

Axiomatizing Eventual Common Knowledge

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Abstract. Common knowledge plays a central role in modeling coordination in multi-agent systems, yet classical results show that even arbitrarily small temporal uncertainty prevents it from ever being attained in practical systems. This has motivated a range of approximations to common knowledge, including ε -common knowledge and eventual common knowledge, which relax the requirement that all levels of mutual knowledge hold simultaneously. These notions aim to retain the logical structure of common knowledge while making it compatible with temporal imprecision. In this paper, we investigate the logic of eventual common knowledge and provide a sound and complete axiomatization.

Keywords: Eventual common knowledge · Fixed point · Axiomatization.

1 Introduction

Beyond its foundational logical interest, common knowledge plays a central role in modeling coordination and agreement in multi-agent systems [4,7,12]. In their seminal paper [7], Halpern and Moses showed that common knowledge is a necessary condition for certain forms of coordinated action in distributed environments. A key insight of their analysis is that common knowledge is extremely fragile: even arbitrarily small uncertainty about message delivery times or clock synchronization suffices to prevent agents from ever attaining common knowledge. As a result, in practical systems with temporal imprecision, facts that intuitively ought to become “publicly known” never formally become common knowledge. This phenomenon—often referred to as the Halpern–Moses paradox [13,6]—highlights that the standard notion of common knowledge implicitly relies on idealized assumptions about simultaneity and timing.

In many settings, coordination requirements are inherently approximate, and small temporal discrepancies in agents’ higher-order knowledge are inconsequential for successful joint action. Relaxing the simultaneity requirement therefore yields epistemic notions that are both attainable under temporal uncertainty and sufficient for the forms of coordination that arise in practice. This has motivated a line of work on *approximations* to common knowledge that weaken the classical definition accordingly. Examples include *eventual common knowledge*, which is studied in this paper, as well as ε -common knowledge [7,4], *probabilistic common knowledge* [2,9], and *concurrent common knowledge* [14].

Notions of this kind are employed in particular for modelling applications in distributed computing. The Firing Rebels with Relay problem [5], which is a simplified version of the consistent broadcasting primitive introduced by Srikanth and Toueg in [15], is closely related to eventual common hope, an epistemic attitude that lies in a certain sense between belief and knowledge. Distributed ledger protocols, such as Bitcoin, have been analyzed using Δ - \square -common knowledge [8] and probabilistic common knowledge [10]. Related epistemic analyses of blockchain protocols are, *e.g.*, [3,11].

Just as common knowledge is defined as a fixed point of the mutual knowledge operator, eventual common knowledge is defined as a fixed point of the *eventual mutual knowledge* operator. A group G of agents has eventual mutual knowledge of a fact φ , and we say that $E_G^\diamond \varphi$ holds, if for each agent i there exists some time at which i knows φ . In contrast to standard mutual knowledge,

no simultaneity requirement is imposed: the agents' knowledge of φ need not hold at the same point in time. While common knowledge and eventual common knowledge are defined analogously as fixed points, it turns out that they behave differently when viewed as infinite conjunctions. In particular, our result shows that eventual common knowledge is strictly stronger than the infinite conjunction of iterated eventual mutual knowledge operators.

Since our main goal here is to axiomatize eventual common knowledge, we work in a simplified framework compared to the runs-and-systems setting in which the operator was originally introduced [4,7]. In particular, in order to apply the standard completeness proof via canonical model construction, we use Kripke models with an epistemic accessibility relation for each agent and a temporal relation, rather than interpreted systems. This simplification is primarily technical and preserves the essential behavior of the original operator. A somewhat surprising feature of our framework is that we model time as an equivalence relation. This choice is motivated by the semantics of eventual mutual knowledge in the runs-and-systems framework. There, a system is represented as a non-empty set of runs, where each run r is a function mapping each point m in time to a global state of the system, and formulas are evaluated at points (r, m) . Eventual mutual knowledge is interpreted as follows:

$$\mathcal{I}, r, m \models E_G^\diamond \varphi \text{ iff } \forall i \in G, \exists m_i \text{ such that } \mathcal{I}, r, m_i \models K_i \varphi.$$

In this definition, the quantification ranges over points within a single run, regardless of their temporal order. Therefore, modeling time as an equivalence relation preserves the distinctions that are relevant to eventual common knowledge.

Paper organization: In Section 2.1, we introduce the formal languages as well as the relevant class of Kripke models. We then present the semantics in Sections 2.2 and investigate a few expected validities in Section 2.3. In Section 2.4, we study approximations of eventual common knowledge. We present a sound and complete axiomatization in Section 3 and conclude with further discussions in Section 4.

2 Preliminaries

2.1 Languages and Models

Throughout this paper, we consider a set $\mathcal{A} = \{1, \dots, n\}$ of agents and a non-empty countable set Prop of atomic propositions. We define the languages \mathcal{L}_C and \mathcal{L}_C^x .

Definition 1 (Languages). *We define the language \mathcal{L}_C by the grammar*

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \wedge \varphi) \mid K_a \varphi \mid \Box \varphi \mid C_G^\diamond \varphi$$

where $a \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$, and $p \in \text{Prop}$. Considering a single variable x , we also define the language \mathcal{L}_C^x by the grammar

$$\psi ::= \varphi \mid x \mid \neg \psi \mid (\psi \wedge \psi) \mid K_a \psi \mid \Box \psi$$

where $a \in \mathcal{A}$ and $\varphi \in \mathcal{L}_C$. In addition to the logical constants and propositional connectives expressed via standard notational abbreviations, we also use $\diamond \psi := \neg \Box \neg \psi$, $E_G^\diamond \psi := \bigwedge_{a \in G} \diamond K_a \psi$, $(E_G^\diamond)^0 \psi := \psi$, and $(E_G^\diamond)^{k+1} \psi := E_G^\diamond ((E_G^\diamond)^k \psi)$ for any $\emptyset \neq G \subseteq \mathcal{A}$ and $k \geq 0$.

Before we formally introduce the structures on which we interpret any formula $\varphi \in \mathcal{L}_C$, we define its closure $cl(\varphi)$.

Definition 2 (Subformula). *The set $\text{Sub}(\varphi)$ of subformulas of $\varphi \in \mathcal{L}_C$ is defined by induction on the construction of φ :*

$$\begin{aligned} \text{Sub}(p) &:= \{p\}; \\ \text{Sub}(\neg\varphi) &:= \{\neg\varphi\} \cup \text{Sub}(\varphi); \\ \text{Sub}(\varphi_1 \wedge \varphi_2) &:= \{\varphi_1 \wedge \varphi_2\} \cup \text{Sub}(\varphi_1) \cup \text{Sub}(\varphi_2); \\ \text{Sub}(K_a\varphi) &:= \{K_a\varphi\} \cup \text{Sub}(\varphi); \\ \text{Sub}(\Box\varphi) &:= \{\Box\varphi\} \cup \text{Sub}(\varphi); \\ \text{Sub}(C_G^\diamond\varphi) &:= \{C_G^\diamond\varphi\} \cup \text{Sub}(\varphi). \end{aligned}$$

Definition 3. *For a formula $\varphi \in \mathcal{L}_C$, we define the following sets.*

- $cl_0(\varphi)$ is the smallest set closed under the following conditions:
 1. $\{\varphi\} \cup \{K_a\top \mid a \in \mathcal{A}\} \subset cl_0(\varphi)$;
 2. if $\chi \in cl_0(\varphi)$, then $\text{Sub}(\chi) \subseteq cl_0(\varphi)$;
 3. if $C_G^\diamond\chi \in cl_0(\varphi)$, then $E_G^\diamond(\chi \wedge C_G^\diamond\chi) \in cl_0(\varphi)$.
- $K_a^\varphi := \{K_a\chi \mid K_a\chi \in cl_0(\varphi)\}$.
- $cl_1(\varphi) := cl_0(\varphi) \cup \{\neg\chi \mid \chi \in cl_0(\varphi)\}$.
- $cl_2(\varphi)$ is the smallest set closed under the following conditions:
 1. $cl_1(\varphi) \subseteq cl_2(\varphi)$;
 2. if $\chi \in cl_2(\varphi)$, then $\text{Sub}(\chi) \subseteq cl_2(\varphi)$;
 3. if $\emptyset \neq \Delta \subseteq K_a^\varphi$, then $\{\Box(\bigwedge_{K_a\chi \in \Delta} K_a\chi), \Box\neg(\bigwedge_{K_a\chi \in \Delta} K_a\chi)\} \subset cl_2(\varphi)$.
- $cl_3(\varphi) := cl_2(\varphi) \cup \{\neg\chi \mid \chi \in cl_2(\varphi)\}$.
- $cl_4(\varphi) := cl_3(\varphi) \cup \{\Box\Box\chi \mid \Box\chi \in cl_3(\varphi)\} \cup \{\Box\neg\Box\chi \mid \neg\Box\chi \in cl_3(\varphi)\}$.
- $cl(\varphi) := cl_4(\varphi) \cup \{\neg\chi \mid \chi \in cl_4(\varphi)\}$.

We have $cl_0(\varphi) \subset cl_1(\varphi) \subseteq cl_2(\varphi) \subset cl_3(\varphi) \subset cl_4(\varphi) \subset cl(\varphi)$.

Lemma 4. *For any formula $\varphi \in \mathcal{L}_C$, the set $cl(\varphi)$ is finite. If $\chi \in cl(\varphi)$, then $\text{Sub}(\chi) \subseteq cl(\varphi)$. Moreover, if $\chi \in cl(\varphi)$ is not a negation, then $\neg\chi \in cl(\varphi)$.*

As mentioned in Section 1, we consider Kripke models with an epistemic accessibility relation for each agent and a temporal relation. While we require an equivalence relation for time, we allow arbitrary relations for knowledge.

Definition 5 (Kripke models). *A Kripke model \mathcal{M} is a tuple $(W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$ consisting of*

- a non-empty set W of possible worlds, which we sometimes refer to as the domain $\mathcal{D}(\mathcal{M})$ of \mathcal{M} ,
- a valuation function $\pi: \text{Prop} \rightarrow 2^W$,
- and binary relations \mathcal{K}_a for each $a \in \mathcal{A}$ and \sim on W .

Definition 6 (Class). *The class \mathcal{C}_{eck} consists of all models of the form $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$, where the relation \sim is an equivalence relation.*

2.2 Semantics

In this section, we define the semantics of eventual common knowledge.

Definition 7. *Given a Kripke model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$, we define $f_\psi: 2^W \rightarrow 2^W$ for each $\psi \in \mathcal{L}_C^x$ by induction on a suitable size measure for \mathcal{L}_C^x -formulas.*

1. $f_p(A) := \pi(p)$ for $p \in \text{Prop}$,
2. $f_x(A) := A$,

3. $f_{\neg\psi}(A) := W \setminus f_{\psi}(A)$,
4. $f_{\psi \wedge \psi'}(A) := f_{\psi}(A) \cap f_{\psi'}(A)$,
5. $f_{K_a\psi}(A) := \{w \mid w' \in f_{\psi}(A) \text{ for all } w' \text{ s.t. } w\mathcal{K}_a w'\}$,
6. $f_{\Box\psi}(A) := \{w \mid w' \in f_{\psi}(A) \text{ for all } w' \text{ s.t. } w \sim w'\}$,
7. $f_{C_G^{\diamond}(\varphi \wedge x)}(A) := \bigcup \{B \subseteq W \mid B \subseteq f_{E_G^{\diamond}(\varphi \wedge x)}(B)\}$ for $\varphi \in \mathcal{L}_C$.

Lemma 8. *Let a Kripke model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$ be given. For any formula $\varphi \in \mathcal{L}_C$ and any group $\emptyset \neq G \subseteq \mathcal{A}$ of agents, the function $f_{E_G^{\diamond}(\varphi \wedge x)}$ is monotone increasing.*

Proof. For any set $A \subseteq W$, we have

$$\begin{aligned}
f_{E_G^{\diamond}(\varphi \wedge x)}(A) &= \bigcap_{a \in G} f_{\diamond K_a(\varphi \wedge x)}(A), \\
f_{\diamond K_a(\varphi \wedge x)}(A) &= W \setminus f_{\Box \neg K_a(\varphi \wedge x)}(A), \\
f_{\Box \neg K_a(\varphi \wedge x)}(A) &= \{w \mid w' \in f_{\neg K_a(\varphi \wedge x)}(A) \text{ for all } w' \text{ s.t. } w \sim w'\}, \\
f_{\neg K_a(\varphi \wedge x)}(A) &= W \setminus f_{K_a(\varphi \wedge x)}(A), \\
f_{K_a(\varphi \wedge x)}(A) &= \{w'' \mid w''' \in f_{\varphi \wedge x}(A) \text{ for all } w''' \text{ s.t. } w''\mathcal{K}_a w'''\} \\
&= \{w'' \mid w''' \in f_{\varphi}(A) \cap A \text{ for all } w''' \text{ s.t. } w''\mathcal{K}_a w'''\}.
\end{aligned}$$

Consider sets $A \subseteq B \subseteq W$. It is not difficult to see that if $f_{\varphi}(A) \subseteq f_{\varphi}(B)$, then we have

$$\begin{aligned}
f_{\varphi}(A) \cap A &\subseteq f_{\varphi}(B) \cap B, \\
f_{K_a(\varphi \wedge x)}(A) &\subseteq f_{K_a(\varphi \wedge x)}(B), \\
f_{\neg K_a(\varphi \wedge x)}(A) &\supseteq f_{\neg K_a(\varphi \wedge x)}(B), \\
f_{\Box \neg K_a(\varphi \wedge x)}(A) &\supseteq f_{\Box \neg K_a(\varphi \wedge x)}(B), \\
f_{\diamond K_a(\varphi \wedge x)}(A) &\subseteq f_{\diamond K_a(\varphi \wedge x)}(B), \\
f_{E_G^{\diamond}(\varphi \wedge x)}(A) &\subseteq f_{E_G^{\diamond}(\varphi \wedge x)}(B).
\end{aligned}$$

Therefore, it is enough to show that $f_{\varphi}(A) \subseteq f_{\varphi}(B)$ holds. However, we are going to show that in fact $f_{\varphi}(A) = f_{\varphi}(B)$ by induction on the structure of $\varphi \in \mathcal{L}_C$.

For $\varphi = p \in \text{Prop}$: We have $f_{\varphi}(A) = \pi(p) = f_{\varphi}(B)$.

For $\varphi = \neg\chi$: We have $f_{\varphi}(A) = W \setminus f_{\chi}(A) =_{\text{IH}} W \setminus f_{\chi}(B) = f_{\varphi}(B)$.

For $\varphi = \chi_1 \wedge \chi_2$: We have $f_{\varphi}(A) = f_{\chi_1}(A) \cap f_{\chi_2}(A) =_{\text{IH}} f_{\chi_1}(B) \cap f_{\chi_2}(B) = f_{\varphi}(B)$.

For $\varphi = K_a\chi$: We have

$$\begin{aligned}
f_{\varphi}(A) &= \{w \mid w' \in f_{\chi}(A) \text{ for all } w' \text{ s.t. } w\mathcal{K}_a w'\} \\
&=_{\text{IH}} \{w \mid w' \in f_{\chi}(B) \text{ for all } w' \text{ s.t. } w\mathcal{K}_a w'\} \\
&= f_{\varphi}(B).
\end{aligned}$$

For $\varphi = \Box\chi$: We have

$$\begin{aligned}
f_{\varphi}(A) &= \{w \mid w' \in f_{\chi}(A) \text{ for all } w' \text{ s.t. } w \sim w'\} \\
&=_{\text{IH}} \{w \mid w' \in f_{\chi}(B) \text{ for all } w' \text{ s.t. } w \sim w'\} \\
&= f_{\varphi}(B).
\end{aligned}$$

For $\varphi = C_G^{\diamond}\chi$: We have $f_{\varphi}(A) = \bigcup \{C \subseteq W \mid C \subseteq f_{E_G^{\diamond}(\chi \wedge x)}(C)\} = f_{\varphi}(B)$.

In the proof of Lemma 8, we showed that $f_{\varphi}(A)$ does not depend on $A \subseteq W$ for any formula $\varphi \in \mathcal{L}_C$. Therefore, we omit the argument and simply write $f_{\varphi}(\star)$ for $\varphi \in \mathcal{L}_C$. \square

Lemma 9. *For any Kripke model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$ and any formula $\varphi \in \mathcal{L}_C$,*

$$f_{E_G^{\diamond}(\varphi \wedge x)}(f_{C_G^{\diamond}(\varphi \wedge x)}(\star)) = f_{C_G^{\diamond}(\varphi \wedge x)}(\star).$$

Proof. \supseteq : For an arbitrary $C \in \{B \subseteq W \mid B \subseteq f_{E_G^\circ(\varphi \wedge x)}(B)\}$, we have $C \subseteq f_{C_G^\circ \varphi}(\star)$ by definition.

Therefore, $f_{E_G^\circ(\varphi \wedge x)}(C) \subseteq f_{E_G^\circ(\varphi \wedge x)}(f_{C_G^\circ \varphi}(\star))$ because $f_{E_G^\circ(\varphi \wedge x)}$ is monotone increasing by Lemma 8. It follows that $C \subseteq f_{E_G^\circ(\varphi \wedge x)}(f_{C_G^\circ \varphi}(\star))$. Since C is arbitrary, we conclude that $f_{C_G^\circ \varphi}(\star) \subseteq f_{E_G^\circ(\varphi \wedge x)}(f_{C_G^\circ \varphi}(\star))$.

\subseteq : From the previous item and the fact that $f_{E_G^\circ(\varphi \wedge x)}$ is monotone increasing, it follows that $f_{E_G^\circ(\varphi \wedge x)}(f_{C_G^\circ \varphi}(\star)) \in \{B \subseteq W \mid B \subseteq f_{E_G^\circ(\varphi \wedge x)}(B)\}$. This entails the other inclusion. \square

Lemma 10. *For any Kripke model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$ and any formulas $\psi \in \mathcal{L}_C^x$ and $\varphi \in \mathcal{L}_C$,*

$$f_\psi(f_\varphi(\star)) = f_{\psi[x/\varphi]}(\star).$$

Proof. We proceed by induction on the structure of $\psi \in \mathcal{L}_C^x$.

For $\psi \in \mathcal{L}_C$: In this case, there is no occurrence of x in ψ . Therefore, we have $\psi[x/\varphi] = \psi$ and $f_\psi(f_\varphi(\star)) = f_\psi(\star) = f_{\psi[x/\varphi]}(\star)$.

For $\psi = x$: We have $\psi[x/\varphi] = \varphi$ and $f_\psi(f_\varphi(\star)) = f_\varphi(\star) = f_{\psi[x/\varphi]}(\star)$.

For $\psi = \neg\chi$: We have $f_\psi(f_\varphi(\star)) = W \setminus f_\chi(f_\varphi(\star)) =_{\text{IH}} W \setminus f_{\chi[x/\varphi]}(\star) = f_{\psi[x/\varphi]}(\star)$.

For $\psi = \chi_1 \wedge \chi_2$: We have

$$\begin{aligned} f_\psi(f_\varphi(\star)) &= f_{\chi_1 \wedge \chi_2}(f_\varphi(\star)) \\ &= f_{\chi_1}(f_\varphi(\star)) \cap f_{\chi_2}(f_\varphi(\star)) \\ &=_{\text{IH}} f_{\chi_1[x/\varphi]}(\star) \cap f_{\chi_2[x/\varphi]}(\star) \\ &= f_{\chi_1[x/\varphi] \wedge \chi_2[x/\varphi]}(\star) \\ &= f_{(\chi_1 \wedge \chi_2)[x/\varphi]}(\star) \\ &= f_{\psi[x/\varphi]}(\star). \end{aligned}$$

For $\psi = K_a\chi$: We have

$$\begin{aligned} f_\psi(f_\varphi(\star)) &= f_{K_a\chi}(f_\varphi(\star)) \\ &= \{w \mid w' \in f_\chi(f_\varphi(\star)) \text{ for all } w' \text{ s.t. } w\mathcal{K}_a w'\} \\ &=_{\text{IH}} \{w \mid w' \in f_{\chi[x/\varphi]}(\star) \text{ for all } w' \text{ s.t. } w\mathcal{K}_a w'\} \\ &= f_{K_a(\chi[x/\varphi])}(\star) \\ &= f_{(K_a\chi)[x/\varphi]}(\star) \\ &= f_{\psi[x/\varphi]}(\star). \end{aligned}$$

For $\psi = \Box\chi$: We have

$$\begin{aligned} f_\psi(f_\varphi(\star)) &= f_{\Box\chi}(f_\varphi(\star)) \\ &= \{w \mid w' \in f_\chi(f_\varphi(\star)) \text{ for all } w' \text{ s.t. } w \sim w'\} \\ &=_{\text{IH}} \{w \mid w' \in f_{\chi[x/\varphi]}(\star) \text{ for all } w' \text{ s.t. } w \sim w'\} \\ &= f_{\Box(\chi[x/\varphi])}(\star) \\ &= f_{(\Box\chi)[x/\varphi]}(\star) \\ &= f_{\psi[x/\varphi]}(\star). \end{aligned}$$

\square

Definition 11 (Truth and validity). *A formula $\varphi \in \mathcal{L}_C$ is true in a world w of a given model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$ and we write $\mathcal{M}, w \vDash \varphi$ iff $w \in f_\varphi(\star)$. If $\mathcal{M}, w \vDash \varphi$ for all $w \in \mathcal{D}(\mathcal{M})$, we write $\mathcal{M} \vDash \varphi$ and say that φ is valid in \mathcal{M} . If $\mathcal{M} \vDash \varphi$ for any model $\mathcal{M} \in \mathcal{C}_{eck}$, then we write $\mathcal{C}_{eck} \vDash \varphi$ and say that φ is valid.*

Remark 12. It is not difficult to check that Definition 11 coincides with the standard semantics for the atomic propositions p , the connectives \neg and \wedge , and the modal operators K_a and \Box .

2.3 Some Validities and Counterexamples

We now investigate a few expected principles and their validity with respect to the semantics introduced in Section 2.2. First, Lemma 13 shows that the Fixed Point Axiom is valid.

Lemma 13. *For any formula $\varphi \in \mathcal{L}_C$, we have $\mathcal{C}_{eck} \models C_G^\diamond \varphi \leftrightarrow E_G^\diamond(\varphi \wedge C_G^\diamond \varphi)$.*

Proof. For an arbitrary $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$, we have

$$f_{C_G^\diamond \varphi}(\star) \stackrel{=_{\text{Lem.9}}}{=} f_{E_G^\diamond(\varphi \wedge \varphi)}(f_{C_G^\diamond \varphi}(\star)) \stackrel{=_{\text{Lem.10}}}{=} f_{E_G^\diamond(\varphi \wedge C_G^\diamond \varphi)}(\star).$$

Therefore, $\mathcal{C}_{eck} \models C_G^\diamond \varphi \leftrightarrow E_G^\diamond(\varphi \wedge C_G^\diamond \varphi)$. \square

Lemma 14. *For any formulas $\varphi, \psi \in \mathcal{L}_C$ and any group $\emptyset \neq G \subseteq \mathcal{A}$:*

$$\mathcal{C}_{eck} \models E_G^\diamond(\varphi \wedge \psi) \rightarrow (E_G^\diamond \varphi \wedge E_G^\diamond \psi).$$

Proof. Let $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$ and $w \in W$ be such that $\mathcal{M}, w \models E_G^\diamond(\varphi \wedge \psi)$. Then $\mathcal{M}, w \models \bigwedge_{a \in G} \diamond K_a(\varphi \wedge \psi)$. Fix an arbitrary $a \in G$. We have $\mathcal{M}, w \models \diamond K_a(\varphi \wedge \psi)$, that is $\mathcal{M}, w_a \models K_a(\varphi \wedge \psi)$ for some w_a with $w \sim w_a$. It follows that $\mathcal{M}, w' \models \varphi \wedge \psi$ for any w' with $w_a \mathcal{K}_a w'$. Therefore, $\mathcal{M}, w' \models \varphi$ and $\mathcal{M}, w' \models \psi$ for any w' with $w_a \mathcal{K}_a w'$. Hence, we obtain $\mathcal{M}, w_a \models K_a \varphi$ and $\mathcal{M}, w_a \models K_a \psi$, which imply that $\mathcal{M}, w \models \diamond K_a \varphi$ and $\mathcal{M}, w \models \diamond K_a \psi$. Since $a \in G$ is arbitrary, it follows that $\mathcal{M}, w \models \bigwedge_{a \in G} \diamond K_a \varphi$ and $\mathcal{M}, w \models \bigwedge_{a \in G} \diamond K_a \psi$. Thus $\mathcal{M}, w \models E_G^\diamond \varphi \wedge E_G^\diamond \psi$. \square

The following proposition states that eventual common knowledge is not factive even if the epistemic relations \mathcal{K}_a are equivalence relations.

Proposition 15. *There exists a model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$, a world $w \in W$, and a formula $\varphi \in \mathcal{L}_C$ such that $\mathcal{K}_1, \dots, \mathcal{K}_n$ are equivalence relations and $\mathcal{M}, w \not\models C_G^\diamond \varphi \rightarrow \varphi$ for any group $\emptyset \neq G \subseteq \mathcal{A}$.*

Proof. Consider a proposition $\varphi = p \in \text{Prop}$ and a model \mathcal{M} consisting of two worlds w_1 and w_2 , where $\mathcal{K}_a := \{(w_1, w_1), (w_2, w_2)\}$ for all $a \in \mathcal{A}$, $\sim := \{(w_1, w_2), (w_2, w_1), (w_1, w_1), (w_2, w_2)\}$, and $\pi(p) := \{w_2\}$. Then, for any $\emptyset \neq G \subseteq \mathcal{A}$ we have $\mathcal{M}, w_1 \models C_G^\diamond p$, but $\mathcal{M}, w_1 \not\models p$. \square

Similarly, Proposition 16 shows that eventual common knowledge fails the normality axiom.

Proposition 16. *There exists a model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$, a world $w \in W$, and formulas $\varphi, \psi \in \mathcal{L}_C$ such that $\mathcal{K}_1, \dots, \mathcal{K}_n$ are equivalence relations and for any group $\emptyset \neq G \subseteq \mathcal{A}$ we have $\mathcal{M}, w \not\models C_G^\diamond(\varphi \rightarrow \psi) \wedge C_G^\diamond \varphi \rightarrow C_G^\diamond \psi$.*

Proof. Consider two distinct propositions p and q and a model \mathcal{M} consisting of two worlds w_1 and w_2 , where $\mathcal{K}_a := \{(w_1, w_1), (w_2, w_2)\}$ for all $a \in \mathcal{A}$, $\sim := \{(w_1, w_2), (w_2, w_1), (w_1, w_1), (w_2, w_2)\}$, $\pi(p) := \{w_2\}$, and $\pi(q) := \emptyset$. Then $\mathcal{M}, w_1 \models C_G^\diamond p$ and $\mathcal{M}, w_1 \models C_G^\diamond(p \rightarrow q)$, but $\mathcal{M}, w_1 \not\models C_G^\diamond q$. \square

2.4 Fixed Point Approximations

Eventual common knowledge, defined as a fixed point, is not equivalent to an infinite conjunction. Halpern and Moses established this result in [7] via an analysis of the coordinated attack problem, where assumptions about unreliable communication and other features of distributed systems play an essential role. By contrast, we provide a Kripke model as a counterexample, establishing the result at a purely logical level.

Theorem 17. *1. For any formula $\varphi \in \mathcal{L}_C$ and any group $\emptyset \neq G \subseteq \mathcal{A}$:*

$$\mathcal{C}_{eck} \models C_G^\diamond \varphi \rightarrow (E_G^\diamond)^k \varphi, \text{ for all } k > 0.$$

2. There exist a model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$, a world $w \in W$, and a formula $\varphi \in \mathcal{L}_C$ such that for any group $\emptyset \neq G \subseteq \mathcal{A}$

$$\mathcal{M}, w \models (E_G^\diamond)^k \varphi \text{ for all } k \geq 0, \text{ but } \mathcal{M}, w \not\models C_G^\diamond \varphi.$$

Proof. 1. Let $k > 0$ and consider a model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$ and a world $w \in W$ such that $\mathcal{M}, w \models C_G^\diamond \varphi$. Then we have $\mathcal{M}, w \models E_G^\diamond(\varphi \wedge C_G^\diamond \varphi)$ by Lemma 13. Using the same valid equivalence, we substitute $E_G^\diamond(\varphi \wedge C_G^\diamond \varphi)$ for $C_G^\diamond \varphi$ additional $k - 1$ times and obtain

$$\mathcal{M}, w \models E_G^\diamond(\varphi \wedge E_G^\diamond(\varphi \wedge \dots E_G^\diamond(\varphi \wedge C_G^\diamond \varphi) \dots))$$

where E_G^\diamond occurs k times. It follows from Lemma 14 that

$$\mathcal{M}, w \models E_G^\diamond \varphi \wedge (E_G^\diamond)^2 \varphi \wedge \dots \wedge (E_G^\diamond)^k \varphi \wedge (E_G^\diamond)^k C_G^\diamond \varphi.$$

Consequently, $\mathcal{M}, w \models (E_G^\diamond)^k \varphi$ and we proved that $\mathcal{M}, w \models C_G^\diamond \varphi \rightarrow (E_G^\diamond)^k \varphi$.

2. Consider a model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$ such that
- $W := \{w_{m,n} \mid m \geq 1 \text{ and } 1 \leq n \leq m + 2\}$,
 - $\mathcal{K}_a := \{(w_{m,n}, w_{m,n+1}) \mid m \geq 1 \text{ and } 1 \leq n \leq m + 1\}$ for all $a \in \mathcal{A}$,
 - \sim is the equivalence relation generated by $\{(w_{m,1}, w_{m+1,1}) \mid m \geq 1\}$,
 - $\pi(p) := W \setminus \{w_{m,m+2} \mid m \geq 1\}$.

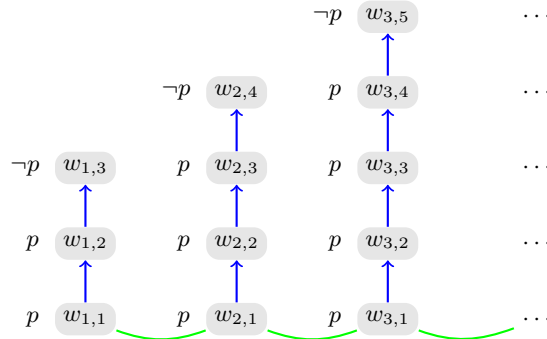


Fig. 1: The green edges depict the (bidirectional) temporal relation and the blue arrows represent the epistemic relation. The reflexive loops are omitted for both relations; additionally, green transitive edges are omitted.

By the definition of \sim , it is clear that $\mathcal{M} \in \mathcal{C}_{eck}$. We observe that $\mathcal{M}, w_{m,m} \models K_a p$ and $\mathcal{M}, w_{m,m+1} \not\models K_a p$ for any $m \geq 1$ and any $a \in \mathcal{A}$. This implies that $\mathcal{M}, w_{m,m} \models E_G^\diamond p$ for any $m \geq 1$ and any group $\emptyset \neq G \subseteq \mathcal{A}$. It also follows that $\mathcal{M}, w_{m,m-l} \models (E_G^\diamond)^{l+1} p$ for any $0 \leq l \leq m - 1$. Consequently, $\mathcal{M}, w_{1,1} \models (E_G^\diamond)^k p$ for any $k \geq 0$.

Next we prove that $\mathcal{M}, w_{1,1} \not\models C_G^\diamond p$.

Assume towards a contradiction that $\mathcal{M}, w_{1,1} \models C_G^\diamond p$. Then we have $\mathcal{M}, w_{1,1} \models E_G^\diamond(p \wedge C_G^\diamond p)$ by Lemma 13, and $\mathcal{M}, w_{1,1} \models E_G^\diamond p \wedge E_G^\diamond C_G^\diamond p$ by Lemma 14. It follows that $\mathcal{M}, w_{1,1} \models E_G^\diamond C_G^\diamond p$, that is $\mathcal{M}, w_{1,1} \models \bigwedge_{a \in G} \diamond K_a C_G^\diamond p$. Fix an arbitrary $a \in G$. Then we have $\mathcal{M}, w_{1,1} \models \diamond K_a C_G^\diamond p$, which implies that $\mathcal{M}, w_{m_a,1} \models K_a C_G^\diamond p$ for some $m_a \geq 1$. By construction, we then obtain $\mathcal{M}, w_{m_a,2} \models C_G^\diamond p$. As with $w_{1,1}$, we can infer that $\mathcal{M}, w_{m_a,2} \models \diamond K_a C_G^\diamond p$. But because $w_{m_a,2}$ is the only \sim -accessible world from $w_{m_a,2}$, it follows that $\mathcal{M}, w_{m_a,2} \models K_a C_G^\diamond p$. Again by construction, we obtain $\mathcal{M}, w_{m_a,3} \models C_G^\diamond p$. Iterating the same reasoning, we establish that $\mathcal{M}, w_{m_a,m_a+1} \models C_G^\diamond p$, which implies that $\mathcal{M}, w_{m_a,m_a+1} \models E_G^\diamond p$. Therefore, we have $\mathcal{M}, w_{m_a,m_a+1} \models \diamond K_a p$. However, because w_{m_a,m_a+1} is the only \sim -accessible world from w_{m_a,m_a+1} , we obtain $\mathcal{M}, w_{m_a,m_a+1} \models K_a p$ and we arrive at a contradiction. \square

Lemma 19 presents a standard result (see, *e.g.*, [1]) that is useful for the completeness proof.

Definition 18. For any formula $\varphi \in \mathcal{L}_C$, we define $\varepsilon_{\varphi,G}^0 := E_G^\diamond \varphi$ and $\varepsilon_{\varphi,G}^{k+1} := E_G^\diamond(\varphi \wedge \varepsilon_{\varphi,G}^k)$ for any $k \geq 0$.

Lemma 19. Consider any formula $\varphi \in \mathcal{L}_C$, any group $G \subseteq \mathcal{A}$, and any Kripke model $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim)$. For any ordinal α , we define subsets $J_{\varphi,G}^\alpha$ of W by

$$J_{\varphi,G}^\alpha := f_{E_G^\diamond(\varphi \wedge x)} \left(\bigcap_{\beta < \alpha} J_{\varphi,G}^\beta \right).$$

Then we have

$$f_{C_G^\diamond \varphi}(\star) = \bigcap_{\alpha} J_{\varphi,G}^\alpha. \quad (1)$$

If W is finite, then there exists some natural number $l \leq \text{card}(W)$ such that

$$f_{C_G^\diamond \varphi}(\star) = \bigcap_{k < l} J_{\varphi,G}^k. \quad (2)$$

Moreover, for any world $w \in W$ and any natural number $k \geq 0$, we have

$$w \in J_{\varphi,G}^k \text{ if and only if } \mathcal{M}, w \models \varepsilon_{\varphi,G}^k. \quad (3)$$

Proof. For equivalence (3), we have $J_{\varphi,G}^0 = f_{E_G^\diamond(\varphi \wedge x)}(W) = f_{E_G^\diamond \varphi}(\star) = f_{\varepsilon_{\varphi,G}^0}(\star)$ and can prove by induction on k that $J_{\varphi,G}^{k+1} \subseteq J_{\varphi,G}^k$ and $J_{\varphi,G}^k = f_{\varepsilon_{\varphi,G}^k}(\star)$. \square

3 Axiomatization

In this section, we present a sound and complete Hilbert-style axiomatization for the logic of eventual common knowledge. The completeness proof follows the standard approach for common knowledge, relying on a formula-specific canonical model construction.

3.1 The System

Definition 20 (Axiomatic system). The axiomatic system ECK consists of the following axioms and rules.

Taut	all propositional tautologies
K^K	$K_a(\varphi \rightarrow \psi) \wedge K_a \varphi \rightarrow K_a \psi$
K[□]	$\Box(\varphi \rightarrow \psi) \wedge \Box \varphi \rightarrow \Box \psi$
T[□]	$\Box \varphi \rightarrow \varphi$
4[□]	$\Box \varphi \rightarrow \Box \Box \varphi$
5[□]	$\neg \Box \varphi \rightarrow \Box \neg \Box \varphi$
Mix	$C_G^\diamond \varphi \rightarrow E_G^\diamond(\varphi \wedge C_G^\diamond \varphi)$
MP	from φ and $\varphi \rightarrow \psi$, infer ψ
Nec^K	from φ , infer $K_a \varphi$
Nec[□]	from φ , infer $\Box \varphi$
Ind	from $\psi \rightarrow E_G^\diamond(\varphi \wedge \psi)$, infer $\psi \rightarrow C_G^\diamond \varphi$

Definition 21 (Derivability). A formula φ is derivable in ECK, written $\vdash \varphi$, iff there is a finite sequence ψ_1, \dots, ψ_k of formulas such that $\psi_k = \varphi$ and each formula in the sequence is either an instance of one of the axioms of ECK or follows from earlier formulas in the sequence by one of the rules of ECK. A formula φ is derivable in ECK from a set Γ of formulas, written $\Gamma \vdash \varphi$, iff $\vdash \bigwedge \Gamma_0 \rightarrow \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.

Theorem 22 (Soundness). *The axiomatic system ECK is strongly sound with respect to \mathcal{C}_{eck} .*

Proof. The proof is standard: one shows that all axioms are valid and all rules preserve validity.

Ind: Let $\mathcal{M} = (W, \pi, \mathcal{K}_1, \dots, \mathcal{K}_n, \sim) \in \mathcal{C}_{eck}$ be such that $\mathcal{M} \models \psi \rightarrow E_G^\diamond(\varphi \wedge \psi)$. Then

$$f_\psi(\star) \subseteq f_{E_G^\diamond(\varphi \wedge \psi)}(\star) \stackrel{\text{Lem.10}}{=} f_{E_G^\diamond(\varphi \wedge x)}(f_\psi(\star)),$$

that is $f_\psi(\star) \in \{B \subseteq W \mid B \subseteq f_{E_G^\diamond(\varphi \wedge x)}(B)\}$. Consequently, we have $f_\psi(\star) \subseteq f_{C_G^\diamond \varphi}(\star)$ and $\mathcal{M} \models \psi \rightarrow C_G^\diamond \varphi$.

The case of **Mix** is given in Lemma 13 and the remaining cases are standard. \square

3.2 Completeness

Definition 23 (Maximal consistent). *Let $\varphi \in \mathcal{L}_C$ be given. A set Γ of formulas is φ -consistent iff $\Gamma \subseteq cl(\varphi)$ and $\Gamma \not\vdash \perp$. Moreover, if no Γ' with $\Gamma \subsetneq \Gamma'$ is φ -consistent, we say that Γ is maximal φ -consistent. We define $Con(\varphi)$ to consist of all maximal φ -consistent sets.*

Lemma 24 (Lindenbaum). *Let $\varphi \in \mathcal{L}_C$ be given. Every φ -consistent set of formulas is a subset of a maximal φ -consistent set of formulas.*

Lemma 25. *Let $\varphi \in \mathcal{L}_C$ be given. For any set $\Gamma \in Con(\varphi)$ and any formulas $\chi, \chi' \in cl(\varphi)$:*

1. *If $\Gamma \vdash \chi$, then $\chi \in \Gamma$.*
2. *If $\chi \in \Gamma$ and $\chi \rightarrow \chi' \in \Gamma$, then $\chi' \in \Gamma$.*
3. *If $\chi \in \Gamma$, then $\neg\chi \notin \Gamma$. Additionally, if $\neg\chi \in cl(\varphi) \setminus \Gamma$, then $\chi \in \Gamma$.*
4. *If $\chi \wedge \chi' \in \Gamma$, then $\chi \in \Gamma$ and $\chi' \in \Gamma$. Additionally, if $\chi \in \Gamma$ and $\chi' \in \Gamma$ and $\chi \wedge \chi' \in cl(\varphi)$, then $\chi \wedge \chi' \in \Gamma$.*
5. *$\{K_a \top \mid a \in \mathcal{A}\} \subset \Gamma$.*
6. *For $a \in \mathcal{A}$, we have $\{K_a \chi \mid K_a \chi \in \Gamma\} \neq \emptyset$, $\diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \in \Gamma$, and $\Box \diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \in \Gamma$.*

Proof. Proofs of Items 1–5 are standard.

Item 6: It follows from Item 5 that $\{K_a \chi \mid K_a \chi \in \Gamma\} \neq \emptyset$. Moreover by Definition 3, we have $\{K_a \chi \mid K_a \chi \in \Gamma\} \subseteq K_a^\varphi$, $\diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \in cl(\varphi)$, and $\Box \diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \in cl(\varphi)$. Since $\vdash \bigwedge_{K_a \chi \in \Gamma} K_a \chi \rightarrow \diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi)$ by **T $^\square$** , it follows from Item 1 that $\diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \in \Gamma$. Finally, given that by **5 $^\square$** we have $\vdash \diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \rightarrow \Box \diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi)$, it follows from Item 1 that $\Box \diamond(\bigwedge_{K_a \chi \in \Gamma} K_a \chi) \in \Gamma$. \square

Definition 26 (Canonical model). *Let a formula $\varphi \in \mathcal{L}_C$ be given. The canonical φ -model $\mathcal{M}^\varphi = (W^\varphi, \pi^\varphi, \mathcal{K}_1^\varphi, \dots, \mathcal{K}_n^\varphi, \sim^\varphi)$ is defined as follows*

- $W^\varphi := Con(\varphi)$
- $\pi^\varphi(p) := \{\Gamma \in W^\varphi \mid p \in \Gamma\}$
- $\mathcal{K}_a^\varphi := \{(\Gamma, \Delta) \mid \forall K_a \chi \in \Gamma, \chi \in \Delta\}$ for all $a \in \mathcal{A}$
- $\sim^\varphi := \{(\Gamma, \Delta) \mid \forall \Box \chi \in \Gamma, \chi \in \Delta\}$

Lemma 27. *For any formula $\varphi \in \mathcal{L}_C$, the canonical φ -model \mathcal{M}^φ from Definition 26 is in the class \mathcal{C}_{eck} . Moreover, W^φ is finite and non-empty.*

Proof. $W^\varphi \neq \emptyset$ follows from the fact that, for any world w of any model $\mathcal{M} \in \mathcal{C}_{eck}$, we have $cl(\varphi) \cap \{\chi \mid \mathcal{M}, w \models \chi\} \in Con(\varphi)$ by Theorem 22. It remains to prove that \sim^φ is an equivalence relation.

Reflexivity: By **T $^\square$** and Item 1 of Lemma 25, $\chi \in \Gamma$ whenever $\Box \chi \in \Gamma$. Hence, $\Gamma \sim^\varphi \Gamma$.

Transitivity: Assume that $\Gamma_1 \sim^\varphi \Gamma_2$ and $\Gamma_2 \sim^\varphi \Gamma_3$. Let $\Box \chi \in \Gamma_1$.

- Assume that $\Box \chi \in cl_3(\varphi) \subset cl(\varphi)$. By Definition 3, we have $\Box \Box \chi \in cl(\varphi)$. By **4 $^\square$** and Item 1 of Lemma 25, it follows that $\Box \Box \chi \in \Gamma_1$. Therefore, we obtain $\Box \chi \in \Gamma_2$ and $\chi \in \Gamma_3$.

- Assume that $\Box\chi \in cl(\varphi) \setminus cl_3(\varphi)$. Then we have two cases:
 - Case $\chi = \Box\theta$ for some $\Box\theta \in cl_3(\varphi)$. Because $\Box\Box\theta \in \Gamma_1$, we have $\Box\theta \in \Gamma_2$. By **4[□]** and Item 1 of Lemma 25, we obtain $\Box\Box\theta \in \Gamma_2$. It follows that $\chi = \Box\theta \in \Gamma_3$.
 - Case $\chi = \neg\Box\theta$ for some $\neg\Box\theta \in cl_3(\varphi)$. Because $\Box\neg\Box\theta \in \Gamma_1$, we have $\neg\Box\theta \in \Gamma_2$. By **5[□]** and Item 1 of Lemma 25, we obtain $\Box\neg\Box\theta \in \Gamma_2$. It follows that $\chi = \neg\Box\theta \in \Gamma_3$.

Euclideanity: Assume that $\Gamma_1 \sim^\varphi \Gamma_2$ and $\Gamma_1 \sim^\varphi \Gamma_3$. Let $\Box\chi \in \Gamma_2$. By Lemma 4, $\neg\Box\chi \in cl(\varphi)$. To show that $\chi \in \Gamma_3$, it suffices to show that $\Box\chi \in \Gamma_1$.

- Assume that $\neg\Box\chi \in cl_3(\varphi) \subset cl(\varphi)$. By Definition 3, we have $\Box\neg\Box\chi \in cl(\varphi)$. Because $\Box\chi \in \Gamma_2$ and $\Gamma_1 \sim^\varphi \Gamma_2$, it follows that $\neg\Box\chi \notin \Gamma_2$ and $\Box\neg\Box\chi \notin \Gamma_1$. If $\neg\Box\chi \in \Gamma_1$, we would have $\Box\neg\Box\chi \in \Gamma_1$ by **5[□]** and Item 1 of Lemma 25. Consequently, $\neg\Box\chi \notin \Gamma_1$ and $\Box\chi \in \Gamma_1$.
- Assume that $\neg\Box\chi \in cl(\varphi) \setminus cl_3(\varphi)$. Then we have two cases:
 - Case $\chi = \Box\theta$ for some $\Box\theta \in cl_3(\varphi)$. By Definition 3, $\Box\neg\Box\theta \in cl(\varphi)$. By **T[□]** and Item 1 of Lemma 25, $\Box\theta \in \Gamma_2$. Consequently $\neg\Box\theta \notin \Gamma_2$ and hence $\Box\neg\Box\theta \notin \Gamma_1$. If $\neg\Box\theta \in \Gamma_1$, we would have $\Box\neg\Box\theta \in \Gamma_1$ by **5[□]** and Item 1 of Lemma 25. Therefore, $\neg\Box\theta \notin \Gamma_1$ and $\Box\theta \in \Gamma_1$. It follows from **4[□]** and Item 1 of Lemma 25 that $\Box\chi = \Box\Box\theta \in \Gamma_1$.
 - Case $\chi = \neg\Box\theta$ for some $\neg\Box\theta \in cl_3(\varphi)$. By Definition 3, $\Box\Box\theta \in cl(\varphi)$. By **T[□]** and Item 1 of Lemma 25, we have $\neg\Box\theta \in \Gamma_2$. Consequently $\Box\theta \notin \Gamma_2$ and hence $\Box\Box\theta \notin \Gamma_1$. If $\Box\theta \in \Gamma_1$, we would have $\Box\Box\theta \in \Gamma_1$ by **4[□]** and Item 1 of Lemma 25. Therefore, $\Box\theta \notin \Gamma_1$ and $\neg\Box\theta \in \Gamma_1$. It follows from **5[□]** and Item 1 of Lemma 25 that $\Box\chi = \Box\neg\Box\theta \in \Gamma_1$. \square

Lemma 28. Let $\varphi \in \mathcal{L}_C$ and $\Delta \in Con(\varphi)$ be given. Assume that $C_G^\diamond\theta \in \Delta$ for some formula θ . Then the set

$$S_\Delta^a := \{\Sigma \in Con(\varphi) \mid \Delta \sim^\varphi \Sigma \text{ and } K_a(\theta \wedge C_G^\diamond\theta) \in \Sigma\}$$

is non-empty for any $a \in \mathcal{A}$.

Proof. Consider any $a \in \mathcal{A}$. By assumption, we have $C_G^\diamond\theta \in cl(\varphi)$. It follows by Definition 3 that $E_G^\diamond(\theta \wedge C_G^\diamond\theta) \in cl(\varphi)$, $\neg\Box\neg K_a(\theta \wedge C_G^\diamond\theta) \in cl(\varphi)$, and $K_a(\theta \wedge C_G^\diamond\theta) \in cl(\varphi)$. Then because $\{K_a(\theta \wedge C_G^\diamond\theta)\}$ is φ -consistent, it can be extended to a maximal φ -consistent set by Lemma 24. Consequently, the set $S_\theta^a := \{\Sigma \in Con(\varphi) \mid K_a(\theta \wedge C_G^\diamond\theta) \in \Sigma\}$ is non-empty.

Assume towards a contradiction that for any $\Sigma \in S_\theta^a$ we do not have $\Delta \sim^\varphi \Sigma$. Then for each $\Sigma \in S_\theta^a$, there is some formula $\chi_\Sigma \in cl(\varphi)$ such that $\Box\chi_\Sigma \in \Delta$ but $\chi_\Sigma \notin \Sigma$. It follows that the set $\{\chi_\Sigma \mid \Sigma \in S_\theta^a\} \cup \{K_a(\theta \wedge C_G^\diamond\theta)\}$ is not φ -consistent, otherwise it would be a subset of some element of S_θ^a and that would lead to a contradiction. Therefore,

$$\vdash \bigwedge_{\Sigma \in S_\theta^a} \chi_\Sigma \rightarrow \neg K_a(\theta \wedge C_G^\diamond\theta).$$

Consequently,

$$\vdash \bigwedge_{\Sigma \in S_\theta^a} \Box\chi_\Sigma \rightarrow \Box\neg K_a(\theta \wedge C_G^\diamond\theta),$$

and by Lemma 25 we obtain $\Box\neg K_a(\theta \wedge C_G^\diamond\theta) \in \Delta$. However, this leads to a contradiction because $C_G^\diamond\theta \in \Delta$ and $\vdash C_G^\diamond\theta \rightarrow \neg\Box\neg K_a(\theta \wedge C_G^\diamond\theta)$. \square

For any finite set Δ of formulas, we denote $\underline{\Delta} := \bigwedge_{\chi \in \Delta} \chi$.

Lemma 29. For any formula $\varphi \in \mathcal{L}_C$,

1. $\vdash \bigvee_{\Gamma \in Con(\varphi)} \underline{\Gamma}$,
2. $\vdash \neg \underline{\Gamma} \vee \neg \underline{\Delta}$ for any distinct $\Gamma, \Delta \in Con(\varphi)$,
3. $\vdash \bigvee_{\Gamma \in S} \underline{\Gamma} \leftrightarrow \bigwedge_{\Delta \in Con(\varphi) \setminus S} \neg \underline{\Delta}$ for any $S \subseteq Con(\varphi)$.

Lemma 30 (Truth). For any formulas $\varphi \in \mathcal{L}_C$ and $\chi \in cl(\varphi)$, for any set $\Gamma \in Con(\varphi)$:

$$\chi \in \Gamma \text{ iff } \mathcal{M}^\varphi, \Gamma \models \chi.$$

Proof. The cases of $p, \neg\theta, \theta_1 \wedge \theta_2, K_a\theta, \Box\theta$ are standard. We proceed for $C_G^\diamond\theta$ with the induction hypothesis:

θ IH : $\theta \in \Gamma$ iff $\mathcal{M}^\varphi, \Gamma \models \theta$.

(\Rightarrow) Assume that $C_G^\diamond\theta \in \Gamma$. Because $Con(\varphi)$ is finite, we can prove that $\mathcal{M}^\varphi, \Gamma \models C_G^\diamond\theta$ by showing that for any natural number k and any $\Delta \in Con(\varphi)$, $C_G^\diamond\theta \in \Delta$ implies that $\Delta \in J_{\theta, G}^k$ where $J_{\theta, G}^k$ is as defined in Lemma 19.

For $k = 0$: Assume that $C_G^\diamond\theta \in \Delta$ and let $a \in G$. Then by Lemma 28, there is some $\Sigma^\diamond \in Con(\varphi)$ such that $\Delta \sim^\varphi \Sigma^\diamond$ and $K_a(\theta \wedge C_G^\diamond\theta) \in \Sigma^\diamond$. Consider any arbitrary $\Sigma^a \in Con(\varphi)$ such that $\Sigma^\diamond \mathcal{K}_a^\varphi \Sigma^a$. Then we have $\theta \wedge C_G^\diamond\theta \in \Sigma^a$ which implies that $\theta \in \Sigma^a$ and $C_G^\diamond\theta \in \Sigma^a$. Consequently $\mathcal{M}^\varphi, \Sigma^a \models \theta$ by θ IH. Because Σ^a is arbitrary, we obtain $\mathcal{M}^\varphi, \Sigma^\diamond \models K_a\theta$ and $\mathcal{M}^\varphi, \Delta \models \Diamond K_a\theta$. Again because $a \in G$ is arbitrary, we have $\mathcal{M}^\varphi, \Delta \models E_G^\diamond\theta$ and $E_G^\diamond\theta = \varepsilon_{\theta, G}^0$ so we can conclude by Lemma 19 that $\Delta \in J_{\theta, G}^0$.

kIH: For any $k \leq l$ and any $\Delta \in Con(\varphi)$, $C_G^\diamond\theta \in \Delta$ implies that $\Delta \in J_{\theta, G}^k$.

For $k = l + 1$: Assume that $C_G^\diamond\theta \in \Delta$ and consider any $a \in G$. Then by Lemma 28, there is some $\Sigma^\diamond \in Con(\varphi)$ such that $\Delta \sim^\varphi \Sigma^\diamond$ and $K_a(\theta \wedge C_G^\diamond\theta) \in \Sigma^\diamond$. Consider any arbitrary $\Sigma^a \in Con(\varphi)$ such that $\Sigma^\diamond \mathcal{K}_a^\varphi \Sigma^a$. Then we have $\theta \wedge C_G^\diamond\theta \in \Sigma^a$ which implies that $\theta \in \Sigma^a$ and $C_G^\diamond\theta \in \Sigma^a$. Consequently $\mathcal{M}^\varphi, \Sigma^a \models \theta$ by θ IH and $\Sigma^a \in J_{\theta, G}^l$ by kIH. It follows from Lemma 19 that $\mathcal{M}^\varphi, \Sigma^a \models \varepsilon_{\theta, G}^l$. Therefore, $\mathcal{M}^\varphi, \Sigma^a \models \theta \wedge \varepsilon_{\theta, G}^l$. Because Σ^a is arbitrary, we obtain $\mathcal{M}^\varphi, \Sigma^\diamond \models K_a(\theta \wedge \varepsilon_{\theta, G}^l)$ and hence $\mathcal{M}^\varphi, \Delta \models \Diamond K_a(\theta \wedge \varepsilon_{\theta, G}^l)$. Again because $a \in G$ is arbitrary, we have $\mathcal{M}^\varphi, \Delta \models E_G^\diamond(\theta \wedge \varepsilon_{\theta, G}^l)$. But because $E_G^\diamond(\theta \wedge \varepsilon_{\theta, G}^l) = \varepsilon_{\theta, G}^{l+1}$, we can conclude by Lemma 19 that $\Delta \in J_{\theta, G}^{l+1}$.

(\Leftarrow) Assume that $\mathcal{M}^\varphi, \Gamma \models C_G^\diamond\theta$. Let $C_\theta := \{\Delta \in Con(\varphi) \mid \mathcal{M}^\varphi, \Delta \models C_G^\diamond\theta\}$ and $C_\theta^\# := Con(\varphi) \setminus C_\theta$.

We first prove that for any $\Delta \in C_\theta$ we have $\vdash \underline{\Delta} \rightarrow E_G^\diamond(\theta \wedge \bigwedge_{\Sigma \in C_\theta^\#} \neg \underline{\Sigma})$.

Consider any arbitrary $\Delta \in C_\theta$. We have $\mathcal{M}^\varphi, \Delta \models E_G^\diamond(\theta \wedge C_G^\diamond\theta)$; that is $\mathcal{M}^\varphi, \Delta \models \Diamond K_a(\theta \wedge C_G^\diamond\theta)$ for each $a \in G$. Fix $a \in G$. We have $\mathcal{M}^\varphi, \Delta_a \models K_a(\theta \wedge C_G^\diamond\theta)$ for some $\Delta_a \in Con(\varphi)$ with $\Delta \sim^\varphi \Delta_a$. Fix such Δ_a . Then we have $\mathcal{M}^\varphi, \Lambda \models \theta$ and $\mathcal{M}^\varphi, \Lambda \models C_G^\diamond\theta$ for any $\Lambda \in Con(\varphi)$ such that $\Delta_a \mathcal{K}_a^\varphi \Lambda$. On the one hand, it follows that $\{\chi \mid K_a\chi \in \Delta_a\} \cup \{\neg\theta\}$ is not φ -consistent; otherwise it can be extended to some $\Lambda \in Con(\varphi)$ by Lemma 24, but by construction we have $\neg\theta \in \Lambda$ and $\Delta_a \mathcal{K}_a^\varphi \Lambda$, the latter implying $\theta \in \Lambda$ by θ IH. Thus we obtain $\vdash \bigwedge_{K_a\chi \in \Delta_a} \chi \rightarrow \theta$ and

$$\vdash \bigwedge_{K_a\chi \in \Delta_a} K_a\chi \rightarrow K_a\theta. \quad (4)$$

On the other hand, if $\Sigma \in C_\theta^\#$, then we do not have $\Delta_a \mathcal{K}_a^\varphi \Sigma$. Therefore, we obtain some $\chi_\Sigma \in \mathcal{L}_C$ such that $K_a\chi_\Sigma \in \Delta_a$ but $\chi_\Sigma \notin \Sigma$ for each $\Sigma \in C_\theta^\#$. For such a set Σ , we have $\neg\chi_\Sigma \in \Sigma$ and

$$\begin{aligned} \vdash \underline{\Sigma} &\rightarrow \neg\chi_\Sigma, \\ \vdash \chi_\Sigma &\rightarrow \neg\underline{\Sigma}, \\ \vdash K_a\chi_\Sigma &\rightarrow K_a\neg\underline{\Sigma}, \end{aligned}$$

$$\vdash \bigwedge_{K_a\chi \in \Delta_a} K_a\chi \rightarrow K_a\neg\underline{\Sigma}. \quad (5)$$

It follows from (4) and (5) that $\vdash \bigwedge_{K_a\chi \in \Delta_a} K_a\chi \rightarrow K_a(\theta \wedge \bigwedge_{\Sigma \in C_\theta^\#} \neg \underline{\Sigma})$ and

$$\vdash \Diamond \bigwedge_{K_a\chi \in \Delta_a} K_a\chi \rightarrow \Diamond K_a \left(\theta \wedge \bigwedge_{\Sigma \in C_\theta^\#} \neg \underline{\Sigma} \right). \quad (6)$$

By Item 6 of Lemma 25, we have $\Box \diamond \bigwedge_{K_a \chi \in \Delta_a} K_a \chi \in \Delta_a$. Because $\Delta \sim^\varphi \Delta_a$ and \sim^φ is an equivalence relation, it follows that $\diamond \bigwedge_{K_a \chi \in \Delta_a} K_a \chi \in \Delta$. Therefore,

$$\vdash \underline{\Delta} \rightarrow \diamond K_a \left(\theta \wedge \bigwedge_{\Sigma \in C_\theta^\#} \neg \underline{\Sigma} \right). \quad (7)$$

Because $a \in G$ is arbitrary, we then obtain

$$\vdash \underline{\Delta} \rightarrow E_G^\diamond \left(\theta \wedge \bigwedge_{\Sigma \in C_\theta^\#} \neg \underline{\Sigma} \right). \quad (8)$$

Using Item 3 of Lemma 29 and the fact that $\Delta \in C_\theta$ is arbitrary, we obtain

$$\vdash \bigvee_{\Delta \in C_\theta} \underline{\Delta} \rightarrow E_G^\diamond \left(\theta \wedge \bigvee_{\Delta \in C_\theta} \underline{\Delta} \right). \quad (9)$$

By **Ind**, we have $\vdash \bigvee_{\Delta \in C_\theta} \underline{\Delta} \rightarrow C_G^\diamond \theta$. Consequently, we have $\vdash \underline{\Gamma} \rightarrow C_G^\diamond \theta$ and we can conclude following Item 1 of Lemma 25 that $C_G^\diamond \theta \in \Gamma$. \square

Theorem 31 (Completeness). *The axiomatic system ECK is complete with respect to the class \mathcal{C}_{eck} .*

Proof. Let $\varphi \in \mathcal{L}_C$ be such that $\not\vdash \varphi$. Then $\{\neg\varphi\}$ is φ -consistent. By Lemma 24, $\neg\varphi \in \Gamma$ for some $\Gamma \in \text{Con}(\varphi)$. It follows from Lemma 30 that $\mathcal{M}^\varphi, \Gamma \models \neg\varphi$. Thus $\not\vdash \varphi$. \square

3.3 Decidability

As shown in Section 3.2, the canonical φ -model \mathcal{M}^φ provides a finite countermodel for any $\varphi \in \mathcal{L}_C$ with $\not\vdash \varphi$. Therefore, we can immediately conclude the following standard results.

Proposition 32. *The logic ECK has the finite model property.*

Proposition 33. *The logic ECK is decidable.*

4 Conclusion

In this paper we investigated the logic of eventual common knowledge and provided a sound and complete Hilbert-style axiomatization. The completeness proof relies on a canonical model construction based on an appropriate closure condition on formulas. As shown, this construction yields finite countermodels, which in turn implies the finite model property and decidability of the logic.

We also established that eventual common knowledge differs substantially from standard common knowledge: it is not equivalent to an infinite conjunction of iterated operators, is not factive, and fails the normality axiom. Finally, although the framework developed here allows arbitrary accessibility relations for individual knowledge operators, the completeness proof can be straightforwardly adapted to more restricted frame classes.

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