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Library of Congress ISSN: 1095-5054

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Justification Logic *First published Wed Jun 22, 2011; substantive revision Wed Jul 17, <sup>2024</sup>*

You may say, "I know that Abraham Lincoln was a tall man. " In turn you may be asked how you know. You would almost certainly not replysemantically, Hintikka-style, that Abraham Lincoln was tall in all<br>sitestings consetible with seen beambake Jacter Learn world was likely situations compatible with your knowledge. Instead you would more likely say, "I read about Abraham Lincoln's height in several books, and I haveseen photographs of him next to other people. " One certifies knowledge o other people. " One certifies knowledge<br>stification. Hintikka semantics captures<br>ification logics supply the missing third<br>zation of knowledge as *justified* true belief.<br>?<br><br>?<br><br>n<br>ic Tradition<br>by<br>f Justification Logic<br>Lo by providing a reason, a justification. Hintikka semantics captures a justification. Hintikka semantics captures<br>Justification logics supply the missing third<br>cterization of knowledge as *justified* true belief.<br>ogic?<br>dition knowledge as true belief.Justification logics supply the missing third acterization of knowledge as *justified* true belief.<br>
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- 1.3 Hyperintensionality<br>• 2. The Basic Components of Justification Logic  $\circ$  1.2 Mathematical Logic Tradition<br>  $\circ$  1.3 Hyperintensionality<br>
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- f Plato's characterization of knowledge as *justified* true belief.<br>
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	- 2.5 Factivity
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# • 7. Self-referentiality of justifications<br>
• 8. Quantifiers in Justification Logic<br>
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1. Why Justification Logic?<br>
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<sup>o</sup> 3.6 Connections with Awareness Modes<br>
• 4. Realization Theorems<br>
• 5. Generalizations<br>
• 5.1 Mixing Explicit and Implicit Knowledge<br>
• 5.2 Multi-Agent Possible World Justification Models<br>
• 6. Russell's Example: Induce Justification logics are epistemic logics which allow knowledge and belief modalities to be 'unfolded' into *justification terms*: instead of  $\Box X$  one *n* terms: instead of  $\Box X$  one writes  $t : X$ , and reads it as "*x* is justified by reason *t*". One may think of the details it as "*X* is justified by reason *t*". One may think of writes  $t : X$ , and reads it as "X is justified by reason  $t$ " traditional modal operators as *implicit* modalities, and d justification terms<br>al logics with finer-<br>ification terms has<br>es rise to different<br>their modalities can<br>prm. In this respect as their *explicit* elaborations whichh supplement modal logics with finer-<br>he family of justification terms has<br>of operations gives rise to different<br>in epistemic logics their modalities can<br>licit justification form. In this respect<br>es the explicit, but hidde grained epistemic machinery. The family of justification terms has grained epistemic machinery. The family of justification terms has structure and operations. Choice of operations gives rise to different 9 3.6 Connections with Awareness M<br>
• 4. Realization Theorems<br>
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7. Self-referentiality of justificat tely unfolded into explicit justification form. In this respect<br>n Logic reveals and uses the explicit, but hidden, content of<br>Epistemic Modal Logic.<br>n logic originated as part of a successful project to provide a<br>resonanti  $t : X$ The roots of justification<br>
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o provide a constructive semantics for intuitionistic

abstracted away all but the most basic features of mathematical proofs. Proofs are justifications in perhaps their purest form. Subsequently<br>justification logics were introduced into formal epistemology. This article<br>presents the general range of justification logics as currently understood. I Proofs are justifications in perhaps their purest form. Subsequently<br>justification logics were introduced into formal epistemology. This article presents the general range of justification logics as currently understood. It f justification logics as currently understood. It<br>with conventional modal logics. In addition to<br>ticle examines in what way the use of explicit<br>ght on a number of traditional philosophical<br>whole is still under active deve discusses their relationships with conventional modal logics. In addition to technical machinery, the article examines in what way the use of explicit Proofs are justifications in perhaps their purest form.<br>justification logics were introduced into formal epistemolog<br>presents the general range of justification logics as currently<br>discusses their relationships with conven justification terms sheds light on a number of traditional philosophical problems. The subject as a whole is still under active development.

Frequentiality of justifications<br>
8. Quantifiers in Justification Logic<br>
9. Historical Notes<br>
9. Historical Notes<br>
1.1 Epistemic Tradition logic can be traced by<br>
8. Quantifiers in Justification Logic<br>
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8. Quantifiers in Justification Logic<br>
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9 The roots of justification logic can be traced back to many different sources, two of which are discussed in detail: epistemology andmathematical logic.

# 1.1 Epistemic Tradition

5.2 Multi-Agent Possible World Justification Models<br>
Simester include Tacking in the second is the second internal interna The properties of knowledge and belief have been a subject for formal logic at least since von Wright and Hintikka, (Hintikka 1962, von Wright 1951). Knowledge and belief are both treated as modalities in a way that is now very familiar—*Epistemic Logic*. But of Plato's three criteria for problems. The subject as a w<br>The roots of justification 1<br>sources, two of which a<br>mathematical logic.<br>1.1 Epistemic Tradition<br>The properties of knowledg<br>logic at least since von Wrig<br>1951). Knowledge and belie<br>f now very f *belief*, (Gettier 1963, Hendricks 2005), epistemic logic really works with only two of them. Possible worlds and<br> indistinguishability model belief—one believes what is so under all circumstances thought possible. Factivity brings a trueness component into <sup>p</sup>lay—if something is not so in the actual world it cannot be known, ns gives rise to different only believed. But there is no representation for the justification condition.<br>
Ionics their modalities can<br>
Nonetheless the modal approach has been remarkably successful in Nonetheless, the modal approach has been remarkably successful in permitting the development of a rich mathematical theory and applications, (Fagin, Halpern, Moses, and Vardi 1995, van Ditmarsch, van der Hoek, and Kooi 2007). Still, it is not the whole picture.

> The modal approach to the logic of knowledge is, in a sense, built around is known in a situation if <sup>X</sup> is true in *all*

situations indistinguishable from that one. Justifications, on the other hand, bring an existential quantifier into the picture:  $X$ *there exists* a iustification for  $X$ dichotomy is a familiar one to logicians—in formal logics there exists a<br>next formal logical solution is  $X$  is there in all models for the local proof for a formula X if and only if X is true in all models for the logic. proof for a formula  $X$  if and only if  $X$  is true in all models for the logic.<br>One thinks of models as inherently non-constructive, and proofs as constructive things. One will not go far wrong in thinking ofin general as much like mathematical proofs. Indeed, the first justification dichotomy is a familiar one to logicians—in formal logics there exists a<br>
proof for a formula X if and only if X is true in all models for the logic.<br>
One thinks of models as inherently non-constructive, and proofs as<br>
in

logic was explicitly designed to capture mathematical proofs in arithmetic, something which will be discussed further in Section 1.2.

In Justification Logic, in addition to the category of formulas, there is a second category of *justifications*. Justifications are formal terms, built up In Justification Logic, in addition to the category of formulas, there is a<br>second category of *justifications*. Justifications are formal terms, built up<br>from constants and variables using various operation symbols. Cons from constants and variables using various operation symbols. Constants constructive things. One will not go far wrong in thinking of justifications<br>
in general as much like mathematical proofs. Indeed, the first justification<br>
logic was explicitly designed to capture mathematical proofs in ar In Justification Logic, in addition to the category of formulas, there is a that the second category of *justifications*. Justifications are formal terms, built up me p from constants and variables using various operation differ on which operations are allowed (and also in other ways too). If  $t$  is a justification term and X is a formula,  $t : X$  is a formula, and is intended *there exists* a justification for  $X$  in that situation. This universal/existential dichotomy is a familiar one to logicians—in formal logics there exists a proof for a formula  $X$  if and only if  $X$  is true in all model to be read:is a justifications. Different justification for a<br>is a formula,  $t : X$  is a formula, and is intended<br>is a justification for X. differ on which operations are allowed (and also in other ways<br>a justification term and  $X$  is a formula,  $t : X$  is a formula, and<br>to be read:<br> $t$  is a justification for  $X$ .<br>One operation, common to all justification logi bring an existential quantiher into the picture:  $X$  is *there exists* a justification for  $X$  in that situation. Thi dichotomy is a familiar one to logicians—in forma proof for a formula  $X$  if and only if  $X$  is true in X is a formula,  $t : X$  is a formula, and is intend

#### $t$  is a justification for X.

d justifications. Different justification logics<br>
ce allowed (and also in other ways too). If t is<br>
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larn,<br>
a justification for X.<br>
lall justification logics, is *application* One operation, common to all justification logics, is *application*, written *t* is a justification for *X*.<br>One operation, common to all justification logics, is *application*, written<br>like multiplication. The idea is, if *s* is a justification for  $A \rightarrow B$  and *t* is a is a justification for X.<br>
o all justification logics, is<br>
ea is, if s is a justification for<br>  $[s \cdot t]$  is a justification for n. The idea is, if s is a justification for  $A \rightarrow B$  and t is a<br>
A, then  $[s \cdot t]$  is a justification for  $B^{[1]}$ . That is, the<br>
lowing is generally assumed: validity of the following is generally assumed:

(1) 
$$
s:(A \rightarrow B) \rightarrow (t:A \rightarrow [s \cdot t]:B).
$$

(1)  $s:(A \rightarrow B) \rightarrow (t:A \rightarrow [s \cdot t]:B)$ .<br>This is the explicit version of the usual distributivity of knowledge operators, and modal operators generally, across implication:

(2) 
$$
\Box(A \to B) \to (\Box A \to \Box B).
$$

n existential quantifier into the picture: *X* is known in a situation if comulscience. It asserts that an agent knows everything that is implied by the signiture one to logicians — in formal logics there exists a<br>
or a fa x a justification for X in that situation. This universal/existential<br>
x is a familiar one to logicians—in formal logics there exists a<br>
a formula X if and only if X is true in all models for the logic.<br>
as of models as in In fact, formula (2) is behind many of the problems of *logicalomniscience*. It asserts that an agent knows everything that is implied by the agent's knowledge—knowledge is closed under consequence. While knowable-in-principle, knowability, is closed under consequence, the same cannot be said for any plausible version of actual knowledge. Thedistinction between (1) and (2) can be exploited in a discussion of the paradigmatic Red Barn Example of Goldman and Kripke; here is a the agent's knowledge—knowledge is closed under conse<br>knowable-in-principle, knowability, is closed under conse<br>cannot be said for any plausible version of actual<br>distinction between (1) and (2) can be exploited in a q<br>par

to ne. Justifications, on the other hand,  $\frac{1}{2}$  in fact, formula (2) is behind many of the problems of *logical*<br>tractions, on the other hand,  $\frac{1}{2}$  is the other hand,  $\frac{1}{2}$  is the problems of *logical*<br>that s f justifications<br>
in arindymatic Red Barn Example of Goldman and Her structurities<br>
in arithmetic,<br>
in arithmetic,<br>
suppose 1 am driving through a neighborhood<br>
unbeknowns to me, papier-mâché barns are carter<br>
and the obj the first justification<br>
1 proofs in arithmetic,<br>
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1 proofs in arithmetic,<br>
1.2.<br>
2 Suppose I am driving through a neighb<br>
unbeknownst to me, papier-mâché barns are<br>
that the object in front of me the agent's knowledge – knowledge is closed under consequence. We knowable-in-principle, knowability, is closed under consequence, the cannot be said for any plausible version of actual knowledge. distinction between (1) a Suppose I am driving through a neighborhood in which,unbeknownst to me, papier-mâché barns are scattered, and I see that the object in front of me is a barn. Because I have barn-beforeme percepts, I believe that the object in front of me is a barn. Our intuitions suggest that I fail to know barn. But now suppose that the neighborhood has no fake red barns, and I also notice that the object in front of me is red, so I know a red barn is there. This juxtaposition, being a red barn, which I know, entails there being abarn, which I do not, "is an embarrassment".

In the first formalization of the Red Barn Example, logical derivation will be performed in a basic modal logic in which  $\Box$  is interpreted as the 'belief' modality. Then some of the occurrences of  $\Box$  will be externally interpreted as 'knowledge' according to the problem's description. Let Bbe the sentence 'the object in front of me is a barn', and let  $R$  be the sentence 'the object in front of me is red'.

1.  $\Box B$ , 'I believe that the object in front of me is a barn';

2.  $\square (B \wedge R)$ , 'I believe that the object in front of me is a red barn'.

At the metalevel, 2 is actually knowledge, whereas by the problemdescription, 1 is not knowledge.

 $3. \Box(B \land R \to B)$ , a knowledge assertion of a logical axiom.

Within this formalization, it appears that epistemic closure in its modal form (2) is violated: line 2,  $\square (B \wedge R)$ , and line 3,  $\square (B \wedge R \rightarrow B)$  are form (2) is violated: line 2,  $\Box(B \land R)$ , and line 3,  $\Box(B \land R \rightarrow B)$  are cases of knowledge whereas  $\Box B$  (line 1) is not knowledge. The modal language here does not seem to help resolving this issue.

interpreted as 'I **believe**form (2) is violated: line 2,  $\Box(B \land R)$ , and line 3,  $\Box(B \land R \rightarrow B)$  are<br>cases of knowledge whereas  $\Box B$  (line 1) is not knowledge. The modal<br>language here does not seem to help resolving this issue.<br>Next consider the Red B for belief that B, and v, for belief that  $B \wedge R$ . In addition, let Ext consider the Red Barn Example in Justification Logic where  $t : F$  is<br>
erpreted as 'I **believe** F for reason  $t'$ . Let  $u$  be a specific individual<br>
tification for belief that  $B$ , and  $v$ , for belief that  $B \land R$ . In ad assumptions is: $t : F$ F for reason t'. Let u be a specific indivi-<br>t B and v for belief that  $B \wedge B$  In addition justification for belief that *B*, and *v*, for belief that  $B \wedge R$ <br>*a* be a justification for the logical truth  $B \wedge R \rightarrow B$ . I<br>assumptions is:

- 1.  $u : B$ , ' $u$  is a reason to believe that the object in front of me is a barn'; barn';
- 2.  $v : (B \wedge R)$ , 'v is a reason to believe that the object in front of me is a red barn'; a red barn';
- 3.  $a : (B \wedge R \rightarrow B)$ .

6

On the metalevel, the problem description states that 2 and 3 are cases of knowledge, and not merely belief, whereas 1 is belief which is not knowledge. Here is how the formal reasoning goes:2.  $v : (B \wedge R), 'v$  is a reason to believe that t<br>a red barn';<br>3.  $a : (B \wedge R \rightarrow B)$ .<br>On the metalevel, the problem description state<br>knowledge, and not merely belief, whereas<br>knowledge. Here is how the formal reasoning gc<br>4.  $a$ 

4.  $a : (B \wedge R \to B) \to (v : (B \wedge R) \to [a : v] : B)$ , by principle (1);<br> $\overline{b}$ ,  $\cdots$  ( $B \wedge B$ ),  $\cdots$  [a, a),  $B$  from 2 and 4 hyperpositional laster 4.  $a : (B \land R \to B) \to (v : (B \land R) \to [a \cdot v] : B)$ , by principle<br>5.  $v : (B \land R) \to [a \cdot v] : B$ , from 3 and 4, by propositional logic; 5.  $v : (B \wedge R) \rightarrow [a \cdot v] : B$ , from 3 and 4, by pro<br>6.  $[a \cdot v] : B$ , from 2 and 5, by propositional logic.

Notice that conclusion 6 is  $[a \cdot v] : B$ , and not  $u : B$ ; epistemic closure B, and not  $u : B$ ; epistemic closure<br>logic it was concluded that  $[a \cdot v] : B$ <br>we B for grasses  $v : C$ . The fact that is a case of knowledge, i.e., 'I know B for reason  $a \cdot v$ '. The fact that is a case of knowledge, i.e., I know B for reason  $a \cdot v'$ . The fact that  $u : B$  is not a case of knowledge does not spoil the closure principle, since the letter cloims knowledge appearing for  $[a, a] : B$ . Hence, after latter claims knowledge specifically for  $[a \cdot v] : B$ . Hence after<br>remine a not food a Lindsed learn B but this learning has nothing On the metalevel, the problem description states t<br>knowledge, and not merely belief, whereas 1<br>knowledge. Here is how the formal reasoning goes<br>4.  $a : (B \land R \rightarrow B) \rightarrow (v : (B \land R) \rightarrow [a \cdot v] :$ <br>5.  $v : (B \land R) \rightarrow [a \cdot v] : B$ , from 3 and 4, b observing a red façade, I indeed know  $B$ , but this knowledge has nothing to do with 1, which remains a case of belief rather than of knowledge. Theform (2) is violated: line 2,  $\sqcup$ <br>cases of knowledge whereas  $\sqcup$ <br>language here does not seem to<br>Next consider the Red Barn Ex<br>interpreted as 'I **believe** F for<br>justification for belief that B, a<br>a be a justification f justification logic formalization represents the situation fairly.

Tracking justifications represents the structure of the Red Barn Example in<br>a way that is not captured by traditional epistemic modal tools. The<br>Justification Logic formalization models what seems to be happening in<br>such a a way that is not captured by traditional epistemic modal tools. Thea way that is not captured by traditional epistemic modal tools. The<br>Justification Logic formalization models what seems to be happening in such a case; closure of knowledge under logical entailment is maintainedeven though 'barn' is not perceptually known.<sup>[2]</sup>

# 1.2 Mathematical Logic Tradition

Next consider the Red Barn Example in Justification Logic where  $t : F$  is<br>
interpreted as T believe  $F$  for reason to the two the a specific individual<br>
interpreted as T believe that the object in front of me is a<br>
sasumpt **E** *F* for reason *t*'. Let *u* be a specific individual<br>
and *B*, and *t*, for belief that *B*  $\wedge$  *R*. In addition, let<br>
the logical ruth *B*  $\wedge$  *R*  $\rightarrow$  *B*. Then the list of<br>
section in the specific information According to Brouwer, truth in constructive (intuitionistic) mathematicsmeans the existence of a proof, cf. (Troelstra and van Dalen 1988). In 1931–34, Heyting and Kolmogorov gave an informal description of the intended proof-based semantics for intuitionistic logic (Kolmogorov 1932, Heyting 1934), which is now referred to as the *Brouwer-Heyting-Kolmogorov (BHK) semantics*. According to the BHK conditions, a formula is 'true' if it has a proof. Furthermore, a proof of a compoundstatement is connected to proofs of its components in the following way:

- a proof of  $A \wedge B$  consists of a proof of proposition  $A$  and a proof of<br>proposition  $B$ : proposition  $B$ ;
- a proof of  $A \vee B$  is given by presenting either a proof of  $A$  or a proof<br>of  $B$ : of  $B;$
- a proof of  $A \to B$  is a construction transforming proofs of  $A$  into proofs of  $B$ ; proofs of  $B;$
- falsehood  $\perp$  is a proposition which has no proof,  $\neg A$  is shorthand for  $A \to \perp$ .  $A \rightarrow \perp$ .

Kolmogorov explicitly suggested that the proof-like objects in his interpretation ("problem solutions") came from classical mathematics (Kolmogorov 1932). Indeed, from a foundational point of view it does not make much sense to understand the 'proofs' above as proofs in an intuitionistic system which these conditions are supposed to be specifying.

The fundamental value of the BHK semantics is that informally butunambiguously it suggests treating justifications, here mathematical proofs, as objects with operations.

In (Gödel 1933), Gödel took the first step towards developing a rigorous<br>
proof-based semantics for intuitionism. Gödel considered the classical<br>
modal logic 54 to be a calculus describing properties of provability:<br> **Axi** proof-based semantics for intuitionism. Gödel considered the classicalmodal logic S4 to be a calculus describing properties of provability: The fundamental value of the BrK sentances is that informally out<br>
unambiguously it suggests treating justifications, here mathematical<br>
In (Gödel 1933), Gödel took the first step towards developing a rigorous<br>
proof-base

- Axioms and rules of classical propositional logic:
- $\Box(F \to G) \to (\Box F \to \Box G); \ \Box F \to F;$
- $\Box(F\rightarrow G\nonumber\ \Box F\rightarrow F;$
- $\Box F \to F; \ \Box F \to \Box \Box F;$
- $\Box F \rightarrow \Box \Box F;$ <br>Rule of necessitation: if  $\vdash F$ , then  $\vdash \Box F$ .

unambiguously it suggests treating justifications, here mathematical<br>
uncertifications the relative of the context and  $\pi$  of the context and  $\pi$  of the context and  $\pi$ <br>
In (Gödel 1933), Gödel too ke first step towards Based on Brouwer's understanding of logical truth as provability, Gödel  $r(F)$  of the propositional formula F in the intuitionistic language into the language of classical modal logic:  $tr(F)$  is • Axioms and rules of classical propositional logic;<br>
•  $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G);$ <br>
•  $\Box F \rightarrow F;$ <br>
•  $\Box F \rightarrow \Box \Box F;$ <br>
• Rule of necessitation: if  $\vdash F$ , then  $\vdash \Box F$ .<br>
Based on Brouwer's understanding of logical truth as provabil g every subformula of  $F$  with the provability modality □. Informally speaking, when the usual procedure of determining classical truth of a formula is applied to  $tr(F)$ , it will test the provability (not the truth) of each of  $F$ 's subformulas, in agreement with Brouwer's ideas. From Gödel's results and the McKinsey-Tarski work on topological semantics for modal logic, it follows that the translation  $tr(F)$  provides a proper embedding of the Intuitionistic Propositional Calculus, IPC, into<br>
International Calculus, and the contract of the cont S4, i.e., an embedding of intuitionistic logic into classical logic extended by the provability operator. intuitionistic language into the language of classical m<br>obtained by prefixing every subformula of  $F$  with the  $\Box$ . Informally speaking, when the usual procedure of d<br>truth of a formula is applied to tr( $F$ ), it will te

(3)If IPC proves  $F$ , then S4 proves  $\text{tr}(F)$ .

Still, Gödel's original goal of defining intuitionistic logic in terms of classical provability was not reached, since the connection of S4 to the usual mathematical notion of provability was not established. Moreover, Gödel noted that the straightforward idea of interpreting modality  $\Box F$  as

*F* is *provable in a given formal system T* contradicted Gödel's second<br>incompletences theorem Indeed  $\Box(\Box F \rightarrow F)$  can be derived in **S4** by incompleteness theorem. Indeed,  $\square(\square F \rightarrow F)$  can be derived in S4 by the rule of necessitation from the axiom  $\Box F \rightarrow F$ . On the other hand, the rule of necessitation from the axiom  $\Box F \to F$ . On the other han interpreting modality  $\Box$  as the predicate of formal provability in theory interpreting modality  $\Box$  as the predicate of formal provability in theory  $T$  and  $F$  as contradiction, converts this formula into a false statement that the consistency of  $T$  is internally provable in  $T$ .  $F \rightarrow F$ ) can be derived in S4<br>om  $\Box F \rightarrow F$ . On the other has

The situation after (Gödel 1933) can be described by the following figure where ' $X \hookrightarrow Y$ ' should be read as 'X is interpreted in Y'

 $\hookrightarrow Y$ ' should be read as 'X is interpreted in Y<br>IPC  $\hookrightarrow$  S4  $\hookrightarrow$ ?  $\hookrightarrow$  CLASSICAL PROC

 $\text{IPC} \hookrightarrow \text{S4} \hookrightarrow \text{CLASSICAL PROOFS}$ <br>In a public lecture in Vienna in 1938, Gödel observed that using the format of explicit proofs:

#### (4) $t$  is s a proof of  $F$ .

Following figure<br>
ed in Y'<br>
PROOFS<br>
ved that using the format<br>
ulus S4 (Gödel 1938).<br>
ained unpublished until<br>
blicit proofs had already<br>
ogic of Proofs LP and<br>
ing it to both S4 and<br>
ustification Logic family.<br>
understood can help in interpreting his provability calculus S4 (Gödel 1938). Unfortunately, Gödel's work (Gödel 1938) remained unpublished until 1995, by which time the Gödelian logic of explicit proofs had already been rediscovered, and axiomatized as the Logic of Proofs LP and supplied with completeness theorems connecting it to both  $\mathsf{S4}$  and classical proofs (Artemov 1995).The Logic of the state of the the Gödelian logic of examining classical and the been rediscovered, and axiomatized as the strouwer's ideas. Supplied with completeness th

The Logic of Proofs LP became the first in the Justification Logic family.<br>
Proof terms in LP are nothing but BHK terms understood as classical<br>
logic extended proofs. With LP, propositional intuitionistic logic received the desired rigorous BHK semantics: The Logic of Proofs LP became the first in the Justification Logic family.

$$
IPC \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP} \hookrightarrow CLASSICAL\ PROOFS
$$

 $\text{IPC} \hookrightarrow \text{SA} \hookrightarrow \text{LP} \hookrightarrow \text{CLASSICAL PROOFS}$ <br>For further discussion of the mathematical logic tradition, see the Section 1 of the supplementary document Some More Technical Matters.

# 1.3 Hyperintensionality

The *hyperintensional paradox* was formulated by Cresswell in 1975.

It is well known that it seems possible to have a situation in whichthere are two propositions  $p$  and  $q$  which are logically equivalent and yet are such that a person may believe the one but not the other. If we regard a proposition as a set of possible worlds thentwo logically equivalent propositions will be identical, and so if  $\hat{x}$ believes that' is a genuine sentential functor, the situation described in the opening sentence could not arise. I call this the paradox of hyperintensional contexts. Hyperintensional contextsare simply contexts which do not respect logical equivalence.

Starting with Cresswell himself, several ways of dealing with this have been proposed. Generally these involve adding more layers to familiar<br>received a second connection of the connection of distinctivity between possible world approaches so that some way of distinguishing between logically equivalent sentences is available. Cresswell suggested that thesyntactic form of sentences be taken into account. Justification Logic, in Justification Logic, in<br>gh its mechanism for<br>cation Logic addresses<br>y and, as a bonus, we<br>odel theory, complexity<br>the informal language effect, takes sentence form into account through its mechanism forhandling justifications for sentences. Thus Justification Logic addresses and yet are such that a person may believe the one but not the<br>other. If we regard a proposition as a set of possible worlds then<br>two logically equivalent propositions will be identical, and so if 'x<br>believes that' is a g s Justification Logic addresses<br>sionality and, as a bonus, we<br>eory model theory complexity some of the central issues of hyperintensionality and, as a bonus, we automatically have an appropriate proof theory, model theory, complexity estimates and a broad variety of applications. *aradax* was formulated by Cresswell in 1975.<br>
To formalize mathematical spectra it is seems possible to have a situation in which<br>
stifties proposition *s* are to provide the one but not the<br>
a proposition as also for po

A good example of a hyperintensional context is the informal language used by mathematicians conversing with each other. Typically when a mathematician says he or she knows something, the understanding is that a proof is at hand. But as the following illustrates, this kind of knowledge is essentially hyperintensional.

Fermat's Last Theorem, FLT, is logically equivalent to  $0 = 0$  since Fermat's Last Theorem, FLT, is logically equivalent to  $0 = 0$  since both are provable, and hence denote the same proposition. However, the context of proofs distinguishes them immediately: aproof  $t$  of  $0 = 0$  is not necessarily a proof of FLT, and vice versa.

choice since  $t:X$  was designed to have characteristics of " $t$  is a proof of  $X$ ."  $X$ ." To formalize mathematical speech the justification logic LP is a natural

The fact that propositions X and Y are equivalent in LP,  $X \leftrightarrow Y$ , does The fact that propositions X and Y are equivalent in LP,  $X \leftrightarrow Y$  not warrant the equivalence of the corresponding justification asse and typically  $t:X$  and  $t:Y$  are not equivalent,  $t:X \leftrightarrow t:Y$ .

To formalize mathematical speech the justification logic LP is a natural<br>choice since t:X was designed to have characteristics of "t is a proof of<br>X."<br>The fact that propositions X and Y are equivalent in LP,  $X \leftrightarrow Y$ , does<br> e equivalence of the corresponding justification assertions<br>:X and t:Y are not equivalent, t:X  $\leftrightarrow$  t:Y.<br>
LP, and Justification Logic in general, is not only<br>
fined to distinguish justification assertions for logically<br>
n Going further LP, and Justification Logic in general, is not only sufficiently refined to distinguish justification assertions for logically equivalent sentences, it provides a flexible machinery to connect justifications proof  $t$  of  $0 = 0$  is not necessarily a proof of FLT, and vice versa.<br>To formalize mathematical speech the justification logic LP is a natural<br>choice since  $t:X$  was designed to have characteristics of " $t$  is a proof of<br> equivalent sentences, it provides a flexible machinery to connect<br>justifications of equivalent sentences and hence to maintain constructive<br>closure properties necessary for a quality logic system. For example, let X<br>and Y proof  $t$  of  $0 = 0$  is not necessarily a proof of<br>To formalize mathematical speech the justificat<br>choice since  $t:X$  was designed to have characte<br> $X$ ."<br>The fact that propositions  $X$  and  $Y$  are equival<br>not warrant the eq closure properties necessary for a quality logic system. For example, let Xand Y be provably equivalent, i.e., there is a proof u of  $X \leftrightarrow Y$ , and so  $w(X \leftrightarrow Y)$  is provable in LP. Suppose also that  $v$  is a proof of X, and so and Y be provably equivalent, i.e., there is a proof u of  $X \leftrightarrow Y$ , and so  $u:(X \leftrightarrow Y)$  is provable in LP. Suppose also that v is a proof of X, and so  $u:(X \leftrightarrow Y)$  is provable in LP. Suppose also that v is a proof of X, and so <br>  $v:X$ . It has already been mentioned that this does not mean v is a proof of  $v:\mathcal{X}$ . It has already been mentioned that this does not mean v is a proof of  $Y$ —this is a hyperintensional context. However within the framework of proof t of  $0 = 0$  is not neces<br>
To formalize mathematical spectoric since t: X was designed<br>
X."<br>
The fact that propositions X and<br>
not warrant the equivalence of<br>
and typically t: X and t: Y are no<br>
Going further LP, and Justification Logic, building on the proofs of X and of  $X \leftrightarrow Y$ , we can Justification Logic, building on the proofs of X and of  $X \leftrightarrow Y$ , construct a proof term  $f(u, v)$  which represents the proof of Y a bonus, we construct a proof term  $f(u, v)$  which represents the proof of Y and so <br>y, complexity  $f(u, v)$ : Y is provable. In this respect, Justification Logic goes beyond Cresswell's expectations: logically equivalent sentences display different but constructively controlled epistemic behavior.Extensionality and, as a bonus, we<br>
2. The Basic Components of Justification Logic goes<br>
of theory, complexity<br>
of theory, complexity<br>  $f(u, v): Y$  is provable. In this respect, Justification Logic goes<br>
Cresswell's expectati Going further LP, and Justification Logic in general, is not only sufficiently refined to distinguish justification assertions for logically  $f(u, v)$ : Y is provable. In this respect, Justification Logic goe

In this section the syntax and axiomatics of the most common systems of

#### 2.1 The Language of Justification Logic

In order to build a formal account of justification logics one must make a *structure and operations on them.* A good example of justifications is provided by formal proofs, which have long been objects of study inmathematical logic and computer science (cf. Section 1.2). Jn order to build a formal account of justification logics one must make a<br>
basic structural assumption: *justifications are abstract objects which have*<br>
structure and operations on them. A good example of justifications In order to build a formal account of justification logics one must make a<br>
basic structural assumption: *justifications are abstract objects which have*<br>
structure and operations on them. A good example of justifications

epistemic assertions  $t : F$ , standing for 't is a justification for  $F'$ <br>Instification Logic does not directly analyze what it means for t to justify  $\mathbf y$  analyze what it means for  $t$  to justify F beyond the format  $t : F$ , but rather attempts to characterize this relation<br>axiomatically. This is similar to the way Boolean logic treats its axiomatically. This is similar to the way Boolean logic treats itsconnectives, say, disjunction: it does not analyze the formula  $p \lor q$  but rather assumes certain logical axioms and truth tables about this formula. provided by formal proofs, which have long been objects of stady in<br>mathematical logic and computer science (cf. Section 1.2).<br>Justification Logic is a formal logical framework which incorporates<br>epistemic assertions  $t : F$  $\frac{\vee}{\mathfrak{m}}$ u

a justification for F'. operation sum. With sum, any two justif<br>it means for t to justify<br>into something with broader scope. If s:<br>haracterize this relation<br>be, the combined evidence  $s + t$  rema-<br>operation logic treats its There are several design decisions made. Justification Logic starts with the<br>
simplest base: *classical Boolean logic*, and for good reasons. Justifications<br>
provide a sufficiently serious challenge on even the simplest le simplest base: *classical Boolean logic*, and for good reasons. Justifications d for good reasons. Justifications<br>on even the simplest level. The<br>man-Kripke, Gettier and others,<br>on Logic. The core of Epistemic<br>lassical Boolean base (K, T, K4,<br>them has been provided with a<br>anion based on Boolean logi n even the simplest level. The paradigmatic examples by Russell, Goldman-Kripke, Gettier and others, can be handled with Boolean Justification Logic. The core of Epistemic sions made. Justification Logic starts with the *ean logic*, and for good reasons. Justifications s challenge on even the simplest level. The core of Epistemic lustell, Goldman-Kripke, Gettier and others, n Justification Logic consists of modal systems with a classical Boolean base (K, T, K4, S4, K45, KD45, S5, etc.), and each of them has been provided with a **Example 1997** of the set of the eigent engine epistemic assertions  $t : F$ , standing for 't is a justifulation Logic does not directly analyze what it mear  $F$  beyond the format  $t : F$ , but rather attempts to characte axiom corresponding Justification Logic companion based on Boolean logic. F beyond the format  $t : F$ , but rather attempts to c<br>axiomatically. This is similar to the way Bo<br>connectives, say, disjunction: it does not analyze<br>rather assumes certain logical axioms and truth tabl<br>There are several de Finally, factivity of justifications is not always assumed. This makes it possible to capture the essence of discussions in epistemology involvingmatters of belief and not knowledge.rather assumes certain logical axioms and truth tables about th<br>There are several design decisions made. Justification Logic s<br>simplest base: *classical Boolean logic*, and for good reasons.<br>provide a sufficiently serious epistemic assertions  $t : F$ , standing for 't is a justification for  $F$ '.<br>Justification Logic does not directly analyze what it means for t to justify<br>F beyond the format  $t : F$ , but rather attempts to characterize this rel of justification logics one must make a<br>  $\lambda$ -calcult (Troelstra and Scending and Scending and Scending and Scending and Scending and Scenarios (The strained in Another common operation on justification constants (The par formal account of justification logies one must make a<br>  $\lambda$ -cactul (Irochistan and Sel enotes which have<br>
umption: *justifications* are abstract objects which have<br>
radioatomy semantics (Theorem . A good example of justi It is a formula proportion and proportion in the matter of  $\eta$  comparison in the sequel of the sequel in the sequel of Austin and proportions is and proportion in Logic is a formal logical finness of Austin Transmatic of

The basic operation on justifications is *application*. The *application* s and t and produces a justification  $s \cdot t$  such such  $\alpha$ that if  $s : (F \to G)$  and  $t : F$ , then  $[s \cdot t] : G$ . Symbolically,<br> $s : (F \to G) \to (t : F \to [s, t] : G)$ 

$$
s:(F\to G)\to (t:F\to [s\cdot t]:G)
$$

λ-calculi (Troelstra and Schwichtenberg 1996), Brouwer-Heyting-<br>*Κ*<sub>λ</sub> Kolmogorov semantics (Troelstra and van Dalen 1988), Kleene realizability (Kleene 1945), the Logic of Proofs LP, etc.

1 The Language of Justification Logic<br>
This is a basic property of justifications assumed in combinatory logic and<br>
Scalenti (Trockstra and Schwichtenberg 1996), Browner-Heyting-<br>
Sic structural assumption: *justification* To build a formal account of justification logics one must make a<br>
retro build a formal account of justifications are abstract objects which have<br>
must make a formal poperations on them. A good example of justifications i basic structural assumption: *justifications are abstract objects which have*<br> *structure and operations on them.* A good example of justifications is<br>
provided by formal proofs, which have long been objects of study in<br> d example of justifications is<br>on geome of interfering the diverse points.<br>
for the section 1.2).<br>
to make explicit the modal logic reasoning (Artemov 1995). He<br>
f. Section 1.2).<br>
Some meaning diversion logics like  $J^-$  ( This is a basic property of justifications assumed in combinatory logic and  $\lambda$ -calculi (Troelstra and Schwichtenberg 1996), Brouwer-Heyting-Kolmogorov semantics (Troelstra and van Dalen 1988), Kleene realizability (Klee Another common operation on justifications is sum: it has been introduced<br>to make explicit the modal logic reasoning (Artemov 1995). However,<br>some meaningful justification logics like  $J^-$  (Artemov and Fitting 2019)<br>or JN to make explicit the modal logic reasoning (Artemov 1995). However, some meaningful justification logics like  $J^-$  (Artemov and Fitting 2019)<br>or JNoC<sup>-</sup> (Faroldi, Ghari, Lehmann, and Studer 2024) do not use the<br>operation sum. With sum, any two justifications can safely be combined or JNoC<sup>−</sup> (Faroldi, Ghari, Lehmann, and Studer 2024) do not use the operation sum. With sum, any two justifications can safely be combined<br>into something with broader scope. If  $s : F$ , then whatever evidence  $t$  may<br>be, the combined evidence  $s + t$  remains a justification for  $F$ . More<br>prop into something with broader scope. If  $s : F$ , then whatever evidence t may<br>be the combined evidence  $s + t$  remains a justification for F. More properly, the operation '+' takes justifications s and t and produces  $s + t$ properly, the operation '+' takes justifications s and t and produces  $s + t$ ,<br>which is a justification for everything justified by s or by t.<br> $s : F \rightarrow [s + t] : F$  and  $t : F \rightarrow [s + t] : F$ <br>As motivation, one might think of s and t as t to make explicit the modal logic reasoning (Artemov 1<br>some meaningful justification logics like  $J^-$  (Artemov ar<br>or  $JNoC^-$  (Faroldi, Ghari, Lehmann, and Studer 2024)<br>operation sum. With sum, any two justifications can saf A-calculi (Troelstra<br>
make a<br>  $h \text{ have}$ <br>  $h \text{ and } h \text{ have}$ <br>  $h \text{ and } h \text{ are } h \text{ and } h \$ For which incorporates<br>
in ustification for F'.<br>
in the position sum. With sum, any two<br>
it means for t to justify<br>
into something with broader scope.<br>
haracterize this relation<br>
be, the combined evidence  $s + t$ <br>
olean log dification logics one must make a<br>
X-chinogrovy semantics (Trochstra and variables), Notice-<br>
Interaction the presentation of putifications is sum: it has been introduced<br>
cood example of justifications is and the measure some meaningful justification logics like J<sup>−</sup> (Artemov and Fitting 2019)  $\mathbf{a}$  $s + t$  remains a justification for  $F$  $s$  or by  $t$ 

$$
s: F \to [s+t] : F \text{ and } t: F \to [s+t] : F
$$

be, the combined evidence  $s + t$  remains a justification for  $F$ . More<br>properly, the operation '+' takes justifications  $s$  and  $t$  and produces  $s + t$ ,<br>which is a justification for everything justified by  $s$  or by  $t$ .<br> $s$ encyclopedia, and  $s + t$  as the set of those two volumes. Imagine that one s, contains a sufficient justification for a proposition  $\overline{F}$ As motivation, one might think of  $s$  encyclopedia, and  $s + t$  as the set of those of the volumes, say  $s$ , contains a sufficient of the volumes, say s, contains a sufficient justification for a proposition  $F$ , i.e.,  $s : F$  is the case. Then the larger set  $s + t$  also contains a sufficient justification for  $F$ ,  $[s + t] : F$ . In the Logic of Proofs LP, S , i.e.,  $s : F$  is the case. Then the larger set  $s + t$  also contains a sufficienties<br>institution for  $F$  is  $s + t$  i.  $F$  In the Logic of Proofs LP. Section 1.2.  $s + t'$  can be interpreted as a concatenation of proofs s and t.  $s + t$ 

#### 2.2 Basic Justification Logic  $J_0$

3. (i.e., s: F is the case. Then the larger set  $s + t$  also contains a sufficient<br>in instification for F,  $[s + t]$ : F. In the Logic of Proofs LP, Section 1.2, '<br> $s + t'$  can be interpreted as a concatenation of proofs s and t.  $a, b, c, \ldots$  (with indices  $i = 1, 2, 3, \ldots$  which are justification constants  $a, b, c, ...$  (with indices  $i = 1, 2, 3, ...$  which are omitted whenever it is safe) by means of the operations '.' and '+'. More elaborate logics considered below also allow additional operations on<br>instifications. Constants denote atomic instifications which the system at the content of them has been provided been based Robelan base (K, T, K4,<br>
5, etc.), and each of them has been provided with a<br>
ification Logic of Proofs S and t.<br>
5, etc.), and each of them has been provided with a<br>
if does not analyze; variables denote unspecified justifications. The Basic d with Boolean Justification Logic. The core of Epistemic<br>
sof modal systems with a classical Boolean base (K, T, K4,<br>
sof modal systems with a classical Boolean base (K, T, K4,<br>
sof modal systems with a classical Boolean  $x, y, z$ Logic of Justifications,  $J_0$  is axiomatized by the following.

 *Classical propositional axioms and the rule Modus Ponens* Application Axiom

 $s : (F \to G) \to (t : F \to [s \cdot t] : G),$ <br>Axioms Sum Axioms

 $s : F \to [s + t] : F, \quad s : F \to [t + s] : F.$  $J_0$  is the logic of absolutely skeptical agent for whom no formula  $J_0$  does not derive  $t : F$  for any  $t$  and  $F$ . Such an agent is, however,<br>canable of drawing *relative justification conclusions* of the form capable of drawing relative justification conclusions of the form Cassical propositional axtoms and the rule modus ronens<br>
Application Axiom<br>  $s: (F \rightarrow G) \rightarrow (t : F \rightarrow [s \cdot t] : G),$ <br>
Sum Axioms<br>  $s : F \rightarrow [s + t] : F, s : F \rightarrow [t + s] : F.$ <br>  $\downarrow_0$  is the logic of general (not necessarily factive) justifications f Classical Logic<br> *Classical propositional axioms and the rule Modus Ponens*<br>
Application Axiom<br>  $s: (F \rightarrow G) \rightarrow (t : F \rightarrow [s \cdot t] : G),$ <br>
Sum Axioms<br>  $s: F \rightarrow [s + t] : F, \quad s : F \rightarrow [t + s] : F.$ <br>  $J_0$  is the logic of general (not necessarily facti  $J_0$  is the logic of general (not necessarily factive) justifications for<br>absolutely skeptical agent for whom no formula is provably justified, i<br> $J_0$  does not derive  $t : F$  for any  $t$  and  $F$ . Such an agent is, howev<br>c

If 
$$
x : A, y : B, \ldots, z : C
$$
 hold, then  $t : F$ .

With this capacity  $J_0$  is able to adequately emulate many other Justification Logic systems in its language.

#### 2.3 Logical Awareness and Constant Specifications

f general (not necessarily factive) justifications for an  $e_n : e_{n-1} : \ldots : e_1 : A(n \geq n)$ <br>
al agent for whom no formula is provably justificat, i.e., where A is an axiom of  $\mathcal{L}$ , and  $e_1, e_2, \ldots, e_n$ <br>
generality justifica 3 Logical Awareness and Constant Specifications<br>
ne Logical Awareness principle states that logical axioms are justified ex<br>
ficio: an agent accepts logical axioms as justified (including the ones<br>
neerning justifications) The *Logical Awareness principle* states that logical axioms are justified *ex* n agent accepts logical axioms as justified (including the ones<br>ng justifications). As just stated, Logical Awareness may be too<br>some epistemic situations. However Justification Logic offers the<br>neghborism of Constant Spec stated, Logical Awareness may be toostrong in some epistemic situations. However Justification Logic offers th n some epistemic situations. However Justification Logic offers the mechanism of Constant Specifications to represent varying shades Classical Logic<br> *Classical propositional axioms and the rule Modus*<br>
Application Axiom<br>  $s: (F \rightarrow G) \rightarrow (t : F \rightarrow [s \cdot t] : G)$ ,<br>
Sum Axioms<br>  $s: F \rightarrow [s + t] : F$ ,  $s: F \rightarrow [t + s] : F$ .<br>  $0$  is the logic of general (not necessarily factive) jus *Feness principle* states that logical axioms are justified *ex* accepts logical axioms as justified (including the ones cations). As just stated, Logical Awareness may be too istemic situations. However Justification Logi of Logical Awareness. Example of drawing *relative justification conclusions* of the form<br>
If  $x : A, y : B, ..., z : C$  hold, then  $t : F$ .<br>
With this capacity  $J_0$  is able to adequately emulate many other<br>
Justification Logic systems in its language.<br>
2. Classical propositional axioms and the rule Modus.<br>
Application Axiom<br>  $s: (F \rightarrow G) \rightarrow (t: F \rightarrow [s \cdot t] : G)$ ,<br>
Sum Axioms<br>  $s: F \rightarrow [s \cdot t] : F$ ,  $s: F \rightarrow [t + s] : F$ .<br>  $\downarrow_0$  is the logic of general (not necessarily factive) justification<br>
a ive) justifications for an<br>
is provably justified, i.e.,<br>
then a agent is, however,<br>
where A is an axiom of C, and<br>
must of the form<br>
with indices 1, 2, ..., n. It<br>
intermediate specifications, i.e., w<br>  $CS$ , then  $e_{n-1}$ 

ie states that logical axioms are justified *ex*<br>al axioms as justified (including the ones<br>ust stated, Logical Awareness may be too<br>ons. However Justification Logic offers the<br>Specifications to represent varying shades<br>b Of course one distinguishes between an assumption and a justified<br>assumption. In Justification Logic constants are used to represent<br>justifications of assumptions in situations where they are not analyzed any<br>further. Sup analyzed anyfurther. Suppose it is desired to postulate that an axiom  $\vec{A}$  is justified for the state of the stat ed to postulate that an axiom A is justified for<br>ostulates  $e_1$ : A for some evidence constant  $e_1$ <br>more, it is desired to postulate that this new the knower. One simply postulates  $e_1 : A$  for some evidence constant the knower. One simply postulates  $e_1$ : A for some evidence constant  $e_1$  (with index 1). If, furthermore, it is desired to postulate that this new capacity  $J_0$  is able to adequately emulate many other<br>on Logic systems in its language.<br>Cal Awareness and Constant Specifications<br>al Awareness principle states that logical axioms are justified ex<br>agent accepts logical Since any specific derivation in Justification Log<br>
Awareness.<br>
one distinguishes between an assumption and a justified<br>
i. In Justification Logic constants are used to represent<br>
as of assumptions in situations where the

constant  $e_2$  (with index 2). And so on. Keeping track of indices is not necessary, but it is easy and helps in decision procedures (Kuznets 2008). The set of all assumptions of this kind for a given logic is called a

of formulas of the form $CS$  for a given justification logic  $\mathcal L$ 

$$
e_n : e_{n-1} : \ldots : e_1 : A(n \geq 1),
$$

a is provably justified, i.e.,<br>
uch an agent is, however,<br>
where A is an axiom of C, and  $e_1, e_2, ...$ <br>
with indices 1, 2, ..., n. It is assumed<br>
then t: F.<br>
then t: F.<br>
with indices 1, 2, ..., it is assumed<br>
intermediate s f drawing *relative justification conclusions* of the form<br>
If  $x : A, y : B, ..., z : C$  hold, then  $t : F$ .<br>
So capacity  $J_0$  is able to adequately emulate many other<br>
There are a number of special containing in the language. Constant Specification. Here is the formal definition:<br>
A **Constant Specification** *CS* for a given justification logic  $\mathcal{L}$  is a s<br>
of formulas of the form<br>  $e_n : e_{n-1} : \dots : e_1 : A(n \ge 1)$ ,<br>
where *A* is an axiom of  $\mathcal{$ A **Constant Specification** *CS* for a given justification logic *L* is a set<br>of formulas of the form<br> $e_n : e_{n-1} : \dots : e_1 : A(n \ge 1)$ ,<br>where *A* is an axiom of *L*, and  $e_1, e_2, \dots, e_n$  are similar constants<br>with indices 1, 2, .  $e_n : e_{n-1} : \dots : e_1 : A(n \ge 1),$ <br>where A is an axiom of  $\mathcal{L}$ , and  $e_1, e_2, \dots, e_n$  are similar constants with indices 1, 2, ...,  $n$ . It is assumed that  $CS$  contains all intermediate specifications, i.e., whenever  $e_n : e_{n-1} : \dots : e_1 : A$  is in  $CS$ , then  $e_{n-1} : \dots : e_1 : A$  is in  $CS$ , too. A is an axiom of L, and  $e_1, e_2, \dots, e_n$  are similar constants<br>ndices 1, 2, ..., n. It is assumed that CS contains all<br>ediate specifications, i.e., whenever  $e_n : e_{n-1} : \dots : e_1 : A$  is in CS, then  $e_{n-1} : \ldots : e_1 : A$  is in CS, too.<br>The area a number of special conditions of formulas of the form<br>  $e_n : e_{n-1} : \dots : e_1 : A(n \ge 1)$ ,<br>
where A is an axiom of C, and  $e_1, e_2, \dots, e_n$  are similar constants<br>
with indices 1, 2, ..., n. It is assumed that CS contains all<br>
intermediate specifications, i.e., *is a finite*  $e_n : e_{n-1} : \dots : e_1 :$ <br>  $e_n : e_{n-1} : \dots : e_1 :$ <br>  $e_n : e_{n-1} : \dots : e_1 : e_1$ <br>  $e_n$  with indices 1, 2, ..., *n*. It is ass<br>
intermediate specifications, i.e., whenev<br> *CS*, then  $e_{n-1} : \dots : e_1 : A$  is in *CS*, to<br>  $e_n : e_{n$ specifications of the formulas of the form<br>
in the form of exact is the form of the form of  $e_n : e_{n-1} : \ldots : e_1 : A(n \ge 1)$ ,<br>  $e_n : e_{n-1} : \ldots : e_1 : A(n \ge 1)$ ,<br>  $\ldots, \ell$  and  $F$ . Such an agent is, however,<br>
intermediate specificat

There are a number of special conditions that have been placed onconstant specifications in the literature. The following are the most common.F.<br>
F.<br>
intermediate specifications, i.e., whenever  $e_n : e_{n-1} : ... : e_1$ <br>  $CS$ , then  $e_{n-1} : ... : e_1 : A$  is in  $CS$ , too.<br>
There are a number of special conditions that have been pl<br>
constant specifications in the literature. The

Empty

 $CS = \varnothing$  . This corresponds to an absolutely skeptical agent. It  $CS = \emptyset$ . This corresponds to an al<br>amounts to working with the logic  $J_0$ .

Finite

 $CS$  is a finite set of formulas. This is a fully representative case, nce any specific derivation in Justification Logic will involve

Axiomatically Appropriate

 Each axiom, including those newly acquired through the constantspecification itself, have justifications. In the formal setting, for each axiom A there is a constant  $e_1$  such that  $e_1 : A$  is in CS, and<br>if  $e_n : e_{n-1} : \ldots : e_1 : A \in CS$ , then  $\lambda$ , then if  $e_n : e_{n-1} : \ldots : e_1 : A \in CS,$ r each  $n \geq 1$ . , for $a_1 : a \in CS$ , for each  $n \geq 1$ .<br>propriate constant specifications are necessary  $\begin{aligned} &\text{if}\qquad \qquad e_n : e_{n-1} : \ldots : e_1 : A \in CS\ &\qquad e_{n+1} : e_n : e_{n-1} : \ldots : e_1 : A \in CS,\qquad \text{for}\ &\qquad \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$ 

for ensuring the Internalization property, discussed at the end ofthis section.

Total

For each axiom A and any constants  $e_1, e_2, \ldots, e_n$ ,

$$
e_n : e_{n-1} : \ldots : e_1 : A \in CS.
$$

 $e_n : e_{n-1} : \ldots : e_1 : A \in CS.$ <br>The name *TCS* is reserved for the total constant a given logic). Naturally, the total constant specification is axiomatically appropriate.

We may now specify:

Let CS be a constant specification.  $J_{CS}$  is the logic  $J_0 + CS$ ; the members of CS and the only logic. axioms are those of  $J_0$  together with the members of  $CS$ , and the only rule of inference is *Modus Ponens*. Note that  $J_0$  is  $J_{\varnothing}$ . a given logic). Naturally, the total constant specification is<br>axiomatically appropriate.<br>
may now specify:<br> **Logic of Justifications with given Constant Specification**:<br>
Let CS be a constant specification.  $J_{CS}$  is the

**J** is the logic  $J_0 + Axi$  **Internalization Rule**. The new rule states:

For each axiom A and any constants  $e_1, e_2, \ldots, e_n$  infer<br> $e_n : e_{n-1} : \ldots : e_1 : A$ .  $e_n : e_{n-1} : \ldots : e_1 : A.$ er embodies the idea  $\epsilon$ 

**Logic of Justifications with given Constant Specification** is<br> **Logic of Justifications with given Constant Specification:**<br> **Logic of Justifications with given Constant Specification:**<br> **Logic of Justifications with giv** The latter embodies the idea of unrestricted Logical Awareness for J. A similar rule appeared in the Logic of Proofs LP, and has also been anticipated in Goldman's (Goldman 1967). Logical Awareness, as expressed by axiomatically appropriate Constant Specifications, is an explicit incorporation of the Necessitation Pule in Model Logic: explicit incarnation of the Necessitation Rule in Modal Logic:  $\vdash F \Rightarrow \vdash \Box F$ , but restricted to axioms. Note that J coincides with  $J_{TCS}$ . We may now specify:<br>
Logic of Justifications with given Constant Specification<br>
Let CS be a constant specification.  $J_{CS}$  is the logic  $J_0 + C.S$ <br>
axioms are those of  $J_0$  together with the members of CS, ar<br>
rule of infe For each axiom A and any constants  $e_1, e_2,$ <br>  $e_n : e_{n-1} : \ldots : e_1 : A$ .<br>
The latter embodies the idea of unrestricted Logical Aware<br>
similar rule appeared in the Logic of Proofs LP, and h<br>
anticipated in Goldman's (Goldman

The key feature of Justification Logic systems is their ability to internalize their own derivations as provable justification assertions within their languages. This property was anticipated in (Gödel 1938).

**Theorem 1:** For each axiomatically appropriate constant specification  $CS, J_{CS}$  enjoys Internalization:

 $\vdash$  F, then  $\vdash$  p : F for some justification term p

If  $\vdash F$ , then  $\vdash p$ : *F* for some justification term *p*.<br>
of. Induction on derivation length. Suppose  $\vdash F$ . If *F* is a member<br>
of o, or a member of *CS*, there is a constant  $e_n$  (where *n* might be 1)<br>
that  $e_n$ : **Proof.** Induction on derivation length. Suppose  $\vdash$  F. If F is a member<br>of  $\vdash$  or a member of CS, there is a constant e, (where n might be 1) of  $J_0$ , or a member of CS, there is a constant  $e_n$  (where n might be 1) such that  $e_n : F$  is in CS, since CS is axiomatically appropriate. Then  $e_n$ : F is derivable. If F is obtained by *Modus Ponens* from  $X \to I$ <br>and X then by the Induction Hypothesis  $\vdash$  s  $\cdot$  ( $X \to F$ ) and  $\vdash$  t · 3  $e_n$ : *F* is derivable. If *F* is obtained by *Modus Ponens* from  $X \to F$  and *X*, then, by the Induction Hypothesis,  $\vdash s : (X \to F)$  and  $\vdash t : X$  for some *s*, *t*. Using the Application Axiom,  $\vdash [s \cdot t] : F$ . and X, then, by the Induction Hypothesis,  $\vdash s : (X \to B)$ <br>for some  $s, t$ . Using the Application Axiom,  $\vdash [s \cdot t] : F$ . such that  $e_n : F$  is in  $CS$ , since  $CS$  is axiomatically<br>  $e_n : F$  is derivable. If F is obtained by *Modus Po*,<br>
and X, then, by the Induction Hypothesis,  $\vdash s : (X$ <br>
for some s, t. Using the Application Axiom,  $\vdash [s \cdot t]$ .<br> **Solution** If  $\vdash F$ , then  $\vdash p : F$  for<br> **Proof.** Induction on derivation of  $J_0$ , or a member of CS, t<br>
such that  $e_n : F$  is in CS, si<br>  $e_n : F$  is derivable. If F is<br>
and X, then, by the Induction<br>
for some s, t. Using th a member of *CS*, there is a constant  $e_n$  (where *n*)<br>  $e_n$ : *F* is in *CS*, since *CS* is axiomatically approximately approximately approximately

See Section 2 of the supplementary document Some More Technical Matters for examples of concrete syntactic derivations in justification logic.

## 2.4<sub>s.c.</sub><br>4 Extending Basic Justification Logic

Sis reserved for the total constant specification (for<br>
c). Naturally, the total constant specification is<br>
and X, then, by the Induction Hypothesis,  $\vdash s : (X \cdot A)$ <br>
and X, then, by the Induction Hypothesis,  $\vdash s : (X \cdot B \cdot B$ 1 constant specification is<br>  $e_n : F$  is derivable. If F is obtained by *Modi*<br>
and X, then, by the Induction Hypothesis,  $\vdash s$ <br>
for some s, t. Using the Application Axiom,  $\vdash$ <br>
for some s, t. Using the Application Axiom e constant specification<br>
erm p.<br>  $\vdash$  F. If F is a member<br>  $_n$  (where n might be 1)<br>
cally appropriate. Then<br>  $s$  Ponens from  $X \rightarrow F$ <br>  $:(X \rightarrow F)$  and  $\vdash t : X$ <br>  $[s \cdot t] : F$ .<br>
Some More Technical<br>
vations in justification<br>
vat s for examples of concrete syntactic derivations in justification<br>xtending Basic Justification Logic<br>asic justification logic  $J_0$ , and its extension with a constant<br>cation  $J_{CS}$ , is an explicit counterpart of the small  $J_0$ , and its extension with a constant  $J_{CS}$ , is an explicit counterpart of the smallest normal modal logic K. A proj for some *s*, *t*. Using the Application Axiom,  $\vdash [s \cdot t] : F$ .<br>
See Section 2 of the supplementary document Some More Technical<br>
Matters for examples of concrete syntactic derivations in justification<br>
logic.<br>
2.4 Extendi because the notion of *realization* is central, but some hints are already apparent at this stage of our presentation. For instance, it was noted inSection 1.1 that (1),  $s:(A \to B) \to (t:A \to [s \cdot t]:B)$ , is an explicit Section 1.1 that (1),  $s:(A \to B) \to (t:A \to [s \cdot t]:B)$ , is an explicit version of the familiar modal principle (2),  $\square(A \to B) \to (\square A \to \square B)$ . version of the familiar modal principle (2),  $\square(A \to B) \to (\square A \to \square B)$ <br>In a similar way the first justification logic LP is an explicit counterpart o and X, then, by the Induction Hypothesis,  $\vdash s : (X \rightarrow F)$  and  $\vdash t : X$ <br>for some s, t. Using the Application Axiom,  $\vdash [s \cdot t] : F$ .<br>that Is alogic  $J_0 + CS$ ; the<br>members of CS, and the only<br>logic.<br>**Rule**. The new rule states:<br> modal S4. It turns out that many modal lo Logic:<br>
itation Rule in Modal Logic:<br>
Note that J coincides with  $J_{TCS}$ . by discussing some very familiar logics, leading up to S4 and LP. Up to this point much of our original motivation applies—we have ification Logic systems is their ability to internalize this point much of our original motivation applies—we have justification<br>as provable justification assertions within their<br>as provable in arithmetic. Then we move on family of modal logics, and the arithmetic motivation is no longer

applicable. The phenomenon of having a modal logic with a justification logic counterpart has turned out to be unexpectedly broad.

In almost all cases, one must add operations to the  $+$  and  $\cdot$  of  $J_0$ , along In almost all cases, one must add operations to the  $+$  and  $\cdot$  of  $J_0$ , along with axioms capturing their intended behavior. The exception is factivity, discussed next, for which no additional operations are required, thoughadditional axioms are. It is always understood that constant specification cover axioms from the enlarged set. We continue using the terminology of Section 2.3; for instance a constant specification is axiomatically 2.3; for instance a constant specification is axiomatically<br>
2.3; for instance if the constant spection as stated there, for all axioms<br>
in Section 2.3.<br>
2.6 Positive Introspection<br>
2.3 continues to apply to our new justi appropriate if it meets the condition as stated there, *for all axioms including any that have been added to the original set.* TheoremSection 2.3 continues to apply to our new justification logics, and with the Section 2.3 continues to apply to our new justification logics, and with the<br>
same proof: if we have a justification logic  $JL_{CS}$  with an axiomatically<br>
appropriate constant specification, Internalization holds.<br>
One of appropriate constant specification, Internalization l Section 2.3 continues to apply to our new justification logics, and with the same proof: if we have a justification logic  $J_{CS}$  with an axiomatically appropriate constant specification, Internalization holds.<br>2.5 Factivi Section 2.3; for instance a constant specification is axiomatically<br>appropriate if it meets the condition as stated there, for all axioms<br>including any that have been added to the original set. Theorem 1 from<br>Section 2.3 discussed next, for which no additional operations are require<br>additional axioms are. It is always understood that constant spe<br>cover axioms from the enlarged set. We continue using the termi<br>Section 2.3; for instance a c JL $_{CS}$ in a diditional operations are required, though<br>
no additional operations are required, though<br>
a always understood that constant specifications<br>  $\bullet$  JT<sub>0</sub> =<br>
a constant specification is axiomatically<br>
the condition as s

# 2.5 Factivity

truth. This is embodied in the following.

# **Factivity Axiom**  $t : F \to F$ .

The Factivity Axiom has a similar motivation to the Truth Axiom ofEpistemic Logic,  $\Box F \to F$ , which is widely accepted as a basic property of knowledge. of knowledge.

systems, which makes them capable of representing both partial and of Proofs LP, Section 1.2, as a principal feature of mathematical proofs. Indeed, in this setting Factivity is clearly valid: if there is a mathematical proof  $t$  of  $F$ , then  $F$  must be true.

However, factivity alone does not warrant knowledge, as has beendemonstrated by the Gettier examples (Gettier 1963).

- $JT_0 = J_0 +$  Factivity;
- $JT_0 = J_0 +$  Factivity.<br> $JT = J +$  Factivity.

**Logic of Factive Justifications:**<br>
•  $JT_0 = J_0 +$  Factivity;<br>
•  $JT = J +$  Factivity.<br>
tems  $JT_{CS}$  corresponding to Constant Specifications *CS* are definection 2.3.<br>
Positive Introspection in Section 2.3. $JT_{CS}$  corresponding to Constant Specifications  $CS$ 

# 2.6 Positive Introspection

Their gamedal logic with a justification<br>
The Newtort, ractivity above does not warrant knowledge, as has been<br>
it to be unexpectedly broad.<br>
However, ractivity above does not warrant knowledge, as has been<br>
identificati since It is always understood that is a specification of the same input is a properties of the constant specification is a statematically reduced that constant specifications is a constant specification is a statematicall Section 2.3 continues to apply to our new justification logics, and with the<br>
same proof: if we have a justification logic  $J_{CS}$  with an axiomatically<br>
appropriate constant specification, Internalization holds.<br>
2.5 Fact The Factivity Axiom is adopted for justifications that lead to knowledge.<br>
However, factivity alone does not warrant knowledge, as has been<br>
demonstrated by the Gettier examples (Gettier 1963).<br> **Logic of Factive Justific** Systems JT<sub>CS</sub> corresponding to Constant Specifications CS are defined as<br>in Section 2.3.<br>2.6 Positive Introspection<br>One of the common principles of knowledge is identifying *knowing* and<br>*knowing that one knows*. In a mo One of the common principles of knowledge is identifying *knowing* and *knowing that one knows*. In a modal setting, this corresponds to $\Box F \rightarrow \Box \Box F$ . This principle has an adequate explicit counterpart: the fact  $\Box F \rightarrow \Box \Box F$ . This principle has an adequate explicit that an agent accepts t as sufficient evidence for F One of the common principles of knowledge is identifying *knowing* and<br> *knowing that one knows*. In a modal setting, this corresponds to<br>  $\Box F \rightarrow \Box \Box F$ . This principle has an adequate explicit counterpart: the fact<br>
that evidence for  $t : F$ . Often such 'meta-evidence' has a physical form: a referee report certifying that a proof in a paper is correct; a computer However, factivity alone does not warra<br>
demonstrated by the Gettier examples (Gettier<br>
given by the Gettier examples (Gettier<br>
gh<br>
gh<br> **Logic of Factive Justifications:**<br>  $\int \Gamma_{\text{C}}$   $\int \Gamma_{\text{D}} = \int_0 + \text{Factivity}$ .<br>  $\int \Gamma_{\text{$  $t$  of  $F$  as an input; a formal proof that t is a proof of  $F$ , etc. A *Positive Introspection* operation '!' may be added to the language for this purpose; one then assumes that given  $t$ , the Fraction 1 from<br>
2.6 Positive Introspection<br>
1 an axiomatically<br>
2.6 Positive Introspection<br>
2.6 Positive Introspection<br>  $\Box P \rightarrow \Box \Box F$ . This principle has<br>
that an agent accepts t as sufficient<br>
evidence for  $t : F$ . Often s !t of  $t : F$  such that  $t : F \rightarrow !t : (t : F)$ .<br>
erational form first anneared in the Logic to the Truth Axiom of<br>
added to the language for this purpose; one then assumes that given t, the<br>
pted as a basic property<br>
positive Introspection in this operational form first appeared in the Logic<br>
form first appeared of Proofs **LP**.

## **Positive Introspection Axiom:**  $t : F \rightarrow !t : (t : F)$ .

- $J4 := J +$  Positive Introspection;
- $J4 := J +$  Positive Introspection;<br>LP :=  $JT +$  Positive Introspection.<sup>[3]</sup>

Logics  $J4_0$ ,  $J4_{CS}$ , LP<sub>0</sub>, and LP<sub>CS</sub> Section 2.3).

In the presence of the Positive Introspection Axiom, one can limit the scope of the Axiom Internalization Rule to internalizing axioms which are not of the form  $e : A$ . This is how it was done in LP: Axiometer in the second the condition of the second  $f(x) = \frac{1}{2}$ . Internalization can then be emulated by using  $\mathcal{L}: (e : (e : A))$  instead of  $e_3 : (e_2 : (e_1 : A)),$  etc. The notion In the presence of the Positive Introspection Axiom, one can limit the<br>scope of the Axiom Internalization Rule to internalizing axioms which are<br>not of the form  $e : A$ . This is how it was done in LP: Axiom<br>Internalization affect the main theorems and applications of Justification Logic. A. This is how it was done in LP<br>be emulated by using  $!!e : (e : (e : A))$ 

## 2.7 Negative Introspection

main theorems and applications of Justification Logic.<br>
gative Introspection<br>
gative Introspection<br>
2006, Rubtsova 2006) considered the *Negative Introspection*<br>
<sup>m</sup><sup>1</sup>? which verifies that a given justification assertion 2.7 Negative Introspection<br>
(Pacuit 2006, Rubtsova 2006) considered the *Negative Introspection*<br>
operation '?' which verifies that a given justification assertion is false. A discuss a single operation '?' which verifies that a given justification assertion is false. A a given justification assertion is false. A<br>
ng such an operation is that the positive<br>
well be regarded as capable of providing<br>
the validity of justification<br>
t a justification for F, such a '!' should<br>
normally the cas possible motivation for considering such an operation is that the positive introspection operation '!' may well be regarded as capable of providing *conclusive* verification judgments about the validity of justification (e<sub>1</sub>: A)), etc. The notion of Constant Specification can also be<br>a ecordingly. Such modifications are minor and they do not<br>main theorems and applications of Justification Logic.<br>20106, Rubtsova 2006) considered the *Neg* assertions  $t : F$ , so when t is not a justification for F, such a '!' should<br>conclude that  $-t \cdot F$ . This is normally the case for computer proof conclude that  $\neg t : F$ . This is normally the case for computer proof eckers in formal theories etc. This motivation is Section 2.3).<br>
In the presence of the Positive Introspection Axiom, one can limit the<br>
scope of the Axiom Internalization Rule to internalizing axioms which are<br>
not of the form  $e : A$ . This is how it was done in LP: Axiom however, nuanced: the examples of proof verifiers and proof checkers work with both  $t$  and  $F$  as inputs, whereas the Pacuit-Rubtsova format ? suggests that the only input for '?' is a justification  $t$ , and the result  $\mathcal{U}$ theories, etc. This motivation is,<br>proof verifiers and proof checkers<br>ereas the Pacuit-Rubtsova format ?t<br>a justification t, and the result ?t is supposed to justify propositions  $\neg t : F$  uniformly for all Fs for which<br> $\tau \cdot F$  does not hold. Such an operation '?' does not exist for formal  $t : F$  does not hold. Such an operation '?' does not exist for formal mathematical proofs since ?*t* ?t should then be a single proof of infinitely<br>which is impossible. The operation '?' was,<br>e that did not fit into the original framework in<br>stract versions of formal proofs. many propositions  $\neg t : F$ , which is impossible. The operation '?' was,  $e_3$ :  $(e_2$ :  $(e_1$ : A)), etc. The<br>simplified accordingly. Suc<br>affect the main theorems and<br>2.7 Negative Introspec<br>(Pacuit 2006, Rubtsova 20<br>operation '?' which verifies<br>possible motivation for consity<br>introspection opera F. This is not a justineation for  $F$ , such a 1 should<br>  $F$ . This is normally the case for computer proof<br>
eckers in formal theories, etc. This motivation is,<br>
the examples of proof verifiers and proof checkers<br>
and  $F$  a which justifications were abstract versions of formal proofs. or the ristolar measuration value to mentimizing exist<br>of the form e: A. This is how it was done in alization can then be emulated by using  $!l\epsilon$ :  $(l\epsilon)$ :  $(e_2 : (e_1 : A))$ , etc. The notion of Constant Specificati<br>lified acc t and F as inputs, whereas the Pacuit-Rubtsova format ?t<br>
conly input for '?' is a justification t, and the result ?t is historically, the first example that did not fit into the original framework in s are defined in the natural way (cf.<br>
We define the systems:<br>  $\cdot$   $45 = 44 + \text{Negative}$  Introspection Axiom, one can limit the<br>  $\cdot$   $1045 = 445 + -t : \bot$ ;<br>
In Rule to internalizing axioms which are<br>  $\cdot$   $1745 = 445 + -t : \bot$ ;<br>
In Ru

**Negative Introspection Axiom**  $\neg t : F \rightarrow ?t : (\neg t : F)$ 

- We define the systems:<br>  $\bullet$  J45 = J4 + Negative Introspection;
	- J45 = J4 + Negative Iı<br>JD45 = J45 + ¬*t* : ⊥ ;
	- $\mathsf{J}\mathsf{D}45 = \mathsf{J}45 + \neg t : \bot$ <br> $\mathsf{J}\mathsf{T}45 = \mathsf{J}45 + \mathsf{Factivity}$

J $45_{CS}$ , JD $45_{CS}$ , and JT $45_{CS}$ 

# 2.8 Geach Logics and More

**are defined in the natural way (cf.** We define the systems:<br>
<br>
stive Introspection Axiom, one can fimit the<br>
lization Rule to internalizing axioms which are<br> **are**  $J145 = J45 + r$   $J15 = J45 + r$  and  $J15 = J45 + r$ .<br>
This is how i d accordingly. Such modifications are minor and they do not<br>
e main theorems and applications of Justification Logic.<br>
<br>
gative Introspection<br>
and the second infinite family of modal logics that have justification<br>
2006, R and naturally extend these definitions to  $J45_{CS}$ ,  $JD45_{CS}$ , and  $JT45_{CS}$ .<br>
2.8 Geach Logics and More<br>
Justification logics involving ? were the first examples that went bey<br>
sublogics of LP. More recently it has been • J45 = J4 + Negative Introspection;<br>• JD45 = J45 +  $\pm t$  :  $\perp$  ;<br>• JT45 = J45 + Factivity<br>and naturally extend these definitions to J45<sub>CS</sub>, JD45<sub>CS</sub>, and JT45<sub>CS</sub>.<br>2.8 Geach Logics and More<br>Justification logics involv sublogics of LP. More recently it has been discovered that there is an  $\mathcal{L}_{\mathcal{L}}$ We define the systems:<br>
•  $J45 = J4 + Negative I$ <br>
•  $JD45 = J45 + \neg t : \bot$ <br>
•  $JT45 = J45 + Factivit$ <br>
and naturally extend these c<br>
2.8 Geach Logics and<br>
Justification logics involvir<br>
sublogics of LP. More recentinative family of modal logical which the connection with arithmetic proofs is weak or missing. We discuss a single case in some detail, and sketch others. ?*infinite* family of modal logics that have justification counterparts, but for

n of Constant Specification can also be<br>
difications are minor and they do not<br>
dification Logic.<br>
Justification logics involving ? were the first example<br>
sublogics of LP. More recently it has been disco-<br>
sublogics of e family of modal logics that have justification counterparts, but for<br>the connection with arithmetic proofs is weak or missing. We<br>ss a single case in some detail, and sketch others.<br>Geach proposed the axiom scheme  $\sqrt{2$ Peter Geach proposed the axiom scheme  $\Diamond \Box X \rightarrow \Box \Diamond X$ . When added to axiomatic S4 it yields an interesting logic known as S4.2. Semantically, which the connection with arithmetic proofs is weak or missing. We<br>discuss a single case in some detail, and sketch others.<br>Peter Geach proposed the axiom scheme  $\sqrt{\square}X \rightarrow \square \sqrt{\lambda}X$ . When added to<br>axiomatic S4 it yields an worlds,  $w_1$  and  $w_2$  are accessible from the same world  $w_0$ , there is a common world  $w_4$  accessible from both  $w_1$  and  $w_2$ . Geach's scheme was generalized in Lemmon and Scott (1977) and a corresponding notation was introduced:  $\mathsf{G}^{k,l,m,n}$  is the scheme  $\Diamond^k \Box^l X \rightarrow \Box^m \Diamond^n X$ , where  $k, l, m, n \geq 0$ . Semantically these schemes correspond to generalized<br>versions of confluence. Some people have begun referring to the schemes  $k, l, m, n \geq 0$ . Semantically these schemes correspond to generalize versions of confluence. Some people have begun referring to the scheme (verification can do be<br>
pecification can slobe<br>
inter and they do not<br>
2.8 Geach Logics and More<br>
cation Logic.<br>
Justification logics involving ? were the first examples that went beyond<br>
sublogics of LP. More recently i as *Geach schemes*, and we will follow this practice. More generally, we will call a modal logic a *Geach* logic if it can be axiomatized by adding a o K. The original Geach scheme is  $G^{1,1,1,1}$ ,<br> $G^{0,1,0,0}$ ,  $\Box X$ ,  $G^{0,1,2,0}$ ,  $\Diamond X$ ,  $G^{0,1}X$ but also note that  $\Box X \rightarrow X$  is  $G^{0,1,0,0}, \Box X \rightarrow \Box \Box X$  is but also note that  $\Box X \rightarrow X$  is  $G^{0,1,0,0}$ ,  $\Box X \rightarrow \Box \Box X$  is  $G^{0,1,2,0}$ ,  $\Diamond X \rightarrow \Box \Diamond X$ <br>is  $G^{1,0,1,1}$ , and  $X \rightarrow \Box \Diamond X$  is  $G^{0,0,1,1}$ , so Geach logics include the most is  $G^{1,0,1,1}$ , and  $X \rightarrow \Box \Diamond X$  is G<br>common of the modal logics. Go of the modal logics ince that  $\Box X \rightarrow X$  is  $G^{0,1,0,0}$ ,  $\Box X \rightarrow \Box \Box X$  is  $G^{0,1,2,0}$ ,  $\Diamond X \rightarrow \Box$ <br>
st example that did not fit into the original framework in<br>
is  $G^{1,0,1,1}$ , and  $X \rightarrow \Box \Diamond X$  is  $G^{0,1,0,0}$ ,  $\Box X \rightarrow \Box \Box X$  is **K**. The original Geach scheme is G<br>  $G_{0,1,0,0}$   $\Box X \rightarrow \Box \Box X$  is  $G_{0,1,2,0}$   $\Diamond X \rightarrow$  $\Box X \!\!\rightarrow \!\! X$  is  $\mathsf{G}^{0,1,0,0}, \Box X \!\!\rightarrow \!\!\Box \Box X$  is  $\mathsf{G}^{0,1,2,0}, \Diamond X \!\!\rightarrow \!\!\Box \Diamond X$ 

Geach logic, with axiom scheme  $G^{1,1,1,1}$ ,  $\Diamond \Box X \rightarrow \Box \Diamond X$  added to a system counterpart for  $S4.2$  axiomatically by starting with LP. Then we add two  $S4.2$  leads to  $\sim$  1.4 LP. function symbols,  $f$  and  $g$ , each two-place, and adopt the following axiom<br>relevant all gradients in the following  $A_0$  $S4$ —the system  $S4.2$  ment scheme, calling the resulting justification logic J4.2.

$$
\neg f(t,u) \mathpunct{:}\neg t \mathpunct{:} X \mathord{\rightarrow} g(t,u) \mathpunct{:}\neg u \mathpunct{:}\neg X
$$

scheme, calling the resulting justification logic  $J4.2$ .<br>  $\rightarrow f(t, u): \rightarrow t: X \rightarrow g(t, u): \rightarrow u: \rightarrow X$ <br>
There is some informal motivation for this scheme. In LP, because of the<br>
axiom scheme  $t: X \rightarrow X$ , we have provability of  $(t: X \land u: \rightarrow X)$ There is some informal motivation for this scheme. In LP, because of the principal stress of  $K \times K$  and the properties of  $(K \times K)$ . axiom scheme  $t: X \to X$ , we have provability of  $(t: X \wedge u: \neg X) \to \bot$  for any axiom scheme  $t: X \to X$ , we have provability of  $(t: X \wedge u: \neg X) \to \bot$  for any  $t$  and  $u$ , and thus provability of  $\neg t: X \vee \neg u: \neg X$ . In any context one of the disjuncts and the hold as The scheme shows is equivalent to disjuncts must hold. The scheme above is equivalent to disjuncts must hold. The scheme above is equivalent to  $f(t, u)$ : $\neg t: X \lor g(t, u)$ : $\neg u: \neg X$ , which informally says that in any context  $f(t, u)$ :¬t: $X \lor g(t, u)$ :¬ $u$ :¬ $X$ , we have means for computing a we have means for computing a justification for the disjunct that holds. It<br>
is a strong assumption, but not implausible at least in some circumstances.<br>
A realization theorem connects S4.2 and J4.2, though it is not known is a strong assumption, but not implausible at least in some circumstances. disjuncts must hold. The scheme above<br>  $f(t, u): \neg t : X \lor g(t, u): \neg u : \neg X$ , which informally sa<br>
we have means for computing a justification for the<br>
is a strong assumption, but not implausible at least in<br>
A realization theorem c Every Geach logic has a justification counterpa<br>
Geach logic, with axiom scheme  $G^{1,1,1,1}$ ,  $\Diamond \Box X \rightarrow$ <br>
for S4-the system S4.2 mentioned above. V<br>
counterpart for S4.2 axiomatically by starting wi<br>
function symbols, f an

this has a constructive proof.

As another example, consider  $\mathsf{G}^{1,2,2,1}$ ,  $\Diamond \Box \Box X \rightarrow \Box \Box \Diamond X$ , or equivalently  $\Box$  $\neg$  $\Box X \vee$   $\Box \Box \neg \Box X$ . It has as a corresponding justification axion  $\square\neg\square\square X \lor \square\square\neg\square X$ . It has as a corresponding justification axion scheme the following, where  $f, g$ , and  $h$  are three-place function symbols. A realization theorem connects **S4.2** and **J4.2**, though it is not k<br>this has a constructive proof.<br>As another example, consider  $G^{1,2,2,1}$ ,  $\Diamond \Box \Box X \rightarrow \Box \Box \Diamond X$ , or equi<br> $\Box \Box \Box X \vee \Box \Box \neg \Box X$ . It has as a corresponding j

 $f(t,u,v)$ :¬t: $u$ : $X \vee g(t,u,v)$ : $h(t,u,v)$ :¬ $v$ :¬ $X$ 

g justification axiom<br>ace function symbols.<br> $): \neg v : \neg X$ <br>lear as it is for  $G^{1,1,1,1}$ , but formally things behave quite well.

An intuitive interpretation for  $f$ ,  $g$ , and  $h$  is not as clear as it is for  $G^{1,1,1,1}$ ,<br>but formally things behave quite well.<br>Even though the Geach family is infinite, these logics do not cover the full<br>range of log Even though the Geach family is infinite, these logics do not cover the full range of logics with justification counterparts. For instance, the normal modal logic using the axiom scheme  $\square(\square X \rightarrow X)$ , sometimes called *shift* Add a one-place function symbol  $k$  to the machinery building up *reflexivity*, is not a Geach logic, but it does have a justification counterpart.

 $k(t)$ :(t:X→X). A  $k(t)$ :  $(t:X \rightarrow X)$ . A Realization Theorem holds; this is shown in Fitting (2014b). We speculate that all logics axiomatized with Sahlquist formulas will have justification counterparts, but this remains a conjecture at this point.

#### 3. Semantics

Every Geach logic has a justification conterpart. Consider the original justification cerms, and adopt the justification axiom scheme<br>
Greach logic, with axiom scheme  $\{x_1, x_2, x_3\}$ .  $\{x_1, x_2, x_3\}$ . A Realization Th for S4. 2 methods above. We build a justification (2014)). We specified als all operation interest in the system state in the system state in the system state in the system of the system method,  $x$  and  $y$  and  $y$  and  $y$ justification terms, and adopt the justification axiom scheme  $k(t)$ :  $(tX-X)$ . A Realization Theorem holds; this is shown in Fitting (2014b). We speculate that all logics axiomatized with Sahlquist formulas will have justif The now-standard semantics for justification logic originates in (Fitting 2005)—the models used are generally called *Fitting models* in the literature, but will be called *possible world justification models* here. Possib 2005)—the models used are generally called *Fitting models* in the literature, but will be called *possible world*2005)—the models used are gener<br>literature, but will be called *possib*<br>Possible world justification models Possible world justification models are an amalgam of the familiar possible world semantics for logics of knowledge and belief, due to Hintikka and Kripke, with machineryintroduced by Mkrtychev in (Mkrtychev 1997), (cf. Section 3.4). specification counterparts, but this remains<br>
(2014b). We speculate that all logics axiomatized with<br>
will have justification counterparts, but this remains<br>
point.<br>
3. Semantics<br>
The now-standard semantics for justificati

*possible world justification models* here.<br> *justification models* here.<br> *logics* of knowledge and belief, due to<br> *nachinery* specific to justification terms,<br> *krtychev* 1997), (cf. Section 3.4).<br> **e World Justificati** Extription (Mixel Specific to justification terms,<br>
krtychev in (Mkrtychev 1997), (cf. Section 3.4).<br>
gent Possible World Justification Models for J<br>
a semantics for J<sub>CS</sub>, where CS is any constant<br>
to be defined. Formall 3.1 Single-Agent Possible World Justification Models for J<br>To be precise, a semantics for J<sub>CS</sub>, where CS is any constant<br>specification, is to be defined. Formally, a *possible world justification*<br>logic model for J<sub>CS</sub> i To be precise, a semantics for  $J_{CS}$ , where  $CS$  is any constant *logic model* for  $J_{CS}$  is a structure  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ . Of this,  $\langle \mathcal{G}, \mathcal{R} \rangle$  is a structure  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ . logic model for  $J_{CS}$  is a structure  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ . Of this,  $\langle \mathcal{G}, \mathcal{R} \rangle$  is a binary standard K frame, where  $\mathcal{G}$  is a set of possible worlds and  $\mathcal{R}$  is a binary standard K frame, where G is a set of possible worlds and R is a binary<br>relation on it.  $V$  is a mapping from propositional variables to subsets of G,<br>specifying stomic truth at possible worlds specifying atomic truth at possible worlds. **EXECUTE:** The any context of the same of the same of the same since the finding ways that in any context fication f specification, is to be defined. Formally, a possible world justification K

*world justification*, is to be defined. Formally, a *possible world justification*<br> *logic model* for  $J_{CS}$  is a structure  $M = \langle G, R, \mathcal{E}, \mathcal{V} \rangle$ . Of this,  $\langle G, R \rangle$  is a<br>
standard K frame, where G is a set of possibl The new item is  $\mathcal{E}$ , an *evidence function*, which originated in (Mkrtychev s justification terms and formulas to sets of worlds. The intuitive idea is, if the possible world  $\Gamma$  is in  $\mathcal{E}(t, X)$ , then t is *relevant* or *admissible* evidence for X at world  $\Gamma$ . One should not think of relevant *admissible* evidence for X at world  $\Gamma$ . One should not think of relevant *admissible* evidence for X at world  $\Gamma$ . One should not think of relevant evidence as conclusive. Rather, think of it as more like evidence that can be admitted in a court of law: this testimony, this document is something a jury should examine, something that is pertinent, but something whose

truth-determining status is yet to be considered. Evidence functions must meet certain conditions, but these are discussed a bit later.

Given a  $J_{CS}$  possible world justification model  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ , truth of formula X at possible world  $\Gamma$  is denoted by  $M, \Gamma \Vdash X$ , and is required to meet the following standard conditions: $J_{CS}$  possible world justification model  $M$ X at possible world  $\Gamma$  is denoted by  $\mathcal{M}, \Gamma \Vdash X$ <br>neet the following standard conditions:

For each  $\Gamma \in \mathcal{G}$ :

1.  $\mathcal{M}, \Gamma \Vdash P$  iff  $\Gamma \in \mathcal{V}(P)$  for P a propositional letter;<br>a it is not the case that  $M, \Gamma \Vdash \bot$ . 1. M,  $\Gamma \Vdash P$  iff  $\Gamma \in \mathcal{V}(P)$  for P a <br>2. it is not the case that  $\mathcal{M}, \Gamma \Vdash \bot$ ;<br>3. M  $\Gamma \Vdash X \to Y$  iff it is not 3.  $M, \Gamma \Vdash X \to Y$  iff it is not the case that  $M, \Gamma \Vdash X$  or  $\mathcal{M}, \Gamma \Vdash Y.$ it is not the case that  $M, \Gamma \Vdash \bot$ ;<br>  $M, \Gamma \Vdash X \to Y$  iff it is not the case that  $M, \Gamma \Vdash X$ <br>  $M, \Gamma \Vdash Y$ .

These just say that atomic truth is specified arbitrarily, and propositional<br>connectives behave truth-functionally at each world. The key item is the<br>next one.<br> $1. M, \Gamma \Vdash (t : X)$  if and only if  $\Gamma \in \mathcal{E}(t, X)$  and, for eve connectives behave truth-functionally at each world. The key item is the next one.

1.  $M, \Gamma \Vdash (t : X)$  if and only if  $\Gamma \in \mathcal{E}(t, X)$  and, for every  $\Delta \in \mathcal{G}$  with<br>  $\Gamma \mathcal{R} \Delta$ , we have that  $M, \Delta \Vdash X$ .  $\Gamma \mathcal{R} \Delta$  , we have that  $\mathcal{M}, \Delta \Vdash X$ .

This condition breaks into two parts. The clause requiring that  $\mathcal{M}, \Delta$ This condition breaks into two parts. The clause requiring that  $\mathcal{M}, \Delta \Vdash X$  for every  $\Delta \in \mathcal{G}$  such that  $\Gamma \mathcal{R} \Delta$  is the familiar Hintikka/Kripke condition for every  $\Delta \in \mathcal{G}$  such that  $\Gamma \mathcal{R} \Delta$  is the familiar Hintikka/Kripke condition for X to be believed, or be believable, at  $\Gamma$ . The clause requiring that for X to be believed, or be believable, at  $\Gamma$ . The clause requiring that  $\Gamma \in \mathcal{E}(t, X)$  adds that t should be relevant evidence for X at  $\Gamma$ . Then, informally  $t \cdot X$  is true at a possible world if X is believable at  $\Gamma \in \mathcal{E}(t, X)$  adds that t should be relevant evidence for X at  $\Gamma$ . Then, informally,  $t : X$  is true at a possible world if X is believable at that world informally,  $t : X$  is true at a possible world if  $X$  is believable at that world<br>in the usual sense of epistemic logic, and  $t$  is relevant evidence for  $X$  at that world.

It is important to realize that, in this semantics, one might not believe something for a particular reason at a world either because it is simply not believable, or because it is but the reason is not appropriate.

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Given a  $J_{CS}$  possible world justification model  $M = \langle G, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ , truth<br>of formula X at possible world  $\Gamma$  is denoted by  $M, \Gamma \Vdash X$ , and is<br>required to meet the following standard conditions:<br>required to me Some conditions must still be placed on evidence functions, and theconstant specification must also be brought into the picture. Suppose one constant specification must also be brought into the picture. Suppose one<br>is given s and t as justifications. One can combine these in two different<br>ways: simultaneously use the information from both; or use the<br>informati is given *s* and *t* as justifications. One can combine these in two different ways: simultaneously use the information from both; or use the information from just one of them, but first choose which one. Each gives rise ways: simultaneously use the information from both; or use the information from just one of them, but first choose which one. Each gives axiomatically in Section 2.2. $s$  and  $t$ rise to a basic operation on justification terms,  $\cdot$  and  $+$ , introduced

m just one of them, but first choose which one. Each gives<br>c operation on justification terms,  $\cdot$  and  $+$ , introduced<br>n Section 2.2.<br>relevant evidence for an implication and  $t$  is relevant<br>the antecedent. Then  $s$  and rise to a basic operation on justification terms,  $\cdot$  and  $+$ , introduced<br>axiomatically in Section 2.2.<br>Suppose s is relevant evidence for an implication and t is relevant<br>evidence for the antecedent. Then s and t togeth Suppose s is relevant evidence for an implication and t is relevant evidence for the antecedent. Then  $s$  and  $t$  together provides relevant evidence for the consequent. The following condition on evidence functions is assumed:

$$
\mathcal{E}(s,X \to Y) \cap \mathcal{E}(t,X) \subseteq \mathcal{E}(s \cdot t,Y)
$$
   
added the validity of

With this condition added, the validity of

$$
s:(X \to Y) \to (t:X \to [s\cdot t]:Y)
$$

is secured.

If  $s$  and  $t$  are items of evidence, one might say that something If s and t are items of evidence, one might say that something is justified<br>
by one of s or t, without bothering to specify which, and this will still be<br>
evidence. The following requirement is imposed on evidence functio by one of  $s$  or  $t$ , without bothering to specify which, and this will still be evidence. The following requirement is imposed on evidence functions.

$$
\mathcal{E}(s,X)\cup\mathcal{E}(t,X)\subseteq\mathcal{E}(s+t,X)
$$

Not surprisingly, both

$$
s:X\to [s+t]:X
$$

and

$$
t:X\to [s+t]:X
$$

now hold.

Recall that constants are intended to represent reasons for basic assumptions that are accepted outright. A model  $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  *meets* Finally, the Constant Specification  $CS$  should be taken into account.  $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ <br>'S then  $\mathcal{E}(c, X) = \mathcal{G}$ *CS* provided: if  $c : X \in CS$  then  $\mathcal{E}(c, X) = \mathcal{G}$ 

conditions listed above, and $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ <br>ting Constant Spec  $\mathop{CS}\nolimits$ 

grained analysis that is not possible with Kripke models. See Section 3 of the supplementary document Some More Technical Matters for moredetails.

# 3.2 Weak and Strong Completeness

A formula X is *valid* in a particular model for  $J_{CS}$  if it is true at all  $\overline{X}$ A formula X is *valid* in a particular model for  $J_{CS}$  if it is true at all possible worlds of the model. Axiomatics for  $J_{CS}$  was given in Sections 2.2 and 2.3. A completeness theorem now takes the expected form.

**Theorem 2:** A formula X is provable in  $J_{CS}$  if and only if X is valid in all  $J_{CS}$  models. in all  $J_{CS}$  models.

Despite their similarities, possible world justification models allow a fine-<br>
grained analysis that is not possible with Kripke models. See Section 30<br>
the applementary document Some More Technical Matters for one<br>
detai The completeness theorem as just stated is sometimes referred to as *weak* completeness. It maybe a bit surprising that it iscompleteness. It maybe a bit surprising that it is significantly easier to prove than completeness for the modal logic K. Comments on this point follow. On the other hand it is very general, working for all Constant prove than completeness for the modal logic K. Comments on this point assumptions that are accepted outright<br>Constant Specification *CS* provided: if<br>**Possible World Justification Me**<br>model for  $J_{CS}$  is a structure A<br>conditions listed above, and meetir<br>Despite their similarities, possible  $J_{CS}$  models.<br>
leteness theorem as just stated is sometime<br>
ess. It maybe a bit surprising that it is s<br>
n completeness for the modal logic K. Con<br>
n the other hand it is very general, work<br>
ions.<br>
2005) a stronger versi

In (Fitting 2005) a stronger version of the semantics was also introduced. A model  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is called *fully explanatory* if it meets the A model  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is called *fully explanatory* if it meets the following condition. For each  $\Gamma \in \mathcal{G}$ , if  $M, \Delta \Vdash X$  for all  $\Delta \in \mathcal{G}$  such that  $\Gamma \mathcal{R} \Delta$ , then  $\mathcal{M}, \Gamma \Vdash t : X$  for some justification term t. Note that the  $\Gamma \in {\cal G}, \text{ if } {\cal M}, \Delta$ <br>for some justific wing condition. For each  $\Gamma \in \mathcal{G}$ , if  $\mathcal{M}, \Delta \Vdash X$  for all  $\Delta \in \mathcal{G}$ <br>  $\Gamma \mathcal{R} \Delta$ , then  $\mathcal{M}, \Gamma \Vdash t : X$  for some justification term t. Note th

condition,  $M, \Delta \Vdash X$  for all  $\Delta \in \mathcal{G}$  such that  $\Gamma \mathcal{R} \Delta$ , is the usual

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condition,  $M, \Delta \Vdash X$  for all  $\Delta \in \mathcal{G}$  such that  $\Gamma \mathcal{R} \Delta$ , is the usual condition for X being believable at  $\Gamma$  in the Hintikka/Kripke sense. So, condition for X being believable at  $\Gamma$  in the Hintikka/Kripke sense. So, fully explanatory really says that if a formula is believable at a possible world, there is a justification for it.

Finally, the Constant Specification CS should be taken into account.<br>
Recall that constants are intended to represent reasons for basic condition for X being believable at T in the Hintikka/Kripke sense. So,<br>
Recall that Constant Specification *CS* provided: if  $c : X \in CS$  then  $\mathcal{E}(c, X) = \mathcal{G}$ .<br> **Possible World Justification Model** A possible world justification<br>
model for  $J_{CS}$  is a structure  $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  satis a justification for it.<br>
hodels meet the fully explanatory condition. Models that  $\log$ <br>
models. If constant specification  $CS$  is rich enough<br>
alization theorem holds, then one has completeness wi<br>
mg models meeting  $CS$ . I Not all weak models meet the fully explanatory condition. Models that doare called *strong* models. If constant specification CS is rich enough so g models. If constant specification *CS* is rich enough so<br>lization theorem holds, then one has completeness with<br>g models meeting *CS*. Indeed, in an appropriate sense<br>ith respect to strong models is equivalent to being a that an Internalization theorem holds, then one has completeness with respect to strong models meeting  $CS$ . Indeed, in an appropriate sense completeness with respect to strong models is equivalent to being able to prove Internalization.Encycle and the methodographs is a methodograph of the completeness with respect to strong models meeting CS<br>
empleteness with respect to strong models<br>
for more<br>
The proof of completeness with respections<br>
The proof of co

d meeting Constant Specification CS.<br>
that an Internalization theorem holds, then one h<br>
the world justification models allow a fine-<br>
completeness with respect to strong models is equiv<br>
Some More Technical Matters for m The proof of completeness with respect to strong models bears a close similarity to the proof of completeness using canonical models for themodal logic K. In turn, strong models can be used to give a semantic proof of the Realization Theorem (cf. Section 4). So See Section 3 of<br>
Simply by adding reflexivity of the section strong models by<br>
similarity to the proof of completeness using canonical mo<br>
modal logic K. In turn, strong models can be used to give a ser<br>
or the Realiz

# 3.3 The Single-Agent Family

So far a possible world semantics for one justification logic has been<br>discussed, for J, the counterpart of K. Now things are broadened to<br>encompass justification analogs of other familiar modal logics.<br>Simply, by adding discussed, for J, the counterpart of K. Now things are broadened to encompass iustification analogs of other familiar modal logics. r familiar modal logi<mark>c</mark>s.

**Possible World Justification Model A possible world justification**  $\sim$  **Not all weak models need the interperty required in the second in the second interperty in the and here the interperty incrediction CS is right and t** n  $R$  to the conditions for a model in Section 3.1, one gains the validity of  $t:X$ conditions for a model in Section 3.1, one gains the validity of  $t:X \to X$ <br>for every t and X, and obtains a semantics for JT, the justification logic the comments of this point for every t and X, and obtains a semantics for JT, the justification logic and, working for all Constant analog of the modal logic T, the weakest logic of knowledge. Indeed, if  $M$ , Γ⊩t:X then, for every  $t$  and  $X$ , and obtains a semantics for  $JT$ , the justification logic analog of the modal logic  $T$ , the weakest logic of knowledge. Indeed, if Simply by adding reflexivity of the accessibility relative the modal logic K. Comments on this point<br>it is very general, working for all Constant<br>it is very general, working for all Constant<br>it is very general, working fo ,  $\mathcal{M}, \Gamma \Vdash X$ . Weak and strong completeness theorems are provable using the same machinerythat applied in the case of J, and a semantic proof of a Realization

Theorem connecting JT and T is also available. The same applies to the logics discussed below. logics discussed below.

For a justification analog of K4 an additional unary operator '!' is added to<br>the term language, see Section 2.5. Recall this operator maps justifications<br>to justifications, where the idea is that if t is a justification the term language, see Section should be a justification for t:X. Semantically this adds conditions to a<br>model  $M = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ , as follows.<br>with **S4.2**. Second, as with ?,  $\mathcal{E}$  must be a *strong* evidence model  $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ , as follows.<br>First, of course, R, should be trans For a justification analog of K4 an additional unary operator '!' is added to<br>
the term language, see Section 2.5. Recall this operator maps justifications<br>
to justifications, where the idea is that if t is a justificatio t is a justification for  $X$ ea is that if t is a justification for X, then  $!t$ <br>t:X. Semantically this adds conditions to a

First, of course,  $\mathcal R$  should be transitive, but not Second, a monotonicity condition on evidence functions is required:

If 
$$
\Gamma \mathcal{R} \Delta
$$
 and  $\Gamma \in \mathcal{E}(t, X)$  then  $\Delta \in \mathcal{E}(t, X)$ 

And finally, one more evidence function condition is needed.

$$
\mathcal{E}(t,X)\subseteq \mathcal{E}(!t,t{:}X)
$$

These conditions together entail the validity of  $t:X \rightarrow !t:t:X$  and produce And finally, one more evidence function condition is needed.<br>  $\mathcal{E}(t, X) \subseteq \mathcal{E}(\ell, t:X)$ <br>
These conditions together entail the validity of  $t:X \rightarrow !t:t:X$  and produce<br>
a semantics for J4, a justification analog of K4, with a R Theorem connecting them. Adding reflexivity leads to a logic that is called conditions together entail the validity of  $t:X \rightarrow !t:t:X$  and produce<br>antics for J4, a justification analog of K4, with a Realization<br>m connecting them. Adding reflexivity leads to a logic that is called<br>historical reasons. LP for historical reasons. And finally, one more evidence function condition is needed.<br>  $\mathcal{E}(t, X) \subseteq \mathcal{E}(\ell; t, t; X)$ <br>
These conditions together entail the validity of  $t: X \rightarrow \ell t : t : X$  and provide a semantics for J4, a justification analog of K4, wi

 $, -1,$ corresponding to sublogics of the modal logic **S4**. The first examples the to sublogics of the modal logic S4. The first examples that P were those discussed in Section 2.7, involving a negative perator, '?'. Models for justification logics that include this area conditions. First P is symmetric. went beyond LP were those discussed in Section 2.7, involving a negative LP for historical reasons.<br>We have discussed justification logics that a<br>corresponding to sublogics of the modal logic S4.<br>went beyond LP were those discussed in Section 2<br>introspection operator, '?'. Models for justificat operator add three conditions. First R is symmetric. Second, one adds acondition that has come to be known as *strong evidence*: *M*, Γ⊩t:*X* for all  $\Gamma \in \mathcal{E}(t, X)$ . Finally, there is a condition on the evidence function: logics discussed below.<br>
For a justification analog of<br>
the term language, see Sect<br>
to justifications, where the<br>
should be a justification for<br>
model  $M = \langle G, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ ,  $\varepsilon$ <br>
First, of course,  $\mathcal{R}$  shoul LP $\mathcal{E}(t, \mathcal{X})$  family has a justification count<br>
unction condition is needed.<br>  $\mathcal{E}(t, \mathcal{X})$  family has a justification count<br>
realization theorem connecting<br>  $\mathcal{E}(t, t, \mathcal{X})$  logic family is infinite, and certa<br>
t dellare transitive, but not necessarily reflexive.<br>
soundness results follow in<br>
suddition on evidence functions is required:<br>
In a similar way every me<br>
family has a justification<br>
efter,  $X$ ) then  $\Delta \in \mathcal{E}(t, X)$ <br>
the below.<br>
below,<br>
neanalge of K4 an additional unary operator "!" is added to<br>
account was shown in (R<br>
inc. e.g. see Section 2.5. Recall this operator maps justifications<br>
where the idea is that if  $t$  is a justification f

$$
\overline{\mathcal{E}(t,X)}\subseteq \mathcal{E}(?t,\neg t{:}X)
$$

If this machinery is added to that for  $J4$  we get the logic  $J45$ , a f K45. Axiomatic soundness and completeness

can be proved. In a similar way, related logics JD45 and JT45 can be<br>formulated concentively: A Declination Theorem taking the cannot a 3 interformulated semantically. A Realization Theorem taking the operator ? into account was shown in (Rubtsova 2006).

Moving to Geach logics as introduced in Section 2.8, a semantic model for**J4.2** can also be specified. Suppose  $G = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is an LP model. We<br>call the following provincing First, the frame was has a group of the in also be specified. Suppose  $G = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is an LP model. We following requirements. First, the frame must be convergent, as 1.2. Second, as with ?,  $\mathcal{E}$  must be a *strong* evidence function. And **J4.2** can also be specified. Suppose  $G = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is an LP model. We add the following requirements. First, the frame must be convergent, as with **S4.2.** Second, as with ?,  $\mathcal{E}$  must be a *strong* evidence function. And third,  $\mathcal{E}(f(t, u), \neg t: X) \cup \mathcal{E}(g(t, u), \neg u: \neg X) = \mathcal{G}$ . Completeness and soundness results follow in the usual way. soundness results follow in the usual way.

the idea is that if t is a justification for X, then the state of the Section Suppose  $G = (g, R, \mathcal{E}, \mathcal{Y})$  is an LP<br>
(on for t.X. Semantically this adds conditions to a<br>
with S4.2. Second, as with ?,  $\mathcal{E}$  musst be a third,  $\mathcal{E}(f(t, u), \neg t : X) \cup \mathcal{E}(g(t, u), \neg u : \neg X) = \mathcal{G}$ . Completeness and<br>soundness results follow in the usual way.<br>In a similar way every modal logic axiomatized by Geach schemes in this<br>family has a justification counte family has a justification counterpart, with a Fitting semantics and a realization theorem connecting the justification counterpart with the realization theorem connecting the justification counterpart with the corresponding modal logic. In particular, this tells us that the justification logic family is infinite, and certainly much broader than it was original corresponding modal logic. In particular, this tells us that the justification whilf 34.2. Second, as whilf  $f$ ;  $\mathcal{E}$  indst be a shong c<br>third,  $\mathcal{E}(f(t, u), \neg t : X) \cup \mathcal{E}(g(t, u), \neg u : \neg X) = \mathcal{G}$ .<br>soundness results follow in the usual way.<br>In a similar way every modal logic axiomatized by<br>family has r than it was originally thought to be. It is also the case that some modal logics not previously considered, and not in this family, haveconsidered, and not in this family, have justification counterparts as well.<br>Investigating the consequences of all this is still work in progress.<br>3.4 Single World Justification Models Investigating the consequences of all this is still work in progress. Frammy has a justification counterpart, while realization theorem connecting the justification corresponding modal logic. In particular, this to logic family is infinite, and certainly much brothought to be. It is also the Finally reflexive.<br>
Similar way  $\mathcal{E}(f(t, u), -t \cdot X) \cup \mathcal{E}(g(t, u), -u \cdot -X) = \mathcal{G}$ . Completeness<br>
soundness results follow in the usual way.<br>
In a similar way every modal logic axiomatized by Geach schemes in<br>
family has a ju  $x$ lt:  $x$  and produce<br>  $x$ lt:  $x$  and produce<br>  $x$  thought to be. It is also the case that some modal logics not<br>
with a Realization considered, and not in this family, have justification counterpa<br>
a logic that is calle is a justification for X, then 1*t*<br>  $\mathbf{H}_2$  can also be specified. Suppose  $G = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is an L<br>
subly this adds conditions to a<br>
add the following requirements. First, the frame must be complete<br>
t

#### 3.4 Single World Justification Models

2.5. Recall this operator maps justifications is a 2.5. Recall this operator payintentions of Section 28.1 average here is a condition of the singular payint and the singlet work of the singular payint and the following r the usual way.<br>
tions is required:<br>
soundness results follow in the usual way.<br>
tions is required:<br>
In a similar way every modal logic axiom<br>
family has a justification counterpart, we<br>
is needed.<br>
some family has a jus –g modal logic. In particular, this tells us that the justification<br>is infinite, and certainly much broader than it was originally<br>e. It is also the case that some modal logics not previously<br>nd not in this family, have jus Single world justification models were developed considerably before the more general possible world justification models we have been discussing, (Mkrtychev 1997). Today they can most simply be thought of as possible world. The completeness proof for J and Single world justification models were developed considerably before the<br>
in Section 2.7, involving a negative<br>
in Section 2.7, involving a negative<br>
in Section 2.7, involving a negative<br>
in Section 2.7, involving a negat lified to establish completeness with respect to single world justification models, though of course this was not the original argument. What completeness with respect to single world us is that information about the possible worldstructure of justification models can be completely encoded by the sourse.  $\pi$  should be transitive, but not necessarily reflexive.<br>
sometheses results follow in the usual way.<br>
in a simulate scan be complete the complete the complete state in the case of the complete transitive of the

admissible evidence function, at least for the logics discussed so far. Mkrtychev used single world justification models to establish decidability vused single world justification models to establish decidability<br>
used of them in setting the of them in setting<br>
informally,  $X \cdot Y = \{F \mid G \rightarrow F \in X \text{ and } G \in Y \text{ for some } G\}$ .<br>
informally,  $X \cdot Y$  is the result of applying *Modus Po* of LP, and others have made fundamental use of them in setting complexity bounds for justification logics, as well as for showing<br>conservativity results for justification logics of belief (Kuznets 2000,<br>Kuznets 2008, Milnikel 2007, Milnikel 2009). Complexity results have<br>further been conservativity results for justification logics of belief (Kuznets 2000, Informally, X · Y is the result of applying *Modus Ponens* once between the problem of logical omniscience (Artemov interpreted as subsets of the se Kuznets 2008, Milnikel 2007, Milnikel 2009). Complexity results have further been used to address the problem of logical omniscience (Artemovand Kuznets 2014).Mkrtychev used single world justification models to establish decidability<br>
of LP, and others have made fundamental use of them in setting<br>
of LP, and others have made fundamental use of them in setting<br>
conservativity re complexity bounds for justification logics, as well as for showing

# 3.5 Ontologically Transparent Semantics

The formal semantics for Justification Logic described above in 3.1–3.4  $(s \cdot t)^* \supseteq s^* \cdot$ <br>defines truth value at a given world  $\Gamma$  the same way it is done in<br>Awareness Models: t:F holds at  $\Gamma$  iff systems require additi  $\Gamma$  the same way it is done in Awareness Models:  $t$ :F holds at  $\Gamma$  iff STRIFIN 2006, MINITERT 2007, MINITERT 2009). Complexity refurther been used to address the problem of logical omniscience and Kuznets 2014).<br>
3.5 Ontologically Transparent Semantics<br>
The formal semantics for Justification

- 1. F holds at all worlds accessible from  $\Gamma$  and
- 2. t is admissible evidence for  $F$  according to the given evidence function.

In addition, there is a different kind of semantics, so-called modular semantics, which focuses on making more transparent the ontologicals receive the 2.  $t$  is admissible evidence for  $F$  according to the given evidence function.<br>In addition, there is a different kind of semantics, so-called modular semantics, which focuses on making more transparent the ontological st as sets of formulas. We retain a classical interpretation  $*$  of the propositional formulas  $Fm$ , which, in the case of a single world, reduces tobe evidence for F according to the given evidence<br>
is a different kind of semantics, so-called modular<br>
is a different kind of semantics, so-called modular<br>
distinguish justification assertions t:F and the sense that<br>
dis

$$
\begin{aligned}\n\ast: Fm \mapsto \{0, 1\} \\
\text{with value 0 (false)}\n\end{aligned}
$$

i.e., each formula gets a truth value 0 (false) or 1 (true), with the usual  $R \times R \times R$ Boolean conditions:  $\mathbb{H}A \to B$  iff  $\mathbb{H}A$  or  $\mathbb{H}B$ , etc. The principal issue is how to interpret justification terms. For *sets of formulas X* and *Y*, we define

$$
X \cdot Y = \{ F \mid G \rightarrow F \in X \text{ and } G \in Y \text{ for some } G \}.
$$

if justification terms. For *sets of formulas*  $X$  and  $Y$ , we<br>  $Y = \{F \mid G \rightarrow F \in X \text{ and } G \in Y \text{ for some } G\}.$ <br>  $Y$  is the result of applying *Modus Ponens* once between all<br>
and of  $Y$  (in that order). Justification terms  $Tm$  are<br> Informally,  $X \cdot Y$  is the result of applying *Modus Ponens* once between all Informally,  $X \cdot Y$  is the result of<br>members of  $X$  and of  $Y$  (in members of  $X$  and of  $Y$  (in that order<br>interpreted as subsets of the set of formulas:

$$
*: Tm \mapsto 2^{Fm}
$$

such that

$$
(s\cdot t)^*\supseteq s^*\cdot t^*\;\; \text{and}\;\; \; (s+t)^*\supseteq s^*\cup t^*.
$$

In that order). Justification terms *Tm* are<br>  $t \text{ of formulas:}$ <br>  $: Tm \mapsto 2^{Fm}$ <br>  $t^*$  and  $(s+t)^* \supseteq s^* \cup t^*$ .<br>
to the basic justification logic J; other<br>
osure properties of \*. Note that whereas<br>
lels are interpreted semantica These conditions correspond to the basic justification logic J; other systems require additional closure properties of  $*$ . Note that whereas propositions in modular models are interpreted semantically, as truth values, j systems require additional closure properties of \*. Note that whereas propositions in modular models are interpreted semantically, as truth interpreted syntactically, as sets of formulas. This mormary,  $X^T$  is the<br>members of  $X$  and of<br>interpreted as subsets of<br>such that<br> $(s \cdot t)^*$ :<br>These conditions corre<br>systems require additic<br>propositions in modula<br>values, justifications are is a principal hyperintensional feature: a modular model may treat distinct formulas F and G as equal in the sense that  $F^* = G^*$ , but still be able to distinguish justification assertions  $f \cdot F$  and  $f \cdot G$  for example when  $F \in f^*$ d *t*:*G*, for example when  $F \in t$ interpreted as subsets of the set of formulas<br>  $* : Tm \mapsto 2$ <br>
such that<br>  $(s \cdot t)^* \supseteq s^* \cdot t^*$  and  $(s \cdot t)$ <br>
These conditions correspond to the bas<br>
systems require additional closure prope<br>
propositions in modular models are distinguish justification assertions  $t$ :F and  $t$ :G, for example when  $F \in t^*$ <br>but  $G \notin t^*$  yielding  $\|\cdot t$ :F but  $\mathbb{1} \notin t$ :G. In the general possible world setting,<br>formulas are interpreted classically as subsets of t but  $G \notin t^*$  yielding  $\vdash t$ :  $F$  but  $\not\vdash t$ :  $G$ . In the general possible world setting, formulas are interpreted classically as subsets of the set  $W$  of possible worlds,gives, as well as 1 by inwrites<br>
sets of belief (Kuznets 2000,<br>
Signis of belief (Kuznets 2000,<br>
2009). Complexity results have<br>
members of X and of Y (in that order). Justification terms Tm are<br>
interpreted as subsets of These conditions correspond to the basic justification logic J; other *G* as equal in the sense that  $F^* = G^*$ , but still be able to<br>fication assertions  $t: F$  and  $t:G$ , for example when  $F \in t^*$ <br>ing  $\vdash t: F$  but  $\nvdash t:G$ . In the general possible world setting

$$
*: Fm \mapsto 2^W,
$$

 $*: Fm \mapsto \ 2^W$ and iustification terms are interpreted syntac each world

$$
*: W \times Tm \mapsto 2^{Fm}.
$$
  
ness of justification Logi

n Logic systems with respect to modular models have been demonstrated in (Artemov 2012; Kuznets and Studer 2012).

# 3.6 Connections with Awareness Models

The logical omniscence problem is that in epistemic logics all tautologies The logical omniscence problem is that in epistemic logics all tautologies<br>are known and knowledge is closed under consequence, which is unreasonable. In Fagin and Halpern (1988) a simple mechanism for avoiding the problems was introduced. One adds to the usual Kripkemodel structure an awareness function  $\mathcal A$  indicating for each world which formulas the agent is aware of at this world. Then a formula is taken to beknown at a possible world  $\Gamma$  if 1) the formula is true at all worlds accessible from  $\Gamma$  (the Kripkean condition for knowledge) and 2) the agent is aware of the formula at  $\Gamma$ . Awareness functions can serve as a practical tool for blocking knowledge of an arbitrary set of formulas. However as logical structures, awareness models can exhibit unusual behavior due to the lack of natural closure properties. For example, the agent can know $A \wedge B$  but be aware of neither A nor B and hence not know either. The logical omniscence problem is that in epistemic logics all tautologies<br>are known and knowledge is closed under consequence, which is<br>unreasonable. In Fagin and Halpern (1988) a simple mechanism for<br>avoiding the proble The logical omniscence problem is that in epistemic logics all tare<br>are known and knowledge is closed under consequence, v<br>unreasonable. In Fagin and Halpern (1988) a simple mechan<br>avoiding the problems was introduced. On a gigan is aware of at ans words. Then a formula is taken to be<br>a possible world  $\Gamma$  if 1) the formula is true at all worlds<br>rom  $\Gamma$  (the Kripkean condition for knowledge) and 2) the agent<br>the formula at  $\Gamma$ . Awareness mic logics all tautologies<br>
onsequence, which is<br>  $\Box F$ , read fo<br>
simple mechanism for<br>
dds to the usual Kripke<br>
may for each world which<br>
a formula is taken to be<br>
la is true at all worlds<br>
weldge) and 2) the agent<br>
Obvi

accessible from  $\Gamma$  (the Kripkean condition for knowledge) and 2) the agent<br>
is aware of the formula at  $\Gamma$ . Awareness functions can serve as a practical<br>
logical structures, awareness models can exhibit unusual behavio reminiscent of the one from the awareness models: for any givent the justification assertion t:F holds at world  $\Gamma$  iff 1) F holds<br>A accessible from  $\Gamma$  and 2) t is admissible evidence for F at at all worlds  $\Delta$  accessible from  $\Gamma$  and 2) t is admissible evidence for F at at all worlds  $\Delta$  accessible from  $\Gamma$  and 2) t is admissible evidence for F at  $\Gamma$ ,  $\Gamma \in \mathcal{E}(t, F)$ . The principal difference is in the operations on institutions and corresponding closure conditions on admissible evi d corresponding closure conditions on admissible evidence function  ${\cal E}$  in a dynamic version of awareness models which necessary closure properties specified. This idea has been explored in Sedlár (2013) which specified. This idea has been explored in Sedlár (2013) which the agent is aware of at this world. Then a formula is taken to be<br>t a possible world  $\Gamma$  if 1) the formula is true at all worlds<br>e from  $\Gamma$  (the Kripkean co worked with the language of LP, thinking of it as a multi-agent modal accession from T (the Kripkean contuntion for knowledge) and 2<br>is aware of the formula at  $\Gamma$ . Awareness functions can serve as<br>tool for blocking knowledge of an arbitrary set of formulas. I<br>logical structures, awareness erms as agents (more properly, actions of  $\frac{1}{1}$  the usual  $A \wedge B$  but be aware of neither  $A$  nor  $B$  and hence not know eith<br>Possible world justification logic models use a forcing<br>reminiscent of the one from the awareness models: for<br>justification  $t$  the justification asserti epistemic themes of awareness, group agency and dynamics in a natural way.

#### 4. Realization Theorems

The natural modal epistemic counterpart of the evidence assertion  $t : F$  is  $\Box F$ , read *for some x*, *x*: *F*. This observation leads to the notion of *forgetful*<br>projection which replaces each occurrence of  $t \cdot F$  by  $\Box F$  and hence *projection* which replaces each occurrence of  $t : F$  by  $\Box F$  and hence<br>converts a Instification Logic sentence S to a corresponding Modal Logic converts a Justification Logic sentence  $S$ converts a Justification Logic sentence  $S$  to a corresponding Modal Logic sentence  $S<sup>o</sup>$ . The forgetful projection extends in the natural way from sentences to logics.

Obviously, different Justification Logic sentences may have the same forgetful projection, hence  $S<sup>o</sup>$  loses certain information that was contained in  $S$ . However, it is easily observed that the forgetful projection forgetful projection, hence  $S<sup>o</sup>$  loses certain information that was contained in S. However, it is easily observed that the forgetful projection always maps valid formulas of Justification Logic (e.g., axioms of J) to valid<br>formulas of a corresponding Epistemic Logic (K in this case). The formulas of a corresponding Epistemic Logic  $(K \text{ in this case})$ . The converse also holds: any valid formula of Epistemic Logic is the forgetful projection of some valid formula of Justification Logic. This follows from the Correspondence Theorem 3.

# **Theorem 3:**  $J^o = K$ .

This correspondence holds for other pairs of Justification and Epistemic<br>on<br>systems, for instance J4 and K4, or LP and S4, and many others. In such<br>extended form, the Correspondence Theorem shows that major modal<br>logics su systems, for instance  $J4$  and  $K4$ , or  $LP$  and  $S4$ , and many others. In such extended form, the Correspondence Theorem shows that major modal extended form, the Correspondence Theorem shows that major modallogics such as  $K, T, K4, S4, K45, S5$  and some others have exact Justification Logic counterparts.

> At the core of the Correspondence Theorem is the following RealizationTheorem.

**Theorem 4:** There is an algorithm which, for each modal formula F derivable in K, assigns evidence terms to each occurrence of modality in  $F$  in such a way that the resulting formula  $F<sup>r</sup>$  is derivable in J.<br>Moreover, the realization assigns evidence variables to the negative Moreover, the realization assigns evidence variables to the negative

occurrences of modal operators in  $F$ , thus respecting the existential reading of epistemic modality.

Known realization algorithms which recover evidence terms in modaltheorems use cut-free derivations in the corresponding modal logics.Alternatively, the Realization Theorem can be established semantically by Fitting's method or its proper modifications. In principle, these semantic arguments also produce realization procedures which are based onexhaustive search.reasonable Justification Logic counterpart. For example the logic of formal<br>reasonable Justification Logic counterpart. For example the logic of formal<br>reasonable Justification Logic counterpart. For example the logic of f

It would be a mistake to draw the conclusion that **any** modal logic has a reasonable Justification Logic counterpart. For example the logic of formal provability, **GL**, (Boolos 1993) contains the *Löb Principle*:

(5) 
$$
\Box(\Box F \to F) \to \Box F,
$$
which does not seem to have an aniotomically as

which does not seem to have an epistemically acceptable explicit version. Consider, for example, the case where F is the propositional constant  $\perp$  for *false* If an analogue of Theorem 4 would cover the Löb Principle there for *false*. If an analogue of Theorem 4 would cover the Löb Principle therewould be justification terms s and t such that  $x : (s : \bot \to \bot) \to t : \bot$ <br>But this is intuitively false for feative justification. Indeed, e.  $\bot \to \bot$ method or its proper modifications. In principle, these semantic<br>ts also produce realization procedures which are based on<br>ve search.<br>I be a mistake to draw the conclusion that **any** modal logic has a<br>ble Justification Lo But this is intuitively false for factive justification. Indeed,  $s : \bot \to \bot$  is an instance of the Factivity Axiom. Apply Axiom Internalization to obtain  $f(c:(s:\bot \to \bot))$  for some constant c. This choice of c makes the  $c : (s : \bot \to \bot)$  for some constant c. This choice of c makes the antecedent of  $c : (s : \bot \to \bot) \to t : \bot$  intuitively true and the conclusion  $s : \bot \to \bot$ antecedent of  $c : (s : \bot \to \bot) \to t : \bot$  intuitively true and the conclusion false<sup>[4]</sup>. In particular, the Löb Principle (5) is not valid for the proof interpretation (cf. (Goris 2007) for a full account of which principles of GL are realizable).  $\bot \to \bot) \to t : \bot$ leed,  $s : \bot \to \bot$  $\perp \rightarrow \perp$ <br>n to obt Factivity Axiom. Apply Axiom Internalization to obtain<br>
for some constant c. This choice of c makes the<br>  $s : \bot \rightarrow \bot) \rightarrow t : \bot$  intuitively true and the conclusion<br>
ular, the Löb Principle (5) is not valid for the proof<br>
incor

The Correspondence Theorem gives fresh insight into epistemic modal logics. Most notably, it provides a new semantics for the major modallogics. In addition to the traditional Kripke-style 'universal' reading of  $\Box F$  as *F holds in all possible situations*, there is now a rigorous

'existential' semantics for  $\Box F$  that can be read as *there is a witness (proof, justification) for F*.

or its proper modifications. In principle, these semantic<br>
or its proper modification procedures which are based on<br>
or Dalen 1986) and Gödel's provability reading of 54 (Gö<br>
ch.<br>
1938). In both cases there is a possiblejustification) for r .<br>Justification semantics plays a similar role in Modal Logic to that played by Kleene realizability in Intuitionistic Logic. In both cases, the intended semantics is **existential**: the Brouwer-Heyting-Kolmogorov interpretation of Intuitionistic Logic (Heyting 1934, Troelstra and van Dalen 1988, van Dalen 1986) and Gödel's provability reading of **S4** (Gödel 1933, Gödel 1939) 1938). In both cases there is a possible-world semantics of **universal** character which is a highly potent and dominant technical tool. It does not, however, address the existential character of the intended semantics. It took Kleene realizability (Kleene 1945, Troelstra 1998) to reveal the computational semantics of Intuitionistic Logic and the Logic of Proofs to provide exact BHK semantics of proofs for Intuitionistic and Modal Logic.

Logic.<br>In the epistemic context, Justification Logic and the Correspondence logics ofprovide exact BHK semantics of proofs for Intuitionistic<br>Logic.<br>In the epistemic context, Justification Logic and the Corr<br>Theorem add a new 'justification' component to modal knowledge and belief. Again, this new component was, in fact, an old and central notion which has been widely discussed by mainstreamepistemologists but which remained out of the scope of classical epistemiclogic. The Correspondence Theorem tells us that justifications are hakes the logic. The Correspondence Theorem tells us that justifications are<br>conclusion compatible with Hintikka-style systems and hence can be safely<br>the proof incorporated into the foundation for Epistemic Modal Logic.<br>n compatible with Hintikka-style systems and hence can be safelyincorporated into the foundation for Epistemic Modal Logic.

See Section 4 of the supplementary document Some More Technical Matters for more on Realization Theorems.

#### 5. Generalizations

So far in this article only single-agent justification logics, analogous to so full the single-agent logics of knowledge, have been considered. Justification<br>single-agent logics of knowledge, have been considered. Justification

Logic can be thought of as logic of *explicit* knowledge, related to more conventional logics of *implicit* knowledge. A number of systems beyond those discussed above have been investigated in the literature, involving multiple agents, or having both implicit and explicit operators, or some continuation of these combination of these.

# 5.1 Mixing Explicit and Implicit Knowledge

combination of these.<br>
S.1 Mixing Explicit and Implicit Knowledge<br>
since justification logics provide explicit justifications, while conventional<br>
in the literature as Attentov-Fitting models be<br>
logics of knowledge provi logics of knowledge provide an implicit knowledge operator, it is natural to consider combining the two in a single system. The most common joint logic of explicit and implicit knowledge is **S4LP** (Artemov and Nogina 2005). The language of S4LP is like that of LP, but with an implicit  $k$  power and the value of  $K$  or  $\Box$ . The axiomatics is like knowledge operator added, written either  $\mathbf K$  or  $\Box$  . The axiomatics is like knowledge operator added, written either **K** or  $\Box$ . The axiomatics is like<br>that of **LP**, combined with that of **54** for the implicit operator, together<br>with a connecting axiom  $t: X \to \Box X$  anything that has an explicit with a connecting axiom,  $t : X \to \Box X$ , anything that has an explicit justification is knowable. conventional logics of *implicit* knowledge. A number of systems beyond<br>those discussed above have been investigated in the literature, involving<br>multiple agents, or having both implicit and explicit operators, or some<br>co justification is knowable.<br>Semantically, possible world justification models for LP need no **EVALUAT:**<br> **ENDIGENTMONE INTERTMONE INTERTMONE INTERTMONE IN the set of set of set of set of these discussed above have been investigated in the literature, involving multiple agents, or having both implicit and explicit** and loges of *umplicit* knowledge. A number of systems beyond the prioric is associated above have been investigated in the literature, involving of implicit knowledge examples to these.<br>
times are the interactions, while

 Hintikka/Kripke models. One models the  $\Box$  operator in the usual way, making use of just the accessibility relation y relation, and one models the justification terms as<br>Section 3.1 using both accessibility and the evidence<br>the usual condition for  $\Box X$  being true at a world is one of<br>is of the condition for  $t : X$  being true, this imme described in Section 3.1 using both accessibility and the evidence function. Since the usual condition for  $\Box X$  being true at a world is one of the two clauses of the condition for  $t : X$  being true, this immediately yields the validity of  $t : X \to \Box X$ , and soundness follows easily. yields the validity of  $t : X \to \Box X$ , and soundnet Axiomatic completeness is also rather straightforward. Since justincation logics provide explicit justit<br>logics of knowledge provide an implicit know<br>to consider combining the two in a single syst<br>logic of explicit and implicit knowledge is S<br>2005). The language of S4LP is li

In S4LP both implicit and explicit knowledge is represented, but in e world justification model semantics a single accessibility relation serves for both. This is not the only way of doing it. More generally, an explicit knowledge accessibility relation could be a proper extension of

that for implicit knowledge. This represents the vision of explicit knowledge as having stricter standards for what counts as known than that of implicit knowledge. Using different accessibility relations for explicit and implicit knowledge becomes necessary when these epistemic notionsobey different logical laws, e.g.,  $\overline{55}$  for implicit knowledge and LP for  $smel$  is  $\overline{5}$ . explicit. The case of multiple accessibility relations is commonly known<br>is the literature of Assume Fitting and the hot will be selled world associated in the literature as Artemov-Fitting models, but will be called multi-agent<br>nearly much be calculated from  $(f, S, \omega)$ possible world models here. (cf. Section 5.2). knowledge as having strict<br>of implicit knowledge. Us<br>and implicit knowledge be<br>obey different logical law<br>explicit. The case of mult<br>in the literature as Artemo<br>possible world models her<br>Curiously, while the log<br>Theorem ha

Curiously, while the logic S4LP seems quite natural, a Realization Theorem has been problematic for it: no such theorem can be proved if one insists on what are called *normal* realizations (Kuznets 2010). Realization of implicit knowledge modalities in S4LP by explicit d respect the epistemic structure remains a major challenge in this area.

Interactions between implicit and explicit knowledge can sometimes berather delicate. As an example, consider the following mixed principle of negative introspection (again  $\Box$  should be read as an implicit epistemic operator),

(6) 
$$
\neg t : X \to \Box \neg t : X.
$$

From the provability perspective, it is the right form of negative  $\neg t : X \to \Box \neg t : X$ . introspection. Indeed, let  $\Box F$  be interpreted as *F* is *provable* and  $t : F$  as *t* is *a proof of F* in a given formal theory  $T \cdot e \cdot g$  in Peano Arithmetic PA *is a proof of F* in a given formal theory *T*, e.g., in Peano Arithmetic PA.<br>Then (6) states a provable principle. Indeed, if *t* is not a proof of *F* then Then (6) states a provable principle. Indeed, if  $t$  is not a proof of  $F$  then, since this statement is decidable, it can be established inside  $T$ , hence in  $T$ <br>this septence is proveble. On the other hand, the proof n of 't is not a proof this sentence is provable. On the other hand, the proof p of 't is not a proof of F' depends on both t and  $F, p = p(t, F)$  and cannot be computed given<br>t only. In this respect,  $\Box$  cannot be replaced by any specific proof term t only. In this respect,  $\Box$  cannot be replaced by any specific proof terr Figure 1.1 In this sentence is provable. On the other hand, the proof p of 't is not a proof<br>ication model semantics a single accessibility relation<br>ication model semantics a single accessibility relation<br>is is not the on depending on  $t$  only and  $(6)$  cannot be presented in an entirely explicit  $y$ le format.

The first examples of explicit/implicit knowledge systems appeared in the These may be assumed to be reflexive, transitive, or symmetric, as desired.<br>They are used to model implicit agent knowledge for the family of agents area of provability logic. In (Sidon 1997, Yavorskaya (Sidon) 2001), alogic LPP was introduced which combined the logic of provability GL<br>with the logic of proofs LP, but to ensure that the resulting system had with the logic of proofs  $LP$ , but to ensure that the resulting system had desirable logical properties some *additional operations* from outside the original languages of GL and LP were added. In (Nogina 2006, Nogina<br>2007) a complete logical system. CLA, for proofs and provobility wes 2007) a complete logical system,  $GLA$ , for proofs and provability was offered, in the sum of the *original languages* of **GL** and LP. Both LPP<br>**GLA** enjoy completeness relative to the class of arithmetical models offered, in the sum of the *original languages* of **GL** and LP. Both LPP and **GLA** enjoy completeness relative to the class of arithmetical models, and the validity to the class of arithmetical models, and also desirable logical properties some *additional operations* from outside the<br>
original languages of GL and LP were added. In (Nogina 2006, Nogina 3.3. *V* maps propos<br>
2007) a complete logical system, GLA, for proofs and pr

orelative to the class of possible world justification models.<br>
Substituted to the class of possible world justification models.<br>
The ones of properties to find a proof term  $l(x)$  such that<br>  $x : (\Box F \rightarrow F) \rightarrow l(x) : F$ <br>
A,  $\Gamma \Vdash K$ Another example of a provability principle that cannot be madecompletely explicit is the Löb Principle (5). For each of LPP and GLA, it<br>is easy to find a proof term  $l(x)$  such that is easy to find a proof term  $l(x)$  such that

(7) 
$$
x:(\Box F \to F) \to l(x):F
$$

holds. However, there is no realization which makes all *three*  $\square$  s in (5) explicit. In fact, the set of realizable provability principles is the intersection of  $GL$  and  $S4$  (Goris 2007). In *holds.* However, there is no realization which makes all *th* explicit. In fact, the set of realizable provability primetersection of GL and S4 (Goris 2007).<br>5.2 Multi-Agent Possible World Justification M<br>In *multi-age*  $\rightarrow$  F)  $\rightarrow$   $l(x)$  : F<br>ization which makes all three  $\Box$ 

# 5.2 Multi-Agent Possible World Justification Models

s multiple accessibility relations are employed, with connections between them, (Artemov 2006). The idea is, there are multiple agents, each with an implicit knowledge completely explicit is the Löb Principle (5). For is easy to find a proof term  $l(x)$  such that<br>
(7)  $x : (\Box F \rightarrow F) \rightarrow l(x) :$ <br>
holds. However, there is no realization which m<br>
explicit. In fact, the set of realizable prova<br>
inter operator, and there are justification terms, which each agent understands. Loosely, everybody understands explicit reasons; these amount to*evidence-based common knowledge*.(7)  $x : (\Box F \rightarrow F) \rightarrow l(x) : F$ <br>holds. However, there is no realization which makes all *thre*<br>explicit. In fact, the set of realizable provability princ<br>intersection of GL and S4 (Goris 2007).<br>5.2 Multi-Agent Possible World Justi

*n*-agent possible world justification model is a structure  $\langle G, \mathcal{R}_1, ..., \mathcal{R}_n \rangle$  meeting the following conditions  $G$  is a set of possible  $\mathcal{R}_n, \mathcal{R}, \mathcal{E}, \mathcal{V}$  meeting the following conditions.  $\mathcal{G}$  is a set of possible worlds. Each of  $\mathcal{R}_1, ..., \mathcal{R}_n$  is an accessibility relation, one for each agent. These may be assumed to be reflexive, transitive, or symmetric, as desired. e reflexive, transitive, or symmetric, as desired.<br>
Dicit agent knowledge for the family of agents.<br>
R meets the LP conditions, reflexivity and<br>
the modeling of explicit knowledge.  $\mathcal E$  is an<br>
the same conditions as tho The accessibility relation  $\mathcal R$ The accessibility relation  $\mathcal R$  meets the LP conditions, reflexivity and transitivity. It is used in the modeling of explicit knowledge.  $\mathcal E$  is an explicit was found to provide the modeling of explicit showledge.  $\math$ evidence function, meeting the same conditions as those for LP in Section 3.3.  $V$  maps propositional letters to sets of worlds, as usual. There is a special condition imposed: for each  $i = 1, ..., n, \mathcal{R}_i \subseteq \mathcal{R}$ . They are used to model implicit agent know<br>The accessibility relation  $R$  meets the<br>transitivity. It is used in the modeling<br>evidence function, meeting the same cono<br>3.3.  $V$  maps propositional letters to sets<br>special con

If  $M = \langle \mathcal{G}, \mathcal{R}_1, ..., \mathcal{R}_n, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$  is a multi-agent possible world justification model a truth-at-a-world relation,  $M, \Gamma \Vdash X$ , is defi<br>most of the usual clauses. The ones of particular interest are these: evidence function, meeting the :<br>
3.3. V maps propositional lette<br>
special condition imposed: for e<br>
If  $M = \langle \mathcal{G}, \mathcal{R}_1, ..., \mathcal{R}_n, \mathcal{R}, \mathcal{E}$ <br>
justification model a truth-at-a-<br>
most of the usual clauses. The o<br>
•  $M,$  $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}_1, ..., \mathcal{R}_n, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ <br>ification model a truth-at-a-work  $M, Γ ⊩  
r interes$ 

- $\mathcal{M}, \Gamma \Vdash K_iX$  if and only if, for every  $\Delta \in \mathcal{G}$  with  $\Gamma \mathcal{R}_i\Delta$ , we have that  $\mathcal{M}, \Delta \Vdash X$ .  $\mathcal{M}, \Gamma \Vdash K_iX$  if<br>that  $\mathcal{M}, \Delta \Vdash X$ .
- that  $\mathcal{M}, \Delta \Vdash X.$ <br>  $\mathcal{M}, \Gamma \Vdash t : X$  if and only if  $\Gamma \in \mathcal{E}(t, X)$  and, for every  $\Delta \in \mathcal{G}$  with  $\Gamma \mathcal{R} \Delta,$  we have that  $\mathcal{M}, \Delta \Vdash X.$  $\mathcal{M}, \Gamma \Vdash t : X$  if and only if  $\Gamma \in \Gamma \mathcal{R}\Delta$ , we have that  $\mathcal{M}, \Delta \Vdash X$ .

The condition  $\mathcal{R}_i \subseteq \mathcal{R}$  entails the validity of  $t : X \to K_i X$ , for each The condition  $\mathcal{R}_i \subseteq \mathcal{R}$  entails the validity of  $t : X \to K_iX$ , for each agent *i*. If there is only a single agent, and the accessibility relation for that is reflexive and transitive, this provides another semantics for S4LP. Whatever the number of agents, each agent accepts explicit reasons as establishing knowledge.

e accessibility relation  $R$  meets the LP conditions, reflexivity and<br>nistivity. It is used in the modeling of explicit knowledge.  $\ell$  is an<br>idence function, meeting the same conditions as those for LP in Section<br>i.  $\mathcal$ n model a truth-at-a-world relation,  $M$ ,  $\Gamma \Vdash X$ , is defined with<br>
versual clauses. The ones of particular interest are these:<br>  $\Vdash K_i X$  if and only if, for every  $\Delta \in \mathcal{G}$  with  $\Gamma \mathcal{R}_i \Delta$ , we have<br>  $\mathcal{M}, \Delta \Vdash X$ A version of LP with two agents was introduced and studied in (Yavorskaya (Sidon) 2008), though it can be generalized to them, (Artemov 2006). (Yavorskaya (Sidon) 2008), though it can be generalized to any finite<br>an implicit knowledge<br>ach agent understands. operators, variables, and constants, rather than having a single set for<br>ns; these a number of agents. In this, each agent has its own set of justification with an implicit knowledge<br>ich each agent understands.<br>ich each agent understands.<br>ich each agent understands.<br>operators, variables, and constants, rather than having a single set for<br>reasons; these amount to everybody, a operators, variables, and constants, rather than having a single set for everybody, as above. In addition some limited communication between agents may be permitted, using a new operator that allows one agent to verify the correctness of the other agent's justifications. Versions of both world justification models multiple accessibility<br>with connections between them, (Artemov 2006).<br>aultiple agents, each with an implicit knowledge<br>ustification terms, which each agent understands.<br>derstands explicit reasons single world and more general possible world justification semantics wer since world justification models multiple accessionity<br>are multiple agents selve them, (Artemov 2006). (Yavorskaya (Sidon) 2008), though it can be generalized to any finite<br>are multiple agents, each with an implicit knowl created for the two-agent logics. This involves a straightforward extension of the notion of an evidence function, and for possible world justification  $\mathcal{E}, \mathcal{V}$  meeting the following conditions.  $\mathcal{G}$  is a set of possible created for the two-agent logics. This involves a straightforward extension Each of  $\mathcal{R}_1,...,\mathcal{R}_n$  is an accessibility relation, one for ea

models, using two accessibility relations. Realization theorems have been proved syntactically, though presumably a semantic proof would also work.

Multi-agent models (where each general has is own as to fustification are conditional or errors as interded, which are conditional experimentally analyze zero. In equal where the studients is the set of the studienties of operators) with explicit and implicit knowledge can be used to epistemically analyze zero-knowledge proofs (Lehnherr, Ognjanovic, and Studer 2022). Zero-knowledge proofs are protocols by which one agent(the prover) can prove to another agent (the verifier) that the prover has n prove to another agent (the verifier) that the prover has<br>
dge (e.g., knows a password) without conveying any<br>
symphete mere fact of the prosession of knowledge (e.g.,<br>
axiomatized by two modalities  $\Box$  and  $\bigcirc$ . The<br> certain knowledge (e.g., knows a password) without conveying any information beyond the mere fact of the possession of knowledge (e.g.,without revealing the password). The following formulas can be used todescribe the situation after the execution of the protocol, where the term *s* proved syntactically, though presumably a semantic proof would also<br>
work.<br>
introduce an explic<br>
Multi-agent models (where each agent has its own set of justification<br>
operators) with explicit and implicit knowledge can b justifies the verifier's knowledge that results from the protocol: proved syntactically, though presumably a semantic proof w<br>work.<br>
Multi-agent models (where each agent has its own set of ju<br>
operators) with explicit and implicit knowledge can be<br>
epistemically analyze zero-knowledge pr Studer 2022). Zero-knowledge proofs are protocols by v<br>(the prover) can prove to another agent (the verifier) tha<br>certain knowledge (e.g., knows a password) without<br>information beyond the mere fact of the possession of k<br>

## $s:_{V}K_{P}F,$

meaning the protocol yields a justification s to the verifier V that the<br>prover P knows F; and<br> $\rightarrow s :_V t :_P F$  for any term t,<br>i.e., for no term t the protocol justifies that the verifier could know that t prover  $P$  knows  $F$ ; and s to the verifier  $V$ 

# $\neg s :_V t :_P F \text{ for any term t,}$

 $t$  is to the verifier V that the<br>  $t$  term t,<br>  $t$  the verifier could know that  $t$ <br>  $t$  is the protocol justifies that<br>
any possible evidence for that<br>  $t$  is  $t$ <br>  $t$  is e verifier could know that<br>s, the protocol justifies that<br>y possible evidence for that e of *F*. That is, the protocol justifies that<br>is not justify any possible evidence for that<br>on of the role of public announcements in<br>Renne 2008, Renne 2009).<br>of evidence-based common knowledge in<br>document Some More Techn the prover knows  $F$  but it does not justify any possible evidence for that knowledge.t the protocol justifies that the verifier could know that  $t$  $_{F}$ 

There has been some exploration of the role of public announcements in

There is more on the notion of evidence-based common knowledge inSection 5 of the supplementary document Some More Technical Matters.

Besides multi-agent epistemic logics, there are other justification logics<br>that feature two types of terms. Kuznets, Marin, and Strassburger (2021)<br>introduce an explicit version of constructive modal logic. There, the  $\square$ that feature two types of terms. Kuznets, Marin, and Strassburger (2021) introduce an explicit version of constructive modal logic. There, the  $\Box$ modality is realized by proof terms like in LP. To realize the  $\Diamond$ -modality, a<br>second kind of terms is introduced, which are called witness terms. In second kind of terms is introduced, which are called witness terms. In constructive modal logic, the formula  $\Diamond F$  means *F* is consistent. In its<br>
realization  $s \cdot F$  the witness term *s* represents an abstract witnessing realization  $s : F$ , the witness term  $s$  represents an abstract witnessing model for the formula  $F$ model for the formula  $F$ .

Another example is dyadic deontic logic (DDL), which can beaxiomatized by two modalities  $\square$  and  $\square$ . The formula  $\square F$  means  $F$  is axiomatized by two modalities  $\Box$  and  $\bigcirc$ . The formula  $\Box F$  means *F* is *settled true*, and the conditional  $\bigcirc$ (*F*/*G*) means *F* is *obligatory given G*. Faroldi, Rohani, and Studer (2023) consider an explicit version of DDL. Again,  $\Box F$  is realized by a proof term as in LP, whereas  $\bigcirc(F/G)$  is realized by making use of a new type of terms that represent deoptic realized by making use of a new type of terms that represent deonticreasons.also that feature two types of terms. Kuznets, Marin, and Strassburger (2<br>introduce an explicit version of constructive modal logic. There, the<br>modality is realized by proof terms like in LP. To realize the  $\Diamond$ -modal<br>do<br> bould also that teature two types of terms. Kuznets, Marin, and Strassbure introduce an explicit version of constructive modal logic. The second kind of erms is introduced, which are called winess used to second kind of

6. Russell's Example: Induced Factivity<br>There is a technique for using Justification Logic to analyze different There is a technique for usingjustifications for the same fact, in particular when some of the e technique consider a well-known example:

> If a man believes that the late Prime Minister's last name began with a 'B,' he believes what is true, since the late Prime Minister was Sir Henry Campbell Bannerman<sup>[5]</sup>. But if he believes that Mr. Balfour was the late Prime Minister<sup>[6]</sup>, he will still believe that the late Prime Minister's last name began with a 'B,' yet this belief, though true, would not be thought to constitute knowledge. (Russell 1912)

As in the Red Barn Example, discussed in Section 1.1, here one has todeal with two justifications for a true statement, one of which is correct deal with two justifications for a true statement, one of which is correct<br>
and one of which is not. Let B be a sentence (propositional atom), w be a<br>
designated justification variable for the wrong reason for B and r a<br> and one of which is not. Let  $B$  be a sentence (propositional atom),  $w$  be a designated justification variable for the wrong reason for B and r a<br>designated justification variable for the right (hence factive) reason for B.<br>Then, Russell's example prompts the following set of assumptions<sup>[7]</sup>:<br> $\math$ designated justification variable for the right (hence factive) reason for B.<br>
Then, Russell's example prompts the following set of assumptions<sup>[7]</sup>:<br>  $\mathbf{1} \Vdash w : B$ <br>  $\mathbf{2} \qquad \begin{bmatrix} w & B & w & B & w & D \\ w & & & & \\ w &$ Then, Russell's example prompts the following set of assumptions<sup>[7]</sup>:  $B$  be a sentence (propositional atom),  $w$  able for the wrong reason for  $B$  and *B* and *r* a<br>ason for *B* 

$$
\mathcal{R} = \{w : B, r : B, r : B \to B\}
$$

 $\mathcal{R} = \{w : B, r : B \to B\}$ <br>Somewhat counter to intuition, one can logically deduce factivity of w from  ${\cal R}$ :

1.  $r : B$  (assumption) 2.  $r : B \to B$  (assumption) 2.  $r : B \to B$  (assumption)<br>3. B (from 1 and 2 by Modus Ponens) 4.  $B \to (w : B \to B)$  (propositional axiom) 4.  $B \to (w : B \to B)$  (propositional axiom)<br>5.  $w : B \to B$  (from 3 and 4 by Modus Ponens)

However, this derivation utilizes the fact that  $r$  is a factive e justification for<br>nduced factivity'<br>'real' factivity of<br>oort of evidence-<br>ropriate tool. The<br>**hout**  $r : B$ , i.e.,<br>lerivable from  $S$ .<br> $\mathcal{E}, \mathcal{V}$ ) in which  $S$ B to conclude  $w : B \to B$ , which constitutes a case of 'induced factivity' B to conclude  $w : B \to B$ , which constitutes a case of 'induced factivity' for  $w : B$ . The question is, how can one distinguish the 'real' factivity of for  $w : B$ . The question is, how can one distinguish the 'real' factivity of  $r : B$  from the 'induced factivity' of  $w : B$  ? Some sort of evidence $r : B$  from the 'induced factivity' of  $w : B$ <br>tracking is needed here, and Justification Logi Indeed 4 by Modus Ponens)<br>
Hilizes the fact that r is a factive justification for<br>
, which constitutes a case of 'induced factivity'<br>
, how can one distinguish the 'real' factivity of<br>
factivity' of  $w : B$ ? Some sort of ev natural approach is to consider the set of assumptions **without**  $r : B$ , i.e., j) (propositional axiom)<br>
3 and 4 by Modus Ponens)<br>
3 and 4 by Modus Ponens)<br>
in utilizes the fact that r is a factive justification for<br>  $\forall B$ , which constitutes a case of 'induced factivity<br>
1 is, how can one distinguis It's example prompts the following set of assumptions<sup>17</sup>!<br>  $R = \{v : B, r : B \rightarrow B\}$  since w is admissible evidence<br>
sounter to intuition, one can logically deduce factivity of w<br>
since, according to  $\mathcal{E}, r$  is not a<br>
simplio

$$
S = \{w : B, r : B \to B\}
$$
  
ity of w. i.e., w : B \to B

and establish that factivity of w, i.e.,  $w : B \to B$  is not derivable from  $S$ . and establish that factivity of w, i.e.,  $w : B \to B$  is not derivable from S.<br>Here is a possible world justification model  $M = (\mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V})$  in which S<br>holds but  $w : B \to B$  does not: holds but  $w : B \to B$  does not:

- $\mathcal{G}=\{\mathbf{1}\},$  $\begin{aligned} \mathcal{G} &= \{\mathbf{1}\} \ \mathcal{R} &= \varnothing \;, \end{aligned}$
- 
- $\mathcal{R} = \varnothing$ ,<br>  $\mathcal{V}(B) = \varnothing$  (and so not-1  $\Vdash B$ ),
- $\mathcal{E}(t, F) = \{1\}$  for all pairs  $(t, F)$  except  $(r, B)$ , and
- $\mathcal{E}(r, B) = \varnothing$ .

 $\mathcal{E}(r, B) = \emptyset$ .<br>
It is easy to see that the closure conditions *Application* and *Sum* on  $\mathcal{E}$  are fulfilled. At **1**,  $w : B$  holds, i.e., e that the closure conditions *Application* and *Sum* on  $\mathcal E$  $1, w : B$ 

$$
\mathbf{1}\Vdash w:B
$$

since  $w$  is admissible evidence for  $B$  at  $\bf{1}$  and there are no possible worlds accessible from **1**. Furthermore,  $\mathbf{u} \perp w : B$ <br>w is admissible evidence for B at 1 a<br>sible from 1. Furthermore,

$$
\text{not-1} \Vdash r:B
$$

since, according to  $\mathcal{E}, r$  is not admissible evidence for B at 1. Hence: not-1  $\Vdash r : B$ <br>  $\mathcal{E}, r$  is not admissible evidence for B at 1

$$
\mathbf{1}\Vdash r:B\to B
$$

On the other hand,

$$
\text{not-1} \Vdash w : B \to B
$$

since  $B$  does not hold at **1**.

#### 7. Self-referentiality of justifications

The Realization algorithms sometimes produce Constant Specifications  $\begin{array}{lll}\n\text{not-1} \Vdash w : B \to B \\
\text{justification for} & \text{since } B \text{ does not hold at 1.} \\
\text{duced facility'} & \text{2al'}\n\end{array} \quad\quad \begin{array}{lll}\n\text{since } B \text{ does not hold at 1.} \\
\text{2. Self-referentially of justifications} & \text{1.} \\
\text{for evidence} & \text{The Realization algorithms sometimes produce Constant Spe} \\
\text{out } r : B \text{ is} & \text{containing self-referential justification assertions} & c : A(c),\n\end{array}$  $c : A(c)$ , that is, On the other hand,<br>  $\text{not-1} \Vdash w : B \to B$ <br>
tive justification for<br>
f 'induced factivity'<br>
he 'real' factivity of<br>
e sort of evidence-<br>
absorption algorithms sometimes produce Constant<br>
without r : B, i.e.,<br>  $\text{with} \text{out } r : B$ ,  $c)$  occurs in the asserted proposition (here  $A(c)$ ). SELF-referentiality of  $u : B \rightarrow B$ <br>
Self-referentiality of justifications is a new phenomenon which is not<br>  $B \rightarrow B$ <br>
Self-referentiality of justifications<br>  $u : B \rightarrow B$ <br>
Self-referentiality of justifications<br>  $u : B \rightarrow B$ <br>
Self-refer

tool. The The Realization algorithms sometimes produce Constant Specifications<br> *B*, i.e., containing self-referential justification assertions  $c : A(c)$ , that is,<br>
assertions in which the justification (here c) occurs in t present in the conventional modal language. In addition to being intriguing epistemic objects, such self-referential assertions provide a special challenge from the semantical viewpoint because of the built-in vicious circle. Indeed, to evaluate  $c$  one would expect first to evaluate  $A$ For Realization algorithms sometimes produce Constant Specifications<br>
is to consider the set of assumptions **without**  $r : B$ , i.e.,<br>  $S = \{w : B, r : B \rightarrow B\}$ <br>
factivity of  $w$ , i.e.,  $w : B \rightarrow B$  is not derivable from S.<br>  $\rightarrow B$  does  $r A$  to  $c$ . However, this cannot be done since

 $A$  contains  $c$  which is yet to be evaluated. The question of whether or not A contains c which is yet to be evaluated. The question of v<br>modal logics can be realized without using self-referential was a major open question in this area.

The principal result by Kuznets in (Brezhnev and Kuznets 2006) states S4in LP. The current state of things is given by the following theorem due to<br>V Kuznets:

**Theorem 5**: Self-referentiality can be avoided in realizations of modal logics K and D. Self-referentiality cannot be avoided in realizations of logics K and D. Self-referential<br>modal logics T, K4, D4 and S4.

This theorem establishes that a system of justification terms for  $S4$  will<br>
necessarily be self-referential. This creates a serious, though not directly<br>
visible, constraint on provability semantics. In the Gödelian cont necessarily be self-referential. This creates a serious, though not directly visible, constraint on provability semantics. In the Gödelian context of arithmetical proofs, the problem was coped with by a general method of assigning arithmetical semantics to self-referential assertions  $c : A(c)$ stating that c is a proof of  $A(c)$ . In the Logic of Proofs LP it was dealt **Theorem 5:** Self-referentiality can be avoided in realizations of modal<br>logics K and D. Self-referentiality cannot be avoided in realizations of<br>modal logics T, K4, D4 and S4.<br>This theorem establishes that a system of ju principal resul<br>self-referential<br>P. The current ands:<br>Theorem 5: Se<br>logics K and D<br>modal logics T<br>is theorem estab<br>ssarily be self-<br>ble, constraint<br>metical proofs<br>gning arithmet<br>ng that c is a p<br>by a non-trivia<br>-referentia This theorem establishes that a system of justification terms for S4 will Self-referential justifications<br>
and Kuznets 2006) states<br>
oridable in realization of S4<br>
intuitionistic<br>
oridable in realization of S4<br>
intuitionistic<br>
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proofs in PA<br>
interesting proofs in PA<br>
died in reali

Self-referentiality gives an interesting perspective on Moore's Paradox. See Section 6 of the supplementary document Some More Technical Matters for details.

The question of the self-referentiality of BHK-semantics for intuitionisticlogic IPC has been answered by Junhua Yu (Yu 2014). Extending Kuznets' method, he established

**Theorem 6**: Each LP realization of the intuitionistic law of double negation  $\neg\neg(\neg\neg p \rightarrow p)$  requires self referential constant specifications. specifications.

that self-referentiality of justifications is unavoidable in realization of S4 intuitionistic logic (even just of intuitionistic implication) is interest.<br>
in LP. The current state of things is given by the following theor More generally, Yu has proved that any double negation of a classical<br>  $\frac{1}{2}$ tautology (by Glivenko's Theorem all of them are theorems of  $\textsf{IPC}$ ) needs self-referential constant specifications for its realization in LC. An t specifications for its realization in LC. Another<br>e self-referentiality was found by Yu in the purely<br>of IPC. This suggests that the BHK semantics of<br>m just of intuitionistic implication) is intrinsically<br>ds a fixed-poin example of unavoidable self-referentiality was found by Yu in the purely implicational fragment of IPC. This suggests that the BHK semantics of implicational fragment of IPC. This suggests that the BHK semantics of<br>intuitionistic logic (even just of intuitionistic implication) is intrinsically self-referential and needs a fixed-point construction to connect it to formal self-referential and needs a fixed-point construction to connect it to formal<br>proofs in PA or similar systems. This might explain, in part, why any<br>attempt to build provability BHK semantics in a direct inductive manner<br>wi proofs in PA or similar systems. This might explain, in part, why anyattempt to build provability BHK semantics in a direct inductive manner<br>without self-referentiality was doomed to failure. without self-referentiality was doomed to failure. Self-referential constant specifications for its realization in LC. Another<br>example of unavoidable self-referentiality was found by Yu in the purely<br>implicational fragment of IPC. This suggests that the BHK semantics of<br>in mplicational fragment of IPC. This suggests that the BHK<br>intuitionistic logic (even just of intuitionistic implication) i<br>self-referential and needs a fixed-point construction to con<br>proofs in PA or similar systems. This m

1 justifications<br>
solvelogy (by Givenchois Theorem all of them are theorems of IPC) needs<br>
solvelogy the solvelogy of the material original of them are the content<br>
solvelogy implexions from the HEC. Another<br>
solved imple 8. Quantifiers in Justification Logic<br>While the investigation of propositional Justification Logic is far from While the investigation of propositional Justification Logic is far from complete, there has also been some work on first-order versions. Quantified versions of Modal Logic already offer complexities beyond standard firstcomplete, there has also been some work on first-order versions.<br>Quantified versions of Modal Logic already offer complexities beyond<br>standard first-order logic. Quantification has an even broader field to play<br>when Justif standard first-order logic. Quantification has an even broader field to play Example of unavolute state-including was bouting to the interpare including minimizational fragment of IPC. This suggests that the BHK semantics of intuitionistic logic (even just of intuitionistic implication) is intrinsi when Justification Logics are involved. Classically one quantifies over<br>
"objects," and models are equipped with a domain over which quantifiers<br>
range. Modally one might have a single domain common to all possible<br>
worlds 'objects,' and models are equipped with a domain over which quantifiers 'objects,' and models are equipped with a domain over which quantifiers<br>range. Modally one might have a single domain common to all possible<br>worlds, or one might have separate domains for each world. The role of<br>the Barcan range. Modally one might have a single domain common to all possible worlds, or one might have separate domains for each world. The role of the Barcan formula is well-known here. Both constant and varying domain options are available for Justification Logic as well. In addition there is a VERTIE CONSERVING While the investigation of propositional Justification Logic is<br>
delian context of<br>
ceneral method of<br>
Quantified versions of Modal Logic already offer complexities<br>
sertions c:  $A(c)$ <br>
standard first-ord possibility that has no analog for Modal Logic: one might quantify overjustifications themselves. options are available for Justification Logic as well. In addition the<br>possibility that has no analog for Modal Logic: one might quantif<br>justifications themselves.<br>Initial results concerning the possibility of Quantified J Exercutiality can be avoided in realizations of modal<br>Self-referentiality cannot be avoided in realizations of<br>without self-referentiality was doomed to<br>K4, D4 and S4.<br>In Section and the result of the Gidelian context of<br>

Initial results concerning the possibility of Quantified Justification Logic were notably unfavorable. The arithmetical provability semantics for theLogic of Proofs LP, naturally generalizes to a first-order version with options are available for Justification Logic as well. In addition there is a<br>possibility that has no analog for Modal Logic: one might quantify over<br>swered by Junhua Yu (Yu 2014). Extending<br>lished<br>possibility of Quantifie thers and to a version with quantifiers over proots T Sulf-referentiality of BHK-semantics for intuitionistic<br>
in answered by Junhua Yu (Yu 2014). Extending<br>
established<br>
Initial results concerning the possibility of Quantified Justification Logic<br>
self referential and to a both cases, axiomatizability questions were answered negatively.

enumerable (Artemov and Yavorskaya (Sidon) 2001). The logic of (Yavorsky 2001).

**Theorem 7:** The first-order logic of proofs is not recursively<br>
enumerable (Artemov and Yavorskaya (Sidon) 2001). The logic of<br>
enumerable (Artemov and Yavorskaya (Sidon) 2001). The logic of<br>
1000 with quantifiers over p proofs with quantifiers over proofs is not recursively enumerable<br>
(Yavorskaya (Sidon) (2011). A semantics for FOLP has been developed in<br>
Fitting (2014a).<br>
Fitting (2014a).<br>
Simple world semantics, and an axiomatic proof Although an arithmetic semantics is not possible, in (Fitting 2008) a possible world semantics, and an axiomatic proof theory, was given for aLP with quantifiers ranging over justifications. Soundness and completeness were proved. At this point possible world semantics separates from arithmetic semantics, which may or may not be a cause for alarm. It was also shown that **S4** translating  $\Box Z$  as "there exists a justification x such that  $x : Z^*$ ," where<br>  $Z^*$  is the translation of Z. While this logic is somewhat complicated, it<br>
has found applications, e.g., in (Dean and Kurokawa 2009b) it is is the translation of  $Z$ . While this logic is somewhat complicated, it has found applications, e.g., in (Dean and Kurokawa 2009b) it is used to analyze the Knower Paradox, though objections have been raised to thisanalysis in (Arlo-Costa and Kishida 2009). orld semantics, and an axiomatic proof theory, was given for a<br>LP with quantifiers ranging over justifications. Soundness and<br>ess were proved. At this point possible world semantics<br>rom arithmetic semantics, which may or Although an arithmetic semantics is not possible, in (Fitting<br>possible world semantics, and an axiomatic proof theory, was giv<br>version of LP with quantifiers ranging over justifications. Soundr<br>completeness were proved. A  $\Box Z$  as "there exists a justification x such that  $x: Z^*$ translating  $\Box Z$  as "there exists a justification x such that  $x : Z$ <br> $Z^*$  is the translation of Z. While this logic is somewhat comp Intervalse Simple The signal Variables of the signal Variables of the signal Variables of the signal Variables Communication Contain The initial Justification Logic of the minimization contains and an axiomatic proof theo

A First-Order Logic of Proofs, FOLP, with quantifiers over individual<br>variables, has been presented in Artemov and Yavorskaya (Sidon) (2011).<br>In FOLP proof assertions are represented by formulas of the form  $t:xA$ <br>where X variables, has been presented in Artemov and Yavorskaya (Sidon) (2011). In FOLP proof assertions are represented by formulas of the form  $t:xA$ <br>where X is a finite set of individual variables that are considered global where  $X$  is a finite set of individual variables that are considered glob parameters open for substitution. All occurrences of variables from  $X$  that are free in A are also free in  $t:_{X}A$ . All other free variables of A are considered local and hence bound in  $t: xA$ . For example, if  $A(x, y)$  is an atomic formula, then in  $p:_{\{x\}}A(x, y)$  variable x is free and variable y is<br>bound. Likewise, in  $x: A(x, y)$  both variables are free and in bound. Likewise, in  $p:_{\{x,y\}}A(x, y)$  both variables are free, and in  $p:{{}_{\scriptscriptstyle\emptyset}} A(x,y)$  neither x nor y is free. A First-Order Logic of Proofs, FOLP, with quantifiers over individual A are also free in  $t:xA$ . All other free variables of A ocal and hence bound in  $t:xA$ . For example, if  $A(x, y)$  is ula, then in  $p:x\in A(x, y)$  variable x is free and variable

Proofs (justifications) are represented by proof terms which do not contain individual variables. In addition to LP operations there is one more series of operations on proof terms,  $\text{gen}_x(t)$ , corresponding to generalization over individual variable  $x$ . The new axiom that governs this operation is

 $t: X \to \text{gen}:x(t)_X \forall xA$ , with  $x \notin X$ . The complete list of FOLP principles along with realization of First-Order S4 can be found in Artemov and along with realization of First-Order  $S4$  can be found in Artemov and<br> $S4$ Yavorskaya (Sidon) (2011). A semantics for FOLP has been developed in Fitting (2014a). Fitting (2014a).

#### 9. Historical Notes

version of LP with quantifiers ranging over justifications. Soundness and<br>
completeness were proved. At this point possible world semantics (and the semantics over proved. At this point<br>
separates from arithmetic semantic **Example 18** to the quantified logic by<br>
in the anisotic stablished<br>
in the assumption and Kurokawa 2009b) it is used to<br>
include this<br>
into though objections have been raised to this<br>
were first established for LP. Symbo The initial Justification Logic system, the Logic of Proofs LP, was introduced in 1995 in (Artemov 1995) (cf. also (Artemov 2001)) where such basic properties as Internalization, Realization, arithmetical<br>completeness, were first established. LP offered an intended provability . LP offered an intended provability semantics for Gödel's provability logic S4, thus providing a formalization of Brouwer-Heyting-Kolmogorov semantics for intuitionistic propositional logic. Epistemic semantics and completeness (Fitting 2005) Yavorskaya (Sidon) (201<br>Fitting (2014a).<br>9. Historical Note<br>The initial Justification<br>introduced in 1995 in (.<br>such basic properties<br>completeness, were first<br>semantics for Gödel's pr<br>of Brouwer-Heyting-<br>propositional logic r LP. Symbolic models and decidability for LP are<br>tychev 1997). Complexity estimates first appeared due to Mkrtychev (Mkrtychev 1997). Complexity estimates first appeared due to Mkrtychev (Mkrtychev 1997). Complexity estimates first appeared<br>in (Brezhnev and Kuznets 2006, Kuznets 2000, Milnikel 2007). A<br>comprehensive overview of all decidability and complexity results can be<br>found in (Kuzne in (Brezhnev and Kuznets 2006, Kuznets 2000, Milnikel 2007). <sup>A</sup> comprehensive overview of all decidability and complexity results can befound in (Kuznets 2008). Systems  $J$ ,  $J4$ , and  $JT$  were first considered in found in (Kuznets 2008). Systems J, J4, and JT were first considered in (Brezhnev 2001) under different names and in a slightly different setting.<br>JT45 appeared independently in (Pacuit 2006) and (Rubtsova 2006), and<br>JD45 (Brezhnev 2001) under different names and in a slightly different setting. JT45 appeared independently in (Pacuit 2006) and (Rubtsova 2006), and JD45 in (Pacuit 2006). The logic of uni-conclusion proofs has been found in (Krupski 1997). A more general approach to common knowledge based Fitting (2014a).<br>
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Eventy, was given for a<br>
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The initial Justification Logic system, the Logic dist<br>
interduced in 1995 in (Art on justified knowledge was offered in (Artemov 2006). Game semantics of Justification Logic and Dynamic Epistemic Logic with justifications were Stoon) 2001). The logic of along with realization of Frist-Order 34 can<br>be treativistic, in (Fitting 2008) a<br>Fitting (2014a).<br>Sisible, in (Fitting 2008) a<br>The initial Justification Logic system, the Logic of<br>theory, was g Justification Logic and Dynamic Epistemic Logic with justifications were<br>
studied in (Renne 2008, Renne 2009). Connections between Justification<br>
Logic and the problem of logical omniscience were examined in (Artemov studied in (Renne 2008, Renne 2009). Connectionsstudied in (Renne 2008, Renne 2009). Connections between Justification<br>
Logic and the problem of logical omniscience were examined in (Artemov<br>
proof terms which do not contain<br>
perations there is one more series<br> *Justifi*  and Kuznets 2009, Artemov and Kuznets 2014, Wang 2009). The name $c$  was introduced in (Artemov 2008), in which Kripke, Russell, and Gettier examples were formalized; this formalization has been used for the resolution of paradoxes, verification, hidden assumption ben for substitution. All occurrences of variables from X that<br>
1 are also free in t:xA. All other free variables of A are<br>
in (Krupski 1997). A more general approach to common knowledge based<br>
cal and hence bound in t:xA analysis, and eliminating redundancies. In (Dean and Kurokawa 2009a),

Justification Logic was used for the analysis of Knower and Knowability paradoxes.

(Artemov and Fitting 2019, Kuznets and Studer 2019).

### Bibliography

- two monographs on Justification Logic were published in 2019<br>
wand Fitting 2019, Kuznets and Studer 2019).<br>
Surgic Memov, S. and E. Nogina, 2005. "Introducing justification into epistemic<br>
Surgic and Computation, 15(6): 10 Antonakos, E., 2007. "Justified and Common Knowledge: Limited<br>
Conservativity", in S. Artemov and A. Nerode (eds.), *Logical*<br>
Foundations of Computer Science, International Symposium, LFCS<br>
2007, New York, NY, USA, June 4 Conservativity", in S. Artemov and A. Nerode (eds.), *Logical Foundations of Computer Science, International Symposium, LFCS2007, New York, NY, USA, June 4–7, 2007, Proceedings* (Lecture Notes in Computer Science: Volume 4514), Berlin: Springer, pp. 1–11.1997 liography<br>
liography<br>
liogic<br>
conservativity", in S. Artemov and A. Nerode (eds.), *Logical*<br>
Foundations of Computer Science, International Symposium, LFCS<br>
2007, New York, NY, USA, June 4-7, 2007, Proceedings (Lectu Antonakos, E., 2007. "Justified and Common Knowledge: Lin<br>
Conservativity", in S. Artemov and A. Nerode (eds.), *Lo*<br>
Foundations of Computer Science, International Sympos<br>
2007, New York, NY, USA, June 4–7, 2007, Proceedi Vav York, Ph.D. Program in Computer<br>
Vav York, Ph.D. Program in Computer<br>
Vav York, Ph.D. Program in Computer<br>
Vav York, NY, USA, June 4–7, 2007, Proceedings (Lecture<br>
Vav York, NY, USA, June 4–7, 2007, Proceedings (Lectur
- Arlo-Costa, H. and K. Kishida, 2009. "Three proofs and the Knower in the *Epistemology Workshop / FEW 2009. Proceedings*, Carnegie Mellon University, Pittsburgh, PA, USA.Lishida, 2009. "Three proofs and the Knower in<br>
f Proofs", in *Formal Epistemology Workshop*<br> *Adings*, Carnegie Mellon University, Pittsburgh<br>
erational modal logic", Technical Report MSI<br>
sity.<br>
wability and constructive Quantified Logic of Proofs", in Form<br>FEW 2009. Proceedings, Carnegie M<br>USA.<br>Artemov, S., 1995. "Operational modal log<br>29, Cornell University.<br>—, 2001. "Explicit provability and constron of Symbolic Logic, 7(1): 1–36.<br>—, 20
- Artemov, S., 1995. "Operational modal logic", Technical Report MSI 95–29, Cornell University.
- –––, 2001. "Explicit provability and constructive semantics", *The Bulletin of Symbolic Logic*, 7(1): 1–36. USA.<br>Artemov, S., 1995. "Operational modal logic", Techni<br>29, Cornell University.<br>—, 2001. "Explicit provability and constructive sem<br>*of Symbolic Logic*, 7(1): 1–36.<br>—, 2006. "Justified common knowledge", *Theoretica*<br>357
- **-**, 2006. "Justified common knowledge", *Theoretical Computer Science*, 357 (1–3): 4–22.
- -, 2008. "The logic of justification", The Review of Symbolic Logic, 1(4): 477–513.
- in the Logical Setting." *Studia* in the Logical Setting." *Studia Logica*, 100(1–2): 17–30.
- Artemov, S. and M. Fitting, 2019. Justification Logic: Reasoning with *Reasons*, New York: Cambridge University Press.
- Artemov, S. and R. Kuznets, 2009. "Logical omniscience as acomputational complexity problem", in A. Heifetz (ed.), *Theoretical*

*Aspects of Rationality and Knowledge, Proceedings of the TwelfthConference* (TARK 2009), ACM Publishers, pp. 14–23.

- paradoxes.<br>Conference (TAKK 2009), ACM Tubishes, pp. 14–25.<br>The first two monographs on Justification Logic were published in 2019<br>Analised Assistant and Analised 25. –––, 2014. "Logical omniscience as infeasibility", *Annals of Pure andApplied Logic*, 165(1): 6–25.
	- logic", *Journal of Logic and Computation*, 15(6): 1059–1073.
	- proofs", *Moscow Mathematical Journal*, 1(4): 475–490.
	- –––, 2011. "First-Order Logic of Proofs." TR–2011005, City University of New York, Ph.D. Program in Computer Science.
	- Boolos, G., 1993. *The Logic of Provability*, Cambridge: CambridgeUniversity Press.
	- Artemov, S. and E. Nogina, 2005. "Introducing justification into epistemic<br>
	logic", Journal of Logic and Computation, 15(6): 1059–1073.<br>
	Artemov, S. and T. Yavorskaya (Sidon), 2001. "On first-order logic of<br>
	proofs", Mosco Artemov, S. and T. Yavorskaya (Sidon), 2001. "On first-order logic of proofs", *Moscow Mathematical Journal*, 1(4): 475–490.<br>
	—, 2011. "First-Order Logic of Proofs." TR-2011005, City Universi<br>
	New York, Ph.D. Program in Co Brezhnev, V., 2001. "On the logic of proofs", in K. Striegnitz (ed.), *Proceedings of the Sixth ESSLLI Student Session, 13th European Summer School in Logic, Language and Information* (ESSLLI'01), pp. 35–46.
	- Brezhnev, V. and R. Kuznets, 2006. "Making knowledge explicit: How hard it is", *Theoretical Computer Science*, 357(1–3): 23–34.
	- Cubitt, R. P. and R. Sugden, 2003. "Common knowledge, salience andconvention: A reconstruction of David Lewis' game theory", *Economics and Philosophy*, 19: 175–210.
	- Dean, W. and H. Kurokawa, 2009a. "From the Knowability Paradox to the existence of proofs", *Synthese*, 176(2): 177–225.
	- –––, 2009b. "Knowledge, proof and the Knower", in A. Heifetz (ed.), *Theoretical Aspects of Rationality and Knowledge, Proceedings of the Twelfth Conference* (TARK 2009), ACM Publications, pp. 81–90.
	- Dretske, F., 2005. "Is Knowledge Closed Under Known Entailment? The Case against Closure", in M. Steup and E. Sosa (eds.), *Contemporary Debates in Epistemology*, Oxford: Blackwell, pp. 13–26.
	- Fagin, R., and J. Y. Halpern, 1988. "Belief, Awareness, and Limited Reasoning." Artificial Intelligence, 34: 39–76.
- Fagin, R., J. Halpern, Y. Moses, and M. Vardi, 1995. *Reasoning About Knowledge*, Cambridge, MA: MIT Press.
- Faroldi, F. L. G., M. Ghari, E. Lehmann, and T. Studer, 2024.*of Logic and Computation*, 34(4): 640–664.
- Faroldi, F. L. G., A. Rohani, and T. Studer, 2023. "Conditional Obligations Works: Volume III), Oxford: Oxford University Pre in Justification Logic", in H.H. Hansen, A. Scedrov, R. de Queiroz 113. in Justification Logic", in H.H. Hansen, A. Scedrov, R. de Queiroz (eds), *Logic, Language, Information, and Computation, WoLLIC 2023* (Lecture Notes in Computer Science: Volume 13923), Cham: Springer, pp. 178–193.
- Fitting, M., 2005. "The logic of proofs, semantically", *Annals of Pure andApplied Logic*, 132(1): 1–25.
- —–, 2006. "A replacement theorem for  $\mathbf{LP}$ ", Technical Report TR-2006002, Department of Computer Science, City University of NewYork.Springer, pp. 178–193.<br>
Fitting, M., 2005. "The logic of proofs, semantically", Annals of Pure and<br>
Applied Logic, 132(1): 1–25.<br>
2006. "A replacement theorem for **LP**", Technical Report TR-<br>
2006002, Department of Compute (eas), *Logic*, *Language*, *mjormanon*, *and Computation*, *wo.*<br>2023 (Lecture Notes in Computer Science: Volume 13923),<br>Springer, pp. 178–193.<br>Fitting, M., 2005. "The logic of proofs, semantically", *Annals of*<br>*Applied* Applied Logic, 132(1): 1–25.<br>
—, 2006. "A replacement theorem for **LP**", Technic<br>
2006002, Department of Computer Science, City<br>
York.<br>
—, 2008. "A quantified logic of evidence", Annals of<br>
Logic, 152(1–3): 67–83.<br>
—, 2009
- -, 2008. "A quantified logic of evidence", *Annals of Pure and Applied Logic*, 152(1–3): 67–83.
- —–, 2009. "Realizations and **LP**", *Annals of Pure and Applied Logic*,<br>161(2), 268, 297 161(3): 368–387.
- –––, 2014a. "Possible World Semantics for First Order Logic of Proofs."*Annals of Pure and Applied Logic* 165: 225–40.
- -, 2014b. "Justification Logics and Realization." TR-2014004, City University of New York, Ph.D. Program in Computer Science.
- Gettier, E., 1963. "Is Justified True Belief Knowledge?" *Analysis*, 23: 121–123.
- Girard, J.-Y., P. Taylor, and Y. Lafont, 1989. *Proofs and Types* (Cambridge Tracts in Computer Science: Volume 7), Cambridge: CambridgeUniversity Press.
- Gödel, K., 1933. "Eine Interpretation des intuitionistischen Aussagenkalkuls", *Ergebnisse Math. Kolloq.*, 4: 39–40. Englishtranslation in: S. Feferman *et al*. (eds.), *Kurt Gödel Collected Works*

(Volume 1), Oxford and New York: Oxford University Press andClarendon Press, 1986, pp. 301–303.<br>1938. "We shall in Til. 10.

- –––, 1938. "Vortrag bei Zilsel/Lecture at Zilsel's" (\*1938a), in S. Feferman, J. J. Dawson, W. Goldfarb, C. Parsons, and R. Solovay(eds.), *Unpublished Essays and Lectures* (Kurt Gödel Collected Works: Volume III), Oxford: Oxford University Press, 1995, pp. 86–113.
- Goldman, A., 1967. "A causal theory of meaning", *The Journal ofPhilosophy*, 64: 335–372.
- Goodman, N., 1970. "A theory of constructions is equivalent to arithmetic", in J. Myhill, A. Kino, and R. Vesley (eds.), *Intuitionismand Proof Theory*, Amsterdam: North-Holland, pp. 101–120.
- "Consistency and permission in density in the consistency and permission in deviation logic", *Journal* (eds.), *Unythisked Essays* and *Lectures* (Kurt Gödel Collected di, F. J. G., A. Rohani, and T. Studer, 2023. "Cond Goris, E., 2007. "Explicit proofs in formal provability logic", in S. Artemov and A. Nerode (eds.), *Logical Foundations of Computer Science, International Symposium, LFCS 2007, New York, NY, USA, June 4–7, 2007, Proceedings* (Lecture Notes in Computer Science: Volume 4514), Berlin: Springer, pp. 241–253.
	- Lehnherr, D., Z. Ognjanovic, and T. Studer, 2022. "A logic of interactive proofs", *Journal of Logic and Computation*, 32(8): 1645–1658.
	- Hendricks, V., 2005. *Mainstream and Formal Epistemology*, New York:Cambridge University Press.
	- Heyting, A., 1934. *Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie*, Berlin: Springer.
	- Hintikka, J., 1962. *Knowledge and Belief*, Ithaca: Cornell UniversityPress.
	- Kleene, S., 1945. "On the interpretation of intuitionistic number theory", *The Journal of Symbolic Logic*, 10(4): 109–124.
	- Kolmogorov, A., 1932. "Zur Deutung der Intuitionistischen Logik", *Mathematische Zeitschrift*, 35: 58–65. English translation in V.M. Tikhomirov (ed.), *Selected works of A.N. Kolmogorov. Volume I:Mathematics and Mechanics*, Dordrecht: Kluwer, 1991, pp. 151–158.
- Kreisel, G., 1962. "Foundations of intuitionistic logic", in E. Nagel, P. Suppes, and A. Tarski (eds.), *Logic, Methodology and Philosophy of Science. Proceedings of the 1960 International Congress*, Stanford: Stanford University Press, pp. 198–210.
- –––, 1965. "Mathematical logic", in T. Saaty (ed.), *Lectures in Modern Mathematics III*, New York: Wiley and Sons, pp. 95–195.
- Krupski, V., 1997. "Operational logic of proofs with functionality<br>exactly the contract function in S.A. line and A. Newsle (contract) condition on proof predicate", in S. Adian and A. Nerode (eds.), *Logical Foundations of Computer Science, 4th International Symposium, LFCS'97, Yaroslavl, Russia, July 6–12, 1997, Proceedings* (Lecture Notes in Computer Science: Volume 1234), Berlin: Springer, pp. 167–177.
- Kurokawa, H., 2009. "Tableaux and Hypersequents forLogic", in S. Artemov and A. Nerode (eds.), *Logical Foundations ofComputer Science, International Symposium,Beach, FL, USA, January 3–6, 2009, Proceedings* (Lecture Notes inComputer Science: Volume 5407), Berlin: Springer, pp. 295–308.
- r Justification McCarthy, J., M. Sato, "<br>
ical Foundations of theory of knowledgers and theory of knowledgers and theory of knowledgers and theory of knowledgers are present and Logics", in P.<br>
in the model Logics", in P. *LFCS 2009, Deerfield* Departn<br>
dings (Lecture Notes in Milnikel, R.,<br>
pringer, pp. 295–308. Proofs i<br>
er Science Logic, 14th —, 2009. "Conference of the and A. l<br>
26, 2000, Proceedings Internat<br>
e 1862), Berlin: Springer, Kuznets, R., 2000. "On the Complexity of Explicit Modal Logics", in P. Clote and H. Schwichtenberg (eds.), *Computer Science Logic, 14thInternational Workshop, CSL 2000, Annual Conference of the EACSL, Fischbachau, Germany, August 21–26, 2000, Proceedings*(Lecture Notes in Computer Science: Volume 1862), Berlin: Springer, pp. 371–383.*Justi*<sub>2</sub>, Derlin: Springer, pp. 295–308.<br> *Justi*ber 1995–308.<br> *Justiq* (eds.), *Computer Science Logic*, 14th<br> *Justi*<sub>2</sub>, *Logicon, Annual Conference of the*<br> *Prenany, August* 21–26, 2000, *Proceeding*<br> *Justificatio*
- –––, 2008. *Complexity Issues in*Computer Science Department, City University of New YorkGraduate Center.
- –––, 2010. "A note on the abnormality of realizations of **S4LP**", in K. Brünnler and T. Studer (eds.), *Proof, Computation, Complexity PCC 2010, International Workshop, Proceedings*, IAM Technical ReportsIAM-10-001, Institute of Computer Science and AppliedMathematics, University of Bern.
- Kuznets, R., S. Marin, and L. Strassburger, 2021. "Justification logic for constructive modal logic", *Journal of Applied Logics*, 8(8): 2313–2332.
- Kuznets, R. and T. Studer, 2012. "Justifications, Ontology, and<br>
Conservativity", in *Advances in Modal Logic* (Volume 9), Thomas<br>
Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence Moss<br>
(eds.), London: College Publ Conservativity", in *Advances in Modal Logic* (Volume 9), Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence Moss (eds.), London: College Publications, 437–58.
- –––, 2019. *Logics of Proofs and*Publications.
- Lemmon, E. J., and Dana S. Scott, 1977. *The "Lemmon Notes": An Introduction to Modal Logic* (American Philosophical Quarterly Monograph 11), Oxford: Blackwell.
- *d Justifications*, London: College<br>
cott, 1977. *The "Lemmon Notes"*<br> *gic* (American Philosophical Qua<br>
Blackwell.<br>
Shi, and S. Igarishi, 1978. "On the<br>
chinical Report STAN-CS-78-667<br>
Science, Stanford University.<br>
ty i McCarthy, J., M. Sato, T. Hayashi, and S. Igarishi, 1978. "On the modeltheory of knowledge", Technical Report STAN-CS-78-667, Department of Computer Science, Stanford University.
- Milnikel, R., 2007. "Derivability in certain subsystems of the Logic of Proofs is  $\Pi_2^p$ -complete", *Annals of Pure and Applied Logic*, 145(3): 223–239.
- . "Justification logic for<br>ed Logics, 8(8): 2313–<br>Ontology, and<br>ic (Volume 9), Thomas<br>sand Lawrence Moss<br>58.<br>nodon: College<br>emmon Notes": An<br>losophical Quarterly<br>ii, 1978. "On the model<br>AN-CS-78-667,<br>University.<br>stems of t ––, 2009. "Conservativity for Logics of Justified Belief", in S. Artemov<br>and A. Nerode (eds.), *Logical Foundations of Computer Science*, and A. Nerode (eds.), *Logical Foundations of Computer Science, International Symposium, LFCS 2009, Deerfield Beach, FL, USA, January 3–6, 2009, Proceedings* (Lecture Notes in ComputerScience: Volume 5407), Berlin: Springer, pp. 354–364.
	- Mkrtychev, A., 1997. "Models for the Logic of Proofs", in S. Adian andA. Nerode (eds.), *Logical Foundations of Computer Science, 4th International Symposium, LFCS'97, Yaroslavl, Russia, July 6–12, 1997, Proceedings* (Lecture Notes in Computer Science: Volume 1234), Berlin: Springer, pp. 266–275.
	- Nogina, E., 2006. "On logic of proofs and provability", in *2005 Summer Meeting of the Association for Symbolic Logic, Logic Colloquium'05,*

*Athens, Greece* (July 28–August 3, 2005), *The Bulletin of Symbolic Logic*, 12(2): 356.

–––, 2007. "Epistemic completeness of **GLA**", in *2007 Annual Meeting of the Association for Symbolic Logic, University of Florida, Gainesville, Florida* (March 10–13, 2007), *The Bulletin of Symbolic Logic*, 13(3): 407.

Pacuit, E., 2006. "A Note on Some Explicit Modal Logics", Technical Report PP–2006–29, Institute for Logic, Language and Computation, University of Amsterdam.

Plaza, J., 2007. "Logics of public communications", *Synthese*, 158(2):165–179.

- Renne, B., 2008. *Dynamic Epistemic Logic with*thesis, Computer Science Department, CUNY Graduate Center, New<br>York. NY. USA. York, NY, USA.
- ——, 2009. "Evidence Elimination in Multi-Agent Justification Logic", in n Multi-Agent Justification Logic", in Holland<br>
spects of Rationality and Knowledge, Wang, R.-J.,<br>
nference (TARK 2009), ACM Ono, M<br>
Informa<br>
ulus and realizability", Transactions of 2009, Ta<br>
ciety, 75: 1–19. in Artificat A. Heifetz (ed.), *Theoretical Aspects of Rationality and Knowledge, Proceedings of the Twelfth Conference (TARK 2009)*, ACMPublications, pp. 227–236. Frame, B., 2008. Dynamic Epistemic Logic with thesis, Computer Science Department, CUN<br>
York, NY, USA.<br>
—, 2009. "Evidence Elimination in Multi-Agent<br>
A. Heifetz (ed.), *Theoretical Aspects of Ratia*<br> *Proceedings of the*

Rose, G., 1953. "Propositional calculus and realizability", *Transactions of the American Mathematical Society*, 75: 1–19.

Rubtsova, N., 2006. "On Realization of **S5**-modality by Evidence Terms", *Journal of Logic and Computation*, 16(5): 671–684.

Russell, B., 1912. *The Problems of Philosophy*, Oxford: Oxford UniversityPress.

or. 2013. "Justifications, Awareness and Epistemic Dynamics", in S. Artemov and A. Nerode (eds.), *Logical Foundations ofComputer Science* (Lecture Notes in Computer Science: 7734), Berlin/Heidelberg: Springer, 307–18.

Sidon, T., 1997. "Provability logic with operations on proofs", in S. Adian and A. Nerode (eds.), *Logical Foundations of Computer Science, 4thInternational Symposium, LFCS'97, Yaroslavl, Russia, July 6–12,*

*1997, Proceedings* (Lecture Notes in Computer Science: Volume1234), Berlin: Springer, pp. 342–353.

Troelstra, A., 1998. "Realizability", in S. Buss (ed.), *Handbook of Proof Theory*, Amsterdam: Elsevier, pp. 407–474.

Troelstra, A. and H. Schwichtenberg, 1996. *Basic Proof Theory*, Amsterdam: Cambridge University Press.

 Troelstra, A. and D. van Dalen, 1988. *Constructivism in Mathematics* (Volumes 1, 2), Amsterdam: North–Holland.

 van Dalen, D., 1986. "Intuitionistic logic", in D. Gabbay and F. Guenther(eds.), *Handbook of Philosophical Logic* (Volume 3), Bordrecht: Reidel, pp. 225–340.

 van Ditmarsch, H., W. van der Hoek, and B. Kooi (eds.), 2007. *DynamicEpistemic Logic* (Synthese Library: Volume 337), Berlin: Springer..

- von Wright, G., 1951. *An Essay in Modal Logic*, Amsterdam: North-Holland.
- *h* Justification, Ph. D.<br>
INY Graduate Center, New<br>  $DlyG$  Fracture Center, New<br>
von Wright, G., 1951. An Ess<br>
ent Justification Logic", in<br>
tationality and Knowledge,<br>
ARK 2009), ACM<br>
(Juzability'', Transactions of<br>  $DyG$ Wang, R.-J., 2009. "Knowledge, Time, and Logical Omniscience", in H. Ono, M. Kanazawa, and R. de Queiroz (eds.), *Logic, Language, Information and Computation, 16th International Workshop, WoLLIC 2009, Tokyo, Japan, June 21–24, 2009, Proceedings* (Lecture Notes 5514), Berlin: Springer, pp. 394–Froessta, A. and H. Senwichtenberg,<br>
Amsterdam: Cambridge Universi<br>
Troelstra, A. and D. van Dalen, 1988.<br>
(Volumes 1, 2), Amsterdam: Nor<br>
van Dalen, D., 1986. "Intuitionistic lo.<br>
(eds.), *Handbook of Philosophice*<br>
Reide 407.
	- Yavorskaya (Sidon), T., 2001. "Logic of proofs and provability", *Annals of Pure and Applied Logic*, 113 (1–3): 345–372.
- –––, 2008. "Interacting Explicit Evidence Systems", *Theory of ComputingSystems*, 43(2): 272–293. Soophy, Oxford: Oxford University<br>
Pure and Applied Logic, 113 (1–3): 345–372.<br>
--, 2008. "Interacting Explicit Evidence Systems", Theoremess and Epistemic Dynamics",<br>
--, 2008. "Interacting Explicit Evidence Systems", The
	- Yavorsky, R., 2001. "Provability logics with quantifiers on proofs", *Annals of Pure and Applied Logic*, 113 (1–3): 373–387.
	- Yu, J., 2014. "Self-Referentiality of Brouwer-Heyting-Kolmogorovsemantics", *Annals of Pure and Applied Logic*, 165: 371–388.

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#### Acknowledgments

Beginning with the 2024 update, Thomas Studer has taken responsibilityfor updating and maintaining this entry.only in the 2024 update, Thomas Studer has<br>pdating and maintaining this entry.<br>The More Technical Matters<br>1. Mathematical Logic Tradition<br>2. Logical Awareness and Constant Specifications ated Entries<br>
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2. Logical Awareness

### Some More Technical Matters

- 1. Mathematical Logic Tradition
- 2. Logical Awareness and Constant Specifications
- 3. Single-Agent Possible World Models for J
- 4. Realization Theorems
- 5. Multi-Agent Justification Models
- 6. Self-referentiality of Justifications

# 1. Mathematical Logic Tradition

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PEINAROOP Project (InPhO).<br>
The University a Several well-known mathematical notions which appeared prior toJustification Logic have sometimes been perceived as related to the BHK Several well-known mathematical notions which appeared prior to<br>Justification Logic have sometimes been perceived as related to the BHK idea: Kleene realizability (Troelstra 1998), Curry-Howard isomorphism(Girard, Taylor, and Lafont 1989, Troelstra and Schwichtenberg 1996),Kreisel-Goodman theory of constructions (Goodman 1970, Kreisel 1962,Kreisel 1965), just to name a few. These interpretations have been very instrumental for understanding intuitionistic logic, though none of themSeveral well-known mathematical no<br>Justification Logic have sometimes bee<br>idea: Kleene realizability (Troelstra 19<br>(Girard, Taylor, and Lafont 1989, Tro<br>Kreisel-Goodman theory of constructio<br>Kreisel 1965), just to name a f street this entry.<br>
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Kleene realizability revealed a fundamental *computational content* of formal intuitionistic derivations, however it is still quite different from theintended *BHK* semantics. Kleene realizers are computational programs<br>rather then proofs. The predicate '*r realizes F*, is not decidable, which rather then proofs. The predicate '*r realizes F* ' is not decidable, which<br>leads to some serious deviations from intuitionistic logic. Kleene leads to some serious deviations from intuitionistic logic. Kleene realizability is not adequate for the intuitionistic propositional calculus**IPC.** There are realizable propositional formulas not derivable in IPC  $(P_{\text{OSE}} | 1953)$   $[8]$ (Rose 1953).<sup>[8]</sup>

The Curry-Howard isomorphism transliterates natural derivations in IPCto typed  $\lambda$ -terms thus providing a generic functional reading for logical derivations. However the foundational value of this interpretation is limited since, as proof objects, Curry-Howard  $\lambda$ -terms denote nothing but derivations in IPC itself and thus yield a circular provability semantics for the latter. $\footnotesize{update}$ . Thomas Studer has taken responsibility<br>  $\footnotesize{aligned}$ <br>  $\footnotesize{24 update, Thomas Student has taken responsibility}$ <br>  $\footnotesize{inting this entry.}\footnotesize{inting but derivations in IPC itself and thus yield a circular provability semantics for the latter.\n\n \n- logic Tradition
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An attempt to formalize the *BHK* semantics directly was made by Kreisel<br>in his theory of constructions (Kreisel 1962, Kreisel 1965). The original in his theory of constructions (Kreisel 1962, Kreisel 1965). The originalpropositional level. In (Goodman 1970) this was fixed by introducing a mamtaning uns entry.<br>
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An attempt to formalize the *BHK* semantics directly was made by Kreisel<br> K character

of this semantics. In particular, a proof of  $A \rightarrow B$  was no longer a<br>construction that could be applied to any proof of A of this semantics. In particular, a proof of  $A \rightarrow$ construction that could be applied to any proof of A

## 2. Logical Awareness and

Two examples in  $J$  are presented, showing modal theorems of  $K$ , and realizations for them. In the examples indices on constants have been omitted.

**Example 1.** This shows howw to build a justification of a conjunction from<br>
ts. In traditional modal language, this principle<br>  $\rightarrow \Box(A \land B)$ . In J this idea is expressed in the<br>
mguage.<br>
The complete reading of the result of this a<br>
(propositional a is formalized as  $\Box A \land \Box B \to \Box (A \land B)$ . In J this idea is expressed in the more precise justification language. **Example 1.** This shows how to build a justification of a conjunction from<br>  $y$  and  $y$  are  $y$  and  $y$  are  $y$  are  $y$  are  $y$  are  $y$  and  $y$  and  $y$  are  $y$  and  $y$  are is shows how to build a justincation of a conjunction from<br>
if the conjuncts. In traditional modal language, this principle<br>  $s \Box A \land \Box B \rightarrow \Box (A \land B)$ . In J this idea is expressed in the<br>
ustification language.<br>  $\rightarrow (A \land B)$ ) (  $A \wedge \Box B \rightarrow$ ication lang

- $1. A \rightarrow (B \rightarrow (A \wedge B))$  (propositional axiom)
- 2.  $c : (A \rightarrow (B \rightarrow (A \land B)))$  (from 1 by Axiom Internalization)  $A \rightarrow (B \rightarrow (A \land B)) \ c : (A \rightarrow (B \rightarrow (A \land$
- 2.  $c : (A \rightarrow (B \rightarrow (A \land B)))$  (from 1 by Axiom Internalization)<br>3.  $x : A \rightarrow [c \cdot x] : (B \rightarrow (A \land B))$  (from 2 by Application and Modus Ponens)  $x : A \rightarrow$ <br>Ponens)
- 4.  $x : A \to (y : B \to [[c \cdot x] \cdot y] : (A \land B))$  (from 3 by Application and propositional reasoning)  $x: A \to (y: B \to ||c \cdot s$ <br>propositional reasoning)
- 5.  $x : A \wedge y : B \rightarrow [[c \cdot x] \cdot y] : (A \wedge B)$  (from 5 by propositional reasoning) reasoning)

The derived formula 5 contains the constant  $c$ , which was introduced in line 2, and the complete reading of the result of this derivation is:

$$
x: A \wedge y: B \rightarrow [[c \cdot x] \cdot y]: (A \wedge B),
$$
 given  

$$
c: (A \rightarrow (B \rightarrow (A \wedge B))).
$$
 given

**Example 2.** This example shows how to build a justification of a disjunction from justifications of either of the disjuncts. In the usual modal language this is represented by  $\Box A \lor \Box B \to \Box (A \lor B)$ . Here is the corresponding result in J. corresponding result in **J**.

Constant Specifications<br>
and differential and the state of K, and<br>  $\begin{array}{lcl}\n & 4. \ B + \left(A \vee B\right) & \mbox{(from 1 by Axi)}\n\end{array} & \begin{array}{lcl}\n & 4. \ B + \left(A \vee B\right) & \mbox{(from 4 by Axi)}\n\end{array} & \begin{array}{lcl}\n & 4. \ B + \left(A \vee B\right) & \mbox{(from 5 by Application and Modus Ponens)}\n\end{array} & \begin{array}{lcl}\n &$ Finally construction that could be applied to any proof of A<br>
2. Logical Awareness and Constant Specifications<br>
Two examples in J are presented, showing modal theorems of K, and<br>
realizations for them. In the examples ind  $1.~A \rightarrow (A \vee B)$  (classical logic) 1.  $A \rightarrow (A \vee B)$  (classical logic)<br>2.  $a : (A \rightarrow (A \vee B))$  (from 1 by Axiom Internalization) 3.  $x : A \to [a \cdot x] : (A \vee B)$  (from 2 by Application and Modus  $x : A \rightarrow$ <br>Ponens)  $4. \, B \rightarrow (A \vee B)$  (by classical logic) 4.  $B \to (A \lor B)$  (by classical logic)<br>5.  $b : (B \to (A \lor B))$  (from 4 by Axiom Internalization) 6.  $y : B \to [b \cdot y] : (A \lor B)$  (from 5 by Application and Modus Ponens) 6.  $y : B \to [b \cdot y] : (A \lor B)$  (from 5 by Application and<br>7.  $[a \cdot x] : (A \lor B) \to [a \cdot x + b \cdot y] : (A \lor B)$  (by Sum)<br>8.  $[b \to b] : (A \lor B) \to [a \to b \to b] : (A \lor B)$  (by Sum)  $[7. [a \cdot x] : (A \vee B) \rightarrow [a \cdot x + b \cdot y] : (A \vee B)$  (by Sum)<br>  $[8. [b \cdot y] : (A \vee B) \rightarrow [a \cdot x + b \cdot y] : (A \vee B)$  (by Sum) 8.  $[b \cdot y] : (A \vee B) \rightarrow [a \cdot x + b \cdot y] : (A \vee B)$  (by Sum)<br>9.  $(x : A \vee y : B) \rightarrow [a \cdot x + b \cdot y] : (A \vee B)$  (from 3, 6, 7, and 8 by propositional reasoning)  $(x : A \lor y : B) \rightarrow [a \cdot x]$ <br>propositional reasoning)  $a : (A \rightarrow (A \vee B))$  (from<br>  $x : A \rightarrow [a \cdot x] : (A \vee B)$ <br>
Ponens)  $\begin{align} b: (B \rightarrow (A \vee B))\ y: B \rightarrow [b \cdot y]: (A \vee B) \end{align}$ 

The complete reading of the result of this derivation is:

$$
(x : A \vee y : B) \rightarrow [a \cdot x + b \cdot y] : (A \vee B),
$$
 given  

$$
a : (A \rightarrow (A \vee B)) \text{ and } b : (B \rightarrow (A \vee B)).
$$

# 3. Single-Agent Possible World Models for J

Here are two models in each of which  $x: P \to x: (P \wedge Q)$  is not valid. (*P* and *Q* are atomic and *x* is a justification variable.) It was pointed our<br>that a formula  $t \cdot X$  might fail at a possible world either because *X* is not 3. Single-Agent Possible World Models for J<br>
Here are two models in each of which  $x : P \to x : (P \wedge Q)$  is not valid.<br>
(*P* and *Q* are atomic and *x* is a justification variable.) It was pointed out (*P* and *Q* are atomic and *x* is a justification variable.) It was pointed out that a formula  $t : X$  might fail at a possible world either because *X* is not that a formula  $t : X$  might fail at a possible world either because  $X$  is not believable there (it is false at some accessible world), or because  $t$  is not an appropriate reason for  $X$ . The two models illustrate both versions.  $x : P \to x : (P \land Q)$ <br>ation variable.) It wa

> First, consider the model M having a single state,  $\Gamma$ , accessible to itself, First, consider the model  $M$  having a single state,  $\Gamma$ , accessible to itself, and with an evidence function such that  $\mathcal{E}(x, Z)$  is  $\Gamma$ , for every formula  $Z$ .<br>In this model,  $x$  serves as universal, evidence, He a In this model,  $x$  serves as universal' evidence. Use a valuation such that  $\mathcal{V}(P) = \Gamma$  and  $\mathcal{V}(Q) = \emptyset$ . Then one has  $\mathcal{M}, \Gamma \Vdash$ <br> $x : (P \wedge Q)$  because even though x serves as uni- $\mathcal{V}(P) = \Gamma$  and  $\mathcal{V}(Q) = \emptyset$ . Then one has  $\mathcal{M}, \Gamma \Vdash x : P$  but not- $\mathcal{M}, \Gamma \Vdash x : (P \wedge Q)$  because, even though x serves as universal evidence,  $P \wedge Q$  is not believable at  $\Gamma$  in the Hintikka/Krinke sense because  $\$ is not believable at  $\Gamma$  in the Hintikka/Kripke sense because  $Q$  is not true.

Next consider the model  $N$ , again having a single state  $\Gamma$  accessible to Next consider the model N, again having a single state  $\Gamma$  accessible to itself. This time take  $V$  to be the mapping assigning  $\Gamma$  to every nonpositional letter But also set  $\mathcal{E}(r, P) - \Gamma \mathcal{E}(r, Z) - \varnothing$  for  $Z \neq P$  an propositional letter. But also, set  $\mathcal{E}(x, P) = \Gamma$ ,  $\mathcal{E}(x, Z) = \emptyset$  for  $Z \neq P$ , and propositional letter. But also, set  $\mathcal{E}(x, P) = \Gamma$ ,  $\mathcal{E}(x, Z) = \emptyset$  for  $Z \neq P$ , and otherwise  $\mathcal E$  doesn't matter for this example. Then of course both  $P$  and  $P \wedge Q$  are heliously at  $\Gamma$  but  $\mathcal N$ ,  $\Gamma \vee \dots \vee P$   $P \wedge Q$  are believable at  $\Gamma$ , but  $\mathcal{N}, \Gamma \Vdash x : P$  and not- $\mathcal{N}, \Gamma \Vdash x : (P \wedge Q)$ <br>the latter because x does not serve as evidence for  $P \wedge Q$  at  $\Gamma$ . the latter because x does not serve as evidence for  $P \wedge Q$  at  $\Gamma$ .  $P \wedge Q$  are believable at  $\Gamma$ , but  $\mathcal{N}, \Gamma \Vdash x : P$  and not- $\mathcal{N}, \Gamma \Vdash$ <br>the latter because  $x$  does not serve as evidence for  $P \wedge Q$  at  $\Gamma$ 

In Hintikka/Kripke models, believability and knowability are essentially semantic notions, but the present treatment of evidence is more of asyntactic nature. For example, the model  $\mathcal N$  above also invalidates  $x : P \to x : (P \wedge P)$ . At first glance this is surprising, since in any standard logic of knowledge or belief  $\Box P \rightarrow \Box (P \wedge P)$  is valid But in t first glance this is surprising, since in any<br>dge or belief  $\Box P \rightarrow \Box (P \land P)$  is valid. But, just  $5''$ .<br>nee for P it need not follow that it also serves as  $6. \quad B \rightarrow \Box A \lor B$ standard logic of knowledge or belief  $\Box P\rightarrow \Box (P\wedge P)$  is valid. But, just because x serves as evidence for  $P$ , it need not follow that it also serves as evidence for  $P \wedge P$ . The formulas are syntactically different, and effort is<br>needed to recognize that the later formula is a redundant version of the needed to recognize that the later formula is a redundant version of the former. To take this to an extreme, consider the formula $x: P \to x: (P \wedge P \wedge ... \wedge P)$ , where the consequent has as many<br>conjuncts as there are elementary particles in the universel In brief conjuncts as there are elementary particles in the universe! In brief, Hintikka/Kripke style knowledge is knowledge of *propositions*, but Itself. This time take V to be the mapping assigning I' to every<br>propositional letter. But also, set  $\mathcal{E}(x, P) = \Gamma$ ,  $\mathcal{E}(x, Z) = \emptyset$  for  $Z \neq P$ , and<br>otherwise  $\mathcal{E}$  doesn't matter for this example. Then of course bo justification terms justify *sentences*. the latter because x does not serve as evidence<br>In Hintikka/Kripke models, believability an<br>semantic notions, but the present treatmer<br>syntactic nature. For example, the mode<br> $x : P \rightarrow x : (P \land P)$ . At first glance this<br>standard semantic notions, but the<br>syntactic nature. For exa<br> $x : P \rightarrow x : (P \land P)$ . At<br>standard logic of knowled<sub>g</sub><br>because x serves as eviden<br>evidence for  $P \land P$ . The f<br>needed to recognize that<br>former. To take this<br> $x : P \rightarrow x : (P \land P \land ...$ <br>co  $P \rightarrow x : (P \land P)$ . At first glance the data logic of knowledge or belief  $\Box P$ logic of knowledge or belief  $\Box P \rightarrow \Box (P \land P)$ <br>x serves as evidence for P, it need not follow the correction of the server syntentically different to the formulas are syntentically different to the server of the syntensis o  $P \to x : (P \land P \land ... \land P)$ <br>juncts as there are elemen

## 4. Realization Theorems

Here is an example of an  $S4$ -derivation realized as an LP-derivation in the style of the Realization theorem. There are two columns in the table a Hilbert-style **S4**-derivation of a modal formula  $\Box A \lor \Box B \to \Box (\Box A \lor B)$ . The second column displays corresponding steps of an LP-derivation of a formula:  $\Box A \lor \Box B \to \Box (\Box A \lor B)$ . The seco steps of an LP-derivation of a formula:

$$
x:A\vee y:B\rightarrow (a\cdot !x+b\cdot y):(x:A\vee B)
$$

 $\{a: (x:A\rightarrow x:A\lor B), b:(B\rightarrow x:A\lor B)\}.$ 





Extra steps  $5', 5'', 8'$ , and  $8''$  are needed in the LP case to reconcile different internalized proofs of the same formula:  $(a \cdot !x) : (x : A \vee B)$  and<br>  $(b, x) \cdot (a \cdot A) \vee B$ . The resulting realization respects Skelem's idea that different internalized proofs of the same formula:  $(a \cdot !x) : (x : A \vee B)$  and  $(b \cdot y) : (x : A \vee B)$ . The resulting realization respects Skolem's idea that negative equations of existential quantifiers (here ever proofs hidden in Extra steps 5', 5", 8', and 8" are needed in the LP of different internalized proofs of the same formula:  $(a \cdot !x)$ <br>an LP-derivation in the  $(b \cdot y) : (x : A \vee B)$ . The resulting realization respects S<br>negative occurrences of ex  $A \vee B$ ). The resulting realization respects Skolem's idea that urrences of existential quantifiers (here over proofs hidden in the modality of provability) are realized by free variables whereas positive occurrences are realized by functions of those variables.

> Proof theory plays an important role in the study of Justification Logics.Axiom systems we represented in this article, and a sequent calculus was introduced in (Artemov 1995, Artemov 2001). It has the curious disadvantage that it is cut free, but does not have the subformula property—no version with the subformula property is known. More recently, other

kinds of proof procedures for Justification Logics have been created. Kurokawa uses hypersequents to provide a comprehensive proof-2009), including those which combine implicit and explicit knowledge.These systems are notoriously hard to analyze and Kurokawa's results constitute a remarkable fundamental contribution to this area.

*r* Justification Logics have been created. algorithm, in turn, serves as part of a new algorithm for the Realization<br>
systems of Justification Logic, (Kurokawa<br>
combine implicit and explicit knowledge. Finally, all initi theoretical tractment of major systems of Justification Logic, (Kurokawa<br>
theoretical tractment of major systems are notoriously had to analyze and Kurokawa's results<br>
These systems are notoriously had to analyze and Kuro Realizations have been investigated for their own sake, (Fitting 2009). The basic results all are algorithmic in nature. For example, in Modal Logic aReplacement Theorem holds just as it does classically: if  $X \equiv X'$  is Replacement Theorem holds just as it does classically: if  $X \equiv X'$  is<br>provable in a normal modal logic then so is  $F \equiv F'$  where F is a formula and  $F'$  is like  $F$  except that some subformula occurrence  $X$  has been replaced with  $X'$ . Indeed this can be strengthened to establish that if  $X \to X'$  is provable, and X has a *positive* designated subformula  $X \to X'$  is provable, and X has a *positive* designated subformula<br>occurrence in F, while F' replaces that occurrence with X', then  $F \to F'$ <br>is provable. This does not carry over in a simple way to Justification occurrence in F, while F' replaces that occurrence with X', then  $F \to F$  is provable. This does not carry over in a simple way to Justification o Justification<br>
Somula F of evidence<br>
fication terms, necessari<br>
nto account a<br>
t an algorithm<br>
Moses, a<br>  $\begin{array}{ccc}\n\text{C}X \\
\text{C}X\n\end{array}$ <br>
whereas<br>  $\begin{array}{ccc}\n\text{U}X \\
\text{D}X\n\end{array}$ <br>
whereas<br>  $\begin{array}{ccc}\n\text{U}X \\
\text{D}X\n\end{array}$ <br>
Som Logic. Roughly speaking, an occurrence of  $X$  in a formula  $F$  of Kurokawa uses nypersequents to provide a comprenentsive proof-<br>theoretical treatment of major systems of Justification Logic, (Kurokawa<br>2009), including those which combine implicit and explicit knowledge.<br>These systems a f various justification terms,<br>d' to take into account a<br>not simple, but an algorithm<br>i, Fitting 2009).<br>In a modal sequent calculus<br> $S_2, Y$ and in  $F'$  these will need to be 'adjusted' to take into account a Kurokawa uses hypersequents to provide a comprehensive proof-<br>theoretical treatment of major systems of Justification Logic, (Kurokawa<br>2009), including those which combine implicit and explicit knowledge.<br>These systems ar justification for  $X \to X'$ . It turns out this is not simple, but for doing so has been developed, (Fitting 2006, Fitting 2009). able in a normal modal logic then so is  $F \equiv F'$  where F is  $F'$  is like F except that some subformula occurrence X<br>aced with X' Indeed this can be strengthened to establi Justification Logic may be within the scope of various justification terms, ion for  $X$ 10 provide a compression of common knowledge operator of the same model<br>
is the same of Justification Logic, (Kurokawa<br>
and of common kind of conference has the same model present as the same model provide as the same mod

Here is another result having a similar proof. In a modal sequent calculusargument a typical step is the following.

$$
\frac{S_1 \rightarrow S_2, X \qquad S_1 \rightarrow S_2, Y}{S_1 \rightarrow S_2, X \land Y}
$$

 $S_1 \rightarrow S_2, X \wedge Y$ <br>The notion of a realization easily extends from formulas to sequents. Suppose both of the premise sequents above have realizations, and one would like a realization for the consequent sequent. The problem is, the two premise realizations may be quite different. Algorithmic machinery for merging them has been developed that does exactly this. Thisizations may be quite different. Algorithmic machinery<br>
m has been developed that does exactly this. This<br>
with unlimited iteration of knowledge operators, but is much closer to

algorithm, in turn, serves as part of a new algorithm for the RealizationTheorem itself.

Finally, all initial realization proofs were constructive and were based on some kind of cut-free proof system. But non-constructive, semantic arguments have been developed, which allow the extension of realizationmachinery substantially. For instance, it is now known that the family of Fitting 2019 contains a detailed investigation along these lines. some kind of eac-rice proof system<br>arguments have been developed, wh<br>machinery substantially. For instance<br>modal logics having justification of<br>Fitting 2019 contains a detailed inve<br>5. Multi-Agent Justification 1<br>To simpli

# 5. Multi-Agent Justification Models

modal logics having justification counterparts is infinite. Artemov and<br>Fitting 2019 contains a detailed investigation along these lines.<br>5. Multi-Agent Justification Models<br>To simplify the language, one can use the forge To simplify the language, one can use the forgetful projection whichreplaces explicit knowledge assertions  $t : X$  by  ${\bf J}X$  where  ${\bf J}$  stands for sobook the *JX* where **J** stands for so-<br>  $e$  modality. Modality **J** is a stronger version of common knowledge:  $JX$  states all agents evidence for X. In a formal setting, in Kripke models,  $JX \to CX$ , but not necessarily  $CX \to JX$  (Artemov 2006). necessarily  $CX \rightarrow JX$  (Artemov 2006). iscally: if  $X \equiv X'$  is<br>
where  $F$  is a formula<br>
currence  $X$  has been<br>
d to establish that if<br>
To simplify the language, one can use the forgetful project<br>
esignated subformula<br>
replaces explicit knowledge assertions  $t : X$ 

Informally, the traditional common knowledge modality (Fagin, Halpern,Moses, and Vardi 1995) is represented by the condition

$$
\mathbf{C} X \Leftrightarrow X \wedge EX \wedge E^2 X \wedge \ldots \wedge E^n X \wedge \ldots
$$

 $\Leftrightarrow$   $X \wedge EX \wedge E^2 X \wedge ... \wedge E^n X \wedge ...$ <br>for the iustified common knowledge operator **J** 

$$
JX \Rightarrow X \wedge EX \wedge E^2 X \wedge \dots \wedge E^n X \wedge \dots
$$

in of common knowledge: **JX** states all agents share sufficient<br>
ce for X. In a formal setting, in Kripke models,  $JX \rightarrow CX$ , but not<br>
arily  $CX \rightarrow JX$  (Artemov 2006).<br>
ally, the traditional common knowledge modality (Fagin, Ha McCarthy's common knowledge (McCarthy, Sato, Hayashi, and Igarishi 1978). In (Cubitt and Sugden 2003) a case is made that David Lewis' version of common knowledge (more properly, belief) is not identified with unlimited iteration of knowledge operators, but is much closer to Justified common knowledge has the same modal principles as justified common knowledge. (See the encyclopedia article on Common  $1 \cdot (R \wedge \neg \Box R) \to R$  (logical axiom)<br>Knowledge). A good example of such a Lewis-McCarthy-Artemov  $2 \cdot \Box((R \wedge \neg \Box R) \to R)$  (Necessitation) Knowledge). A good example of such a Lewis-McCarthy-Artemov<br>
justified common knowledge assertion, JX, which is stronger than the<br>  $3. \Box(R \land \neg \Box R) \to R$ , (from 2 by Distribution)<br>  $4. \Box(R \land \neg \Box R) \to \Box R$ , (from 2 by Distributio usual common knowledge,  $\mathbf{C}X$ , is provided by situations following a nullis common second of  $Y(2)$  and  $2007$ ) of the which  $X$  halo at all states public announcement of  $X$  (Plaza 2007) after which  $X$  holds at all states, public announcement of  $X$  (Plaza 2007) after which  $X$  holds at all states, not only at reachable states. Note that public announcements are the usual means for attaining common knowledge, and they lead to justified common knowledge  **rather than the usual common knowledge**  $**C**$ **.** Significantly simpler than that of C. According to (Antonakos 2007), in<br>
significantly simpler than that of C. According to (Antonakos 2007), in<br>
significantly simpler than that of C. According to (Antonakos 2007), in<br>  $\$ 

The axiomatic description of justified common knowledge **J** is<br>
significantly simpler than that of **C**. According to (Antonakos 2007), in<br>
the standard epistemic scenarios, justified common knowledge **J** is<br>
conservative the standard epistemic scenarios, justified common knowledge  $J$  i d epistemic scenarios, justified common knowledge **J** is<br>
e with respect to the usual common knowledge **C** and hence<br>  $\begin{aligned}\n &\text{a. } x: (R \land \neg[c \cdot x] : R) \rightarrow \\
 &\text{b. } x: (R \land \neg[c \cdot x] : R) \rightarrow \\
 &\text{c. } x: (R \land \neg[c \cdot x] : R) \rightarrow \\
 &\text{d. } x: (R \land \$ conservative with respect to the usual common knowledge  $C$  and hence provides a lighter alternative to the latter. $9e$  J  $\bf{C}$  . According to (Antonakos 200'

# 6. Self-referentiality of Justifications

Let us consider an example which was suggested by the well-known*Moore's paradox*:

*It will rain but I don't believe that it will.*

If R stands for *it will rain*, then a modal formalization is:

$$
M=R\wedge \neg \Box R.
$$

It will rain but I don't believe that it will.<br>
If R stands for it will rain, then a modal formalization is:<br>  $M = R \land \neg \Box R$ .<br>
The Moore sentence M is easily satisfiable, hence consistent, e.g., whenever the weather forecast wrongly shows "no rain". However, it isimpossible to know Moore's sentence because $M = R \wedge \neg \Box R$ .<br>Moore sentence M

$$
\neg \Box M = \neg \Box (R \land \neg \Box R)
$$

 $\neg \Box M = \neg \Box (R \land \neg \Box R)$ <br>holds in any modal logic containing T. Here is a derivation.

 $1.\left(R\wedge\neg\Box R\right)\rightarrow R$  (logical axiom) 1.  $(R \wedge \neg \Box R) \rightarrow R$  (logical axiom)<br>2.  $\Box((R \wedge \neg \Box R) \rightarrow R)$  (Necessitation) 2.  $\square((R \wedge \neg \Box R) \rightarrow R)$  (Necessitation)<br>3.  $\square(R \wedge \neg \Box R) \rightarrow \Box R$ , (from 2 by Distribution)  $4. \ \Box(R \land \neg \Box R) \to (R \land \neg \Box R)$  (Factivity in T)  $4. \,\square (R \wedge \neg \square R) \rightarrow (R \wedge \neg \square R)$  (Factivity in T)<br>5.  $\square (R \wedge \neg \square R) \rightarrow \neg \square R$  (from 4 in Boolean logic) 6.  $\neg\Box(R \land \neg \Box R)$  (from 3 and 5 in Boolean logic)  $\Box(R\wedge \neg \Box R)\to \Box R \ \Box(R\wedge \neg \Box R)\to (R\, .$  $\square(R \wedge \neg \Box R) \rightarrow \neg \Box R$ <br> $\neg \Box(R \wedge \neg \Box R)$  (from 3

Furthermore, here is how this derivation is realized in LP.

 $1.\left(R\wedge\neg[c\cdot x]:R\right)\rightarrow R$  (logical axiom)  $1. (R \wedge \neg[c \cdot x] : R) \rightarrow R$  (logical axiom)<br>  $2. c : ((R \wedge \neg[c \cdot x] : R) \rightarrow R)$  (Constant (Constant Specification)<br>  $[x]$ :  $R$  (from 2 by Applic<br>  $\sqrt{-}[c \cdot x] : R$ ) (Factivity<br>  $x] : R$  (from 4 by Bool<br>
3 and 5 in Boolean logi<br>
line 2 is self-referential<br>
gic<br>  $[x \cdot [x, y]$  will be used<br>
th will be avoided when<br>
on of t 2.  $c: ((R \wedge \neg [c \cdot x] : R) \rightarrow R)$  (Constant Specification)<br>
3.  $x: (R \wedge \neg [c \cdot x] : R) \rightarrow [c \cdot x] : R$  (from 2 by Application)<br>  $A \cdot x : (B \wedge \neg [c \cdot x] : R) \rightarrow (B \wedge \neg [c \cdot x] : R)$  (Factivity) 4.  $x:(R \wedge \neg[c \cdot x]: R) \rightarrow (R \wedge \neg[c \cdot x]: R)$  (Factivity)  $4. x : (R \wedge \neg[c \cdot x] : R) \rightarrow (R \wedge \neg[c \cdot x] : R)$  (Factivity)<br>
5.  $x : (R \wedge \neg[c \cdot x] : R) \rightarrow \neg[c \cdot x] : R$  (from 4 by Boolean logic)<br>
6.  $\neg x : (R \wedge \neg[c \cdot x] : R)$  (from 3 and 5 in Boolean logic) 6.  $\neg x$  :  $(R \wedge \neg[c \cdot x] : R)$  (from 3 and 5 in Boolean logic) 3.  $x : (R \land \neg[c \cdot x] : R) \to [c \cdot x] : R$  (from 2 by Applic<br>4.  $x : (R \land \neg[c \cdot x] : R) \to (R \land \neg[c \cdot x] : R)$  (Factivity<br>5.  $x : (R \land \neg[c \cdot x] : R) \to \neg[c \cdot x] : R$  (from 4 by Boole<br>6.  $\neg x : (R \land \neg[c \cdot x] : R)$  (from 3 and 5 in Boolean logic<br>Note that Constan 1.  $(R \wedge \neg[c \cdot x] : R) \rightarrow R$  (logical a<br>
2.  $c : ((R \wedge \neg[c \cdot x] : R) \rightarrow R)$  (Co<br>
3.  $x : (R \wedge \neg[c \cdot x] : R) \rightarrow [c \cdot x] :$ <br>
4.  $x : (R \wedge \neg[c \cdot x] : R) \rightarrow (R \wedge \neg$ <br>
5.  $x : (R \wedge \neg[c \cdot x] : R) \rightarrow \neg[c \cdot x]$ <br>
6.  $\neg x : (R \wedge \neg[c \cdot x] : R)$  (from 3 a<br>
Note that Constant it at Lewis MoCarithy-Artenov<br>
3.  $\Box (R \land \neg \Box R) \rightarrow R)$  (Necessitation)<br>
3.  $\Box (R \land \neg \Box R) \rightarrow \Box R$ , (from 2 by Distribution)<br>
vided by situations following a<br>
4.  $\Box (R \land \neg \Box R) \rightarrow \Box R$ , (from 2 by Distribution)<br>
bile announcements  $x:(R \wedge \neg[c \cdot x]:R) \rightarrow \neg[c \cdot x]:R \neg x:(R \wedge \neg[c \cdot x]:R) \text{ (from 3 and 5)}$ 

1. For better readability brackets '[', ']' will be used in terms, andparentheses '(', ')' in formulas. Both will be avoided when it is safe.

at a mining common knowledge and they lead to justified<br>
and the value of passified common knowledge **3** is<br>  $\frac{3}{2}$ . ( $R \wedge \neg [c \cdot x] : R \rangle \rightarrow R$  (hegiven a knowledge based to  $\mathbf{C}$ ,  $\mathbf{C}$ ,  $\left(R \wedge \neg [c \cdot x] : R \right) \rightarrow R$  (he 2. One could devise a formalization of the Red Barn Example in a bimodal language with distinct modalities for knowledge and belief.However, it seems that such a resolution must involve reproducingn reveals, the truth. Such a bi-modal formalization would distinguish  $u : B$  from [ not because  $[a \cdot v] : B$  not because they have different reasons (which reflects the true epistemic structure of the problem), but rather because the former is labelled 'belief' and the latter 'knowledge.' But what if one needs to keep track of a larger number of different unrelated reasons? By introducing amultiplicity of distinct modalities and then imposing various assumptions

governing the inter-relationships between these modalities, one would essentially end up with a reformulation of theLogic itself (with distinct terms replaced by distinct modalities). This suggests that there may not be a satisfactory 'halfway point' between the modal language and the language of Justification Logic, at least inasmuch as one tries to capture the essential structure of examples involving the deductive nature of knowledge.

3. In our notation, LP can be assigned the name JT4. However, in virtue of the fundamental role played by LP in the history of Justification Logic. of the fundamental role played by LP the name LP has been preserved for this system.

4. To be precise, one must substitute c for x everywhere in s and t.

and the language of Justification Logic, at least inasmuch<br>
capture the essential structure of examples involving the<br>
context of knowledge.<br>
Sergei Artemov, Melvin Fitting, and Thomas Studer<br>
e of knowledge.<br>
In the basi 5. Which was true back in 1912. There is a linguistical problem with this example. The correct spelling of this person's last name is Campbell-Bannerman; strictly speaking, this name begins with a 'C.'

6. Which was false in 1912.

In any of Justification<br>
identication conditions). This<br>
identication conditions, This<br>
interpretation.<br>
In Logic, at least inations<br>
of countples involving the<br>
of Sengei Arenow. Nevia Fitting, and Thomas Studer<br>
or J. H y LP in the history of Justification Logic,<br>for this system.<br>tute c for x everywhere in s and t.<br>There is a linguistical problem with this<br>f this person's last name is Campbell-<br>s name begins with a 'C.'<br>in a member of th 7. Here a possiblee objection is ignored that the justifications 'the late<br>as Sir Henry Campbell Bannerman' and 'Mr. Balfour<br>
Le Minister' are mutually exclusive since there could be<br>
linister at a time. If the reader is not comfortable wit Prime Minister was Sir Henry Campbell Bannerman' and 'Mr. Balfour was the late Prime Minister' are mutually exclusive since there could be only one Prime Minister at a time. If the reader is not comfortable with Russell's example in which 'Prime Minister' 3. In our notation, LP can be assigned the name JT4. In the fundamental role played by LP in the history of the name LP has been preserved for this system.<br>4. To be precise, one must substitute *c* for *x* everywhere 5. W is replaced by 'member of the Cabinet' can be used instead. Thes ' $X$  was the this, a slight modification of Russell's example in which 'Pr<br>is replaced by 'member of the Cabinet' can be used<br>compatibility concern then disappears since justifications member of the late Cabinet' and 'Y was the member of the late Cabinet'<br> $\mathcal{H} = \mathcal{H} \times \mathcal{H} = \mathcal{H} \times \mathcal{H}$ with different  $X$  and  $Y$  are not necessarily incompatible.

Notes to the Supplement

8. Kleene himself denied any connection of his realizability with the BHKinterpretation.

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