathcal{L}^n_{\mathsf{C}}$. We inductively define the subsets $\mathcal{B}^k_{\mathsf{x},\psi_1,\ldots,\psi_l}$ of $\mathcal{L}^n_{\mathsf{C},\mathsf{x}}$ as follows:*

$$\mathcal{B}^1_{\mathsf{x},\psi_1,\ldots,\psi_l} := \{\phi \in \mathcal{L}^n_{\mathsf{C},\mathsf{x}}; \phi = \psi \vee \neg\mathsf{K}_i\psi_1 \vee \ldots \vee \neg\mathsf{K}_i\psi_l \vee \mathsf{K}_i\mathsf{x} \text{ and } \psi \in \mathcal{L}^n_{\mathsf{C}}\}$$
$$\mathcal{B}^{k+1}_{\mathsf{x},\psi_1,\ldots,\psi_l} := \{\phi \in \mathcal{L}^n_{\mathsf{C},\mathsf{x}}; \phi = \psi \vee \mathsf{K}_i\delta[\mathsf{x}] \text{ where } \psi \in dis\hat{\mathcal{L}}^n_{\mathsf{C}} \text{ and } \delta[\mathsf{x}] \text{ is in } \mathcal{B}^k_{\mathsf{x},\psi_1,\ldots,\psi_l}\}$$

Furthermore we define $\mathcal{B}_{\mathsf{x},\psi_1,\ldots,\psi_l}$ as $\bigcup \mathcal{B}^k_{\mathsf{x},\psi_1,\ldots,\psi_l}$ and for $\phi \in \mathcal{B}_{\mathsf{x},\psi_1,\ldots,\psi_l}$ depth(ϕ) as the least k, such that $\phi \in \mathcal{B}^k_{\mathsf{x},\psi_1,\ldots,\psi_l}$.

Lemma 5.7 *Let B be a formula in $\mathcal{B}_{\mathsf{x},\psi_1,\dots,\psi_l}$ and Γ be a finite subset of $\hat{\mathcal{L}}^n_{\mathsf{C}}$. If $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, B[\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l]$, then $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, B[\phi]$.*

PROOF. We prove this claim by induction on $d := \text{depth}(B)$.

$\underline{d = 1:}$ We thus have $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha}} \Gamma, \psi \vee \neg\mathsf{K}_i\psi_1 \vee \dots \vee \neg\mathsf{K}_i\psi_l \vee \mathsf{K}_i(\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l)$ and with iterated applications of Lemma 2.3

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha}} \Gamma, \psi, \neg\mathsf{K}_i\psi_1, \dots, \neg\mathsf{K}_i\psi_l, \mathsf{K}_i(\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l). \tag{13}$$

We show that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, \psi, \neg\mathsf{K}_i\psi_1, \dots, \neg\mathsf{K}_i\psi_l, \mathsf{K}_i\phi$ by induction on α. The base case of $\alpha = 0$ is trivial. Therefore, we assume that the claim holds for all $\alpha' < \alpha$ and distinguish cases, as to whether or not $\mathsf{K}_i(\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l)$ was the distinguished formula of the last inference used to derive (13). If it was the distinguished formula, then we have

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Delta, \neg\psi_1, \dots, \neg\psi_l, \phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l$$

for some suitable set Δ. Hence, with an iteration of Lemma 2.3 we obtain $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Delta, \neg\psi_1, \dots, \neg\psi_l, \phi$ and thus applying (K_i) we arrive at the claim. If $\mathsf{K}_i(\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l)$ was not the distinguished formula, then we distinguish further cases for the last rule applied to obtain (13). In the cases of the rules $(\wedge)$, $(\vee)$, (C^{ω}) and $(\neg\mathsf{C})$ we simply use the induction hypothesis of the inner induction on the premise and apply the same rule again. In the case of rule (K_j) (for any $1 \leq j \leq n$) we see that $\mathsf{K}_i(\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l)$ can only have been obtained with weakening. Thus we may obtain $\mathsf{K}_i\phi$ instead in the same manner.

$\underline{d \to d+1:}$ Thus $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha}} \Gamma, \psi \vee \mathsf{K}_i\delta[\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l]$ and by iteration of Lemma 2.3

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash_{\overline{\alpha}} \Gamma, \psi_1, \dots, \psi_l, \mathsf{K}_i\delta[\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l] \tag{14}$$

for suitable $\psi_1, \dots, \psi_l$. We claim that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, \psi_1, \dots, \psi_l, \mathsf{K}_i\delta[\phi]$ and again distinguish two cases:

(i) $\mathsf{K}_i\delta[\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l]$ was obtained by weakening in the derivation of (14)

(ii) $\mathsf{K}_i\delta[\phi \vee \neg\psi_1 \vee \dots \vee \neg\psi_l]$ was obtained by the rule (K_i) in the derivation of (14).

In both cases we may show the claim as before using the fact that $\Gamma, \psi_1, \dots, \psi_l \subset \hat{\mathcal{L}}^n_{\mathsf{C}}$. Then by an iteration of the rule $(\vee)$, we arrive at $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \Gamma, A[\phi]$ and the Lemma is shown.

□

Definition 5.8 *Let ψ_1 be a formula of $\mathcal{L}^n_{\mathsf{C}}$. We inductively define the subsets $\mathcal{C}^k_{\mathsf{x},\psi_1}$ of $\mathcal{L}^n_{\mathsf{C},\mathsf{x}}$ as follows:*

$\mathcal{C}^1_{\mathsf{x},\psi_1} := \{\phi \in \mathcal{L}^n_{\mathsf{C},\mathsf{x}}; \phi = \psi \vee \neg\mathsf{K}_i\psi_1 \vee \mathsf{K}_i(\mathsf{x} \vee \alpha_1) \vee \ldots \vee \mathsf{K}_i(\mathsf{x} \vee \alpha_p)$ *and* $\psi, \alpha_1, \ldots, \alpha_p \in \mathcal{L}^n_{\mathsf{C}}\}$

$\mathcal{C}^{k+1}_{\mathsf{x},\psi_1} := \{\phi \in \mathcal{L}^n_{\mathsf{C},\mathsf{x}}; \phi = \psi \vee \mathsf{K}_i\delta[\mathsf{x}]$ *where* $\psi \in dis\hat{\mathcal{L}}^n_{\mathsf{C}}$ *and* $\delta[\mathsf{x}]$ *is in* $\mathcal{C}^k_{\mathsf{x},\psi_1}\}$

Furthermore we define $\mathcal{C}_{\mathsf{x},\psi_1}$ *as* $\bigcup \mathcal{C}^k_{\mathsf{x},\psi_1}$ *and for* $\phi \in \mathcal{C}_{\mathsf{x},\psi_1}$ *depth*(ϕ) *as the least* k*, such that* $\phi \in \mathcal{C}^k_{\mathsf{x},\psi_1}$*.*

Lemma 5.9 *Let* C *be a formula in* $\mathcal{C}_{\mathsf{x},\phi}$ *and* Γ *be a finite subset of* $\hat{\mathcal{L}}^n_{\mathsf{C}}$*.* $\overline{C}$ *denotes the formula of* $\mathcal{L}^n_{\mathsf{C}}$ *which results from erasing every disjunct of the form* x *in* C*. If* $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash \Gamma, C[\neg\phi]$*, then* $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash \Gamma, \overline{C}$

PROOF. We prove this claim by induction on $d :=$ depth(C).

<u>$d = 1$:</u> Thus $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \psi \vee \neg\mathsf{K}_i\phi \vee \mathsf{K}_i(\neg\phi \vee \alpha_1) \vee \ldots \vee \mathsf{K}_i(\neg\phi \vee \alpha_p)$ and by repeated applications of Lemma 2.3

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash_{\overline{\alpha}} \Gamma, \psi, \neg\mathsf{K}_i\phi, \mathsf{K}_i(\neg\phi \vee \alpha_1), \ldots, \mathsf{K}_i(\neg\phi \vee \alpha_p) \tag{15}$$

We claim that $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash \Gamma, \psi, \neg\mathsf{K}_i\phi, \mathsf{K}_i\alpha_1, \ldots, \mathsf{K}_i\alpha_p$ by induction on α. The base case of $\alpha = 0$ is trivial. Thus we assume that the claim holds for all $\alpha' < \alpha$ and make a case distinction as to whether or not $\mathsf{K}_i(\neg\phi \vee \alpha_j)$ was the distinguished formula of the last inference used to derive (15) for any $1 \leq j \leq l$. In the first case we then have $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash \Delta, \neg\phi, \neg\phi \vee \alpha_j$ and thus with Lemma 2.3 $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_n} \vdash \Delta, \neg\phi, \alpha_j$. Therefore, using (K_i) we obtain the claim. If $\mathsf{K}_i(\neg\phi \vee \alpha_j)$ was not the distinguished formula for any $1 \leq j \leq l$, then we distinguish further cases for the last rule applied to obtain (15). In the cases of the rules $(\wedge)$, $(\vee)$, (C^ω) and $(\neg\mathsf{C})$ we simply use the induction hypothesis of the inner induction on the premise and apply the same rule again. In the case of rule (K_h) (for any $1 \leq h \leq n$) we see that for every $1 \leq j \leq l$ $\mathsf{K}_i(\neg\phi \vee \alpha_j)$ can only have been obtained with weakening. Thus we may obtain $\mathsf{K}_i\alpha_j$ for every $1 \leq j \leq l$ in the same manner.

<u>$d \to d + 1$:</u> This part of the induction is analogous to the corresponding part in the proof of Lemma 5.7.

□

Definition 5.10 *Let* $\mathbf{R}$ *be a sequence tree over* I *and* $a = (l, \alpha) \in I$*. We define the characteristic set* $C^{\mathbf{R}}_a$ *of* $\mathbf{R}$ *at* a *inductively as follows:*

(1) If a *is a leaf of* I*, then* $C^{\mathbf{R}}_a :=$ *set*$^+(\mathbf{R}_a)$

(2) If a *has successors* $b_1, \ldots, b_m \in I$ *and*

$$b_1 = (p_1, (\alpha, q_1))$$
$$\vdots \quad \vdots$$
$$b_m = (p_m, (\alpha, q_m)),$$

then $C^{\mathbf{R}}_a := set^+(\mathbf{R}_a) \cup \{\mathsf{K}_{p_1} \bigvee C^{\mathbf{R}}_{b_1}\} \cup \ldots \cup \{\mathsf{K}_{p_m} \bigvee C^{\mathbf{R}}_{b_m}\}$.

Lemma 5.11 *If* $\mathbf{R}$ *is an axiomatic sequence tree over* I*, then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. Since $\mathbf{R}$ is axiomatic, there exists a $c \in I$ and some atomic formula p, such that p and $\sim\mathsf{p}$ are both in $C^{\mathbf{R}}_c$. Thus using (ID) we obtain $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_c$. We show that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_b$ for all $b \preccurlyeq c$ by induction inverse to the length of c.

<u>$b = c$:</u> This case is already shown above.

<u>$b \prec c$:</u> Let $b = (k, \beta)$. Then there exists a $d \preccurlyeq c$ such that $d = (i, (\beta, l))$ for some natural numbers i and l. By induction hypothesis $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_d$, thus an iteration of applications of ($\vee$) yields $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \bigvee C^{\mathbf{R}}_d$. Then, applying ($\mathsf{K}_i$), we obtain $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_b$.

Thus the claim holds and since $(0, (0)) \preccurlyeq c$ the Lemma is shown. □

Lemma 5.12 *Let* $\mathbf{R}$ *be a sequence tree with redex* $\phi \vee \psi$ *and* $\mathbf{S}$ *be the type 1 successor of* $\mathbf{R}$*. If* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}}_{(0,(0))}$*, then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. This claim trivially holds since $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}}_{(0,(0))}$ and $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$ are the same set of formulae. □

Lemma 5.13 *Let* $\mathbf{R}$ *be a sequence tree with redex* $\phi \wedge \psi$ *and* $\mathbf{S}, \mathbf{T}$ *be the type 2 successors of* $\mathbf{R}$*. If* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}}_{(0,(0))}$ *and* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{T}}_{(0,(0))}$*, then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. There exists a formula $A \in \mathcal{A}_{\mathsf{x}}$, such that $A[\phi] = \bigvee C^{\mathbf{S}}_{(0,(0))}$ and $A[\psi] = \bigvee C^{\mathbf{T}}_{(0,(0))}$ as well as $A[\phi \wedge \psi] = \bigvee C^{\mathbf{R}}_{(0,(0))}$. Therefore, the claim holds by clause 2 of Lemma 5.5 and iterations of Lemma 2.3. □

Lemma 5.14 *Let* $\mathbf{R}$ *be a sequence tree with redex* $\sim\mathsf{K}_i\phi$ *and* $\mathbf{S}$ *be the type 3 successor of* $\mathbf{R}$*. If* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}}_{(0,(0))}$*, then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. Since $\mathbf{S}$ is the type 3 successor of a sequence tree with redex $\sim\mathsf{K}_i\phi$, there exists a formula $C \in \mathcal{C}_{\mathsf{x},\phi}$ such that $C[\neg\phi] = \bigvee C^{\mathbf{S}}_{(0,(0))}$ and $\overline{C} = \bigvee C^{\mathbf{R}}_{(0,(0))}$. Therefore, the claim holds by Lemma 5.9 and iterations of Lemma 2.3. □

Lemma 5.15 *Let* $\mathbf{R}$ *be a sequence tree with redex* $\mathsf{K}_i\phi$ *and* $\mathbf{S}$ *be the type 4 successor of* $\mathbf{R}$. *If* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}}_{(0,(0))}$, *then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. Since $\mathbf{S}$ is the type 4 successor of a sequence tree with redex $\mathsf{K}_i\phi$, there exist formulae $\psi_1, \ldots, \psi_l$ and a formula $B \in \mathcal{B}_{\mathsf{x},\psi_1,\ldots,\psi_l}$ such that

$$B[\phi \vee \neg\psi_1 \vee \ldots \vee \neg\psi_l] = \bigvee C^{\mathbf{S}}_{(0,(0))}$$

and $B[\phi] = \bigvee C^{\mathbf{R}}_{(0,(0))}$. Therefore, the claim holds by Lemma 5.7 and iterations of Lemma 2.3. □

Lemma 5.16 *Let* $\mathbf{R}$ *be a sequence tree with redex* $\mathsf{C}\phi$ *and* $\mathbf{S}^i$ *where* $i \in \omega$ *be the type 5 successors of* $\mathbf{R}$. *If* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}^i}_{(0,(0))}$ *for all* $i \in \omega$, *then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. There exists a formula $A \in \mathcal{A}_{\mathsf{x}}$, such that $A[\mathsf{E}^k\phi] = \bigvee C^{\mathbf{S}^k}_{(0,(0))}$ and $A[\mathsf{C}\phi] = \bigvee C^{\mathbf{R}}_{(0,(0))}$. Therefore, the claim holds by clause 1 of Lemma 5.5 and iterations of Lemma 2.3. □

Lemma 5.17 *Let* $\mathbf{R}$ *be a sequence tree with redex* $\sim\mathsf{C}\phi$ *and* $\mathbf{S}$ *be the type 6 successor of* $\mathbf{R}$. *If* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{S}}_{(0,(0))}$, *then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. There exists a formula $A \in \mathcal{A}_{\mathsf{x}}$, such that

$$A[\sim\mathsf{C}\phi \vee \neg\mathsf{E}^k\phi] = \bigvee C^{\mathbf{S}}_{(0,(0))}$$

for some $k \in \omega$ and $A[\sim\mathsf{C}\phi] = \bigvee C^{\mathbf{R}}_{(0,(0))}$. Therefore, the claim holds by clause 3 of Lemma 5.5 and iterations of Lemma 2.3. □

Definition 5.18 *Let* $\mathbf{R}$ *be a sequence tree. The deduction tree of* $\mathbf{R}$ *denoted by* $\mathbb{DT}(\mathbf{R})$ *is the set of all deduction chains of* $\mathbf{R}$, *closed under initial segments. For* $\Theta, \Theta' \in \mathbb{DT}(\mathbf{R})$ *we say* $\Theta \lhd \Theta'$ *if and only if* Θ *is a proper initial segment of* Θ'. *For all finite* $\Theta \in \mathbb{DT}(\mathbf{R})$ *we define last*(Θ) *to be the last sequence tree in* Θ.

In order to establish the principal syntactic lemma we require the following consequence of a standard result about the Kleene-Brouwer ordering on a wellfounded tree. Proofs of this result may be found in [4] (Corollary 5.4.18) and [13] (Lemma V.1.3).

Lemma 5.19 *Let* $\mathbf{R}$ *be a sequence tree. If the deduction tree* $\mathbb{DT}(\mathbf{R})$ *contains only finite deduction chains, then there exists an ordinal* α *and a bijective function* $\mathsf{f} : \alpha + 1 \to \mathbb{DT}(\mathbf{R})$, *such that for all ordinals* $\beta, \gamma \leq \alpha$

$$\mathsf{f}(\beta) \lhd \mathsf{f}(\gamma) \implies \gamma < \beta.$$

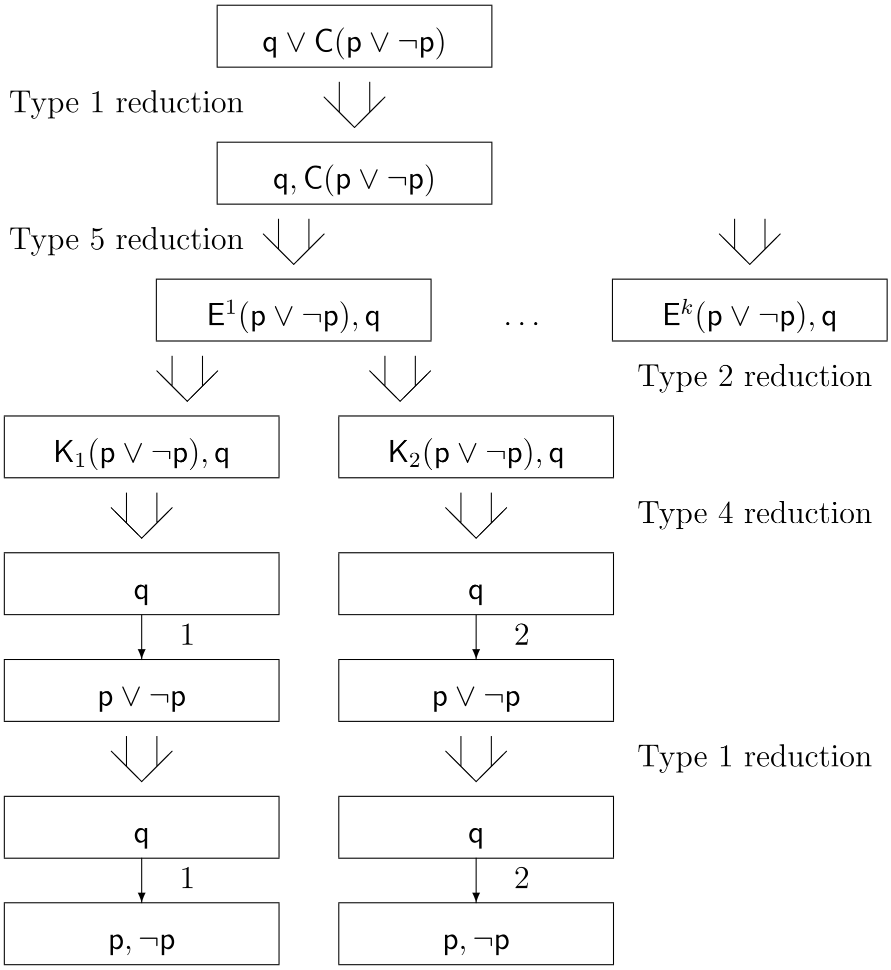


Fig. 8. Example deduction tree

Lemma 5.20 (Principle syntactic lemma) *If every deduction chain of* $\mathbf{R}$ *ends with an axiomatic sequence tree, then* $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{n}} \vdash C^{\mathbf{R}}_{(0,(0))}$.

PROOF. By assumption the deduction tree $\mathbb{DT}(\mathbf{R})$ contains only finite deduction chains. Thus we may apply Lemma 5.19 to obtain a function f and an ordinal α with the described properties. It suffices to show

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{n}} \vdash C^{\text{last}(\mathsf{f}(\beta))}_{(0,(0))} \tag{16}$$

for all $\beta \leq \alpha$, since $\text{last}(\mathsf{f}(\alpha)) = \mathbf{R}$. We prove (16) by transfinite induction on β.

$\underline{\beta = 0}$: By Lemma 5.19 we find that $\mathsf{f}(\beta)$ must be $\sqsubset$-maximal. Thus by assumption $\text{last}(\mathsf{f}(\beta))$ is axiomatic and the claim follows by Lemma 5.11.

$\underline{\text{(16) holds for all } \hat{\beta} < \beta}$: If $\text{last}(\mathsf{f}(\beta))$ is axiomatic, then the claim holds again by Lemma 5.11. Otherwise $\text{last}(\mathsf{f}(\beta))$ has a redex ϕ. We distinguish between the different possibilities for ϕ and use Lemmata 5.12 – 5.17. The case of $\phi = \psi_1 \vee \psi_2$ is given as an example. In this case there exists an $\mathsf{f}(\gamma)$, such that $\text{last}(\mathsf{f}(\gamma))$ is the type 1 successor of $\text{last}(\mathsf{f}(\beta))$, thus $\mathsf{f}(\beta) \lhd \mathsf{f}(\gamma)$. By Lemma 5.19 we have $\gamma < \beta$ and by induction hypothesis $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\text{last}(\mathsf{f}(\gamma))}_{(0,(0))}$. Therefore applying Lemma 5.12 yields $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash C^{\text{last}(\mathsf{f}(\beta))}_{(0,(0))}$. The other cases are treated analogously using the induction hypothesis and applications of Lemmata 5.13 – 5.17.

Thus (16) holds for all $\beta \leq \alpha$ and the claim is shown. □

Combining the principle semantic lemma and the principle syntactic lemma yields completeness for $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$.

Corollary 5.21 (Completeness) *Let ϕ be a formula of $\mathcal{L}^n_{\mathsf{C}}$. If for all Kripke structures $\mathcal{M}$ and all worlds $w \in \mathcal{M}$ we have that $\mathcal{M}, w \models \phi$, then $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \vdash \phi$.*

PROOF. Assume we had $\mathcal{M}, w \models \phi$ for all Kripke structures $\mathcal{M}$ and all worlds $w \in \mathcal{M}$ and ϕ were not provable in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$. By contraposition of the principal syntactic lemma there would need to exist a deduction chain of ϕ which is infinite or ends non-axiomatically. But in this case the principal semantic lemma would supply us with a countermodel for ϕ, contradicting our assumption. Thus ϕ must be provable in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ and indeed the principal syntactic lemma constructs such a proof. □

6 Conclusion

In the current study we have given a syntactic method for proving completeness of the infinitary system $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ as is stated more precisely in Corollary 5.21. In the case of a valid formula ϕ, a proof of ϕ in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ may be reconstructed from the principal syntactic lemma along with Lemmata 5.11 to 5.17 and thus, in this sense, our method is constructive. However, our analysis does not yet provide us with any statements about the length of canonical proofs for valid formulae let alone about whether such proofs are optimal in length. On the semantic side our method also behaves constructively to the extent of providing canonical countermodels for non-valid formulae. This is guaranteed by the principal semantic lemma. It is known from [3] that Logic of Common Knowledge possesses a strong form of the finite model property where the size of a countermodel for a non-valid formula ϕ may be bounded exponentially in the length of ϕ. Currently this result is not reflected in the canonical countermodels constructed by our method, but further refinements should ultimatively lead to the construction of size-optimal countermodels. As mentioned before, the main contribution of this study is the extension of the deduction chain method to Logic of Common Knowledge. In a next step the method could be adapted to other more expressive modal logics with fixed points as well as the modal μ-calculus in its general form [7] and thus contribute to a better proof-theoretical understanding of the area in particular with respect to systematic proof-search and syntactic decision procedures.

An approach similar to the one presented here has recently been undertaken by Tanaka [15] in the framework of predicate common knowledge logic. Let us briefly compare the two studies. Tanaka investigates proof systems for CKL, the predicate common knowledge logic for Kripke frames with constant domain. He introduces an infinitary cut-free deductive system for CKL and proves a completeness theorem about it. Like in our system $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$, Tanaka's rule for introducing the common knowledge operator has infinitely many premises. His deductive system is a kind of tree sequent calculus. That means his system does not derive (sets of) formulae but so-called tree sequents which are finite trees where each node is a sequent and the edges are labeled by symbols for the agents. A formula ϕ is called derivable if the tree sequent which consists only of the root node $\vdash \phi$ is derivable.

There is a relation between Tanaka's approach and the method of deduction chains: the rules of his calculus correspond to the conditions we impose on deduction chains. Hence, a branch of a derivation in Tanaka's system corresponds to a deduction chain in our approach. In order to prove completeness, he only needs to show the analogue of our principal semantic lemma: given a non-derivable tree sequent, it is possible to construct a countermodel. Since we work in the Tait-style system $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$ which derives sets of formulae and not tree sequents, we also need the principal syntactic lemma. This lemma states that if every deduction chain of a formula ϕ ends axiomatically, then it is provable in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$. That could be translated into something like if ϕ is derivable in Tanaka's system, then it is provable in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n}$.

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