# Probabilistic Justification Logic

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**Abstract.** We present a probabilistic justification logic, PPJ, to study rational belief, degrees of belief and justifications. We establish soundness and completeness for PPJ and show that its satisfiability problem is decidable. In the last part we use PPJ to provide a solution to the lottery paradox.

## 1 Introduction

In epistemic modal logic, we use formulas of the form $\Box A$ to express that $A$ *is believed*. Justification logic unfolds the $\Box$-modality into a family of so-called *justification terms* to represent evidence for an agent's belief. That is in justification logic we use $t : A$ to state that $A$ *is believed for reason* $t$.

Originally, Artemov developed the first justification logic, the Logic of Proofs, to give a classical provability semantics for intuitionistic logic [1, 2, 15]. Later, Fitting [7] introduced epistemic models for justification logic. As it turned out this interpretation provides a very successful approach to study many epistemic puzzles and problems [3, 5, 14].

In this paper, we extend justification logic with probability operators in order to accommodate the idea that

different kinds of evidence for $A$
lead to different degrees of belief in $A$. (1)

In [10] we have introduced a first probabilistic justification logic PJ, which features formulas of the form $P_{\geq s}(t : A)$ to state that *the probability of* $t : A$ *is greater than or equal to* $s$. The language of PJ, however, does neither include justification statements over probabilities (i.e. $t : (P_{\geq s}A)$) nor iterated probabilities (i.e. $P_{\geq r}(P_{\geq s}A)$).

In the present paper, we remedy these shortcomings and present the logic $\mathsf{PPJ}$, which supports formulas of the form $t : (P_{\geq s}A)$ as well as $P_{\geq r}(P_{\geq s}A)$. This explains the name $\mathsf{PPJ}$: the two $P$s refer to iterated $P$-operators. We introduce syntax and semantics for $\mathsf{PPJ}$ and establish soundness and completeness. We also show that satisfiability for $\mathsf{PPJ}$ is decidable. In the final part we present an application of $\mathsf{PPJ}$ to the lottery paradox.

**Related work.** The design of $\mathsf{PPJ}$ follows that of $\mathsf{LPP}_1$, which is a probability logic over classical propositional logic [21]. The proofs that we present for $\mathsf{PPJ}$ are extensions of the corresponding proofs for $\mathsf{LPP}_1$. Note, however, that these extensions are non-trivial due to the presence of formulas of the form $t : (P_{\geq s}A)$.

Milnikel [19] proposes a logic with uncertain justifications. We thoroughly study the relationship between Milnikel's logic and our approach in [10] where we show that three of his four axioms are theorems in our logic and that the fourth axiom holds under an additional independence assumption.

In the preprint [9], Ghari presents fuzzy variants of justification logic, in which an agent can have a justification for a statement with certainty between 0 and 1. He introduces fuzzy Fitting models and establishes a graded completeness theorem. Ghari also shows that Milnikel's principles are valid in his fuzzy setting.

Recently, Fan and Liau [6] introduced a possibilistic justification logic, which is an explicit version of a graded modal logic. Their logic includes formulas $t :_r A$ to express that *according to evidence $t$, $A$ is believed with certainty at least $r$*. However, the following principle holds in their logic:

$$s :_r A \wedge t :_q A \rightarrow s :_{\max(r,q)} A.$$

Hence all justifications for a belief yield the same (strongest) certainty, which is not in accordance with our guiding idea (1).

## 2 The Probabilistic Justification Logic PPJ

*Justification terms* are built from countably many constants and countably many variables according to the following grammar:

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t$$

where $c$ is a constant and $x$ is a variable. $\mathsf{Tm}$ denotes the set of all terms and $\mathsf{Con}$ denotes the sets of all constants. For any term $t$ and natural number $n$ we define $!^0 t := t$ and $!^{n+1} t := !\,(!^n t)$.

Let $\mathsf{Prop}$ be a countable set of atomic propositions. We denote the set of rational numbers by $\mathbb{Q}$. Further we set $\mathsf{S} := \mathbb{Q} \cap [0,1]$. The set of formulas $\mathcal{L}$ is defined by the following grammar:

$$A ::= p \mid P_{\geq s} A \mid \neg A \mid A \wedge A \mid t : A$$

where $t \in \mathsf{Tm}, s \in \mathsf{S}$ and $p \in \mathsf{Prop}$. We employ the standard abbreviations for classical connectives. Additionally, we set

$$P_{<s} A \equiv \neg P_{\geq s} A \qquad P_{\leq s} A \equiv P_{\geq 1-s} \neg A$$
$$P_{>s} A \equiv \neg P_{\leq s} A \qquad P_{=s} A \equiv P_{\geq s} A \wedge P_{\leq s} A$$

The axioms of $\mathsf{PPJ}$ are presented in Figure 1.

| | |
|---|---|
| (P) | finitely many schemes in the language of $\mathcal{L}$ axiomatizing classical propositional logic |
| (J) | $\vdash u : (A \to B) \to (v : A \to u \cdot v : B)$ |
| (+) | $\vdash u : A \vee v : A \to u + v : A$ |
| (PI) | $\vdash P_{\geq 0} A$ |
| (WE) | $\vdash P_{\leq r} A \to P_{<s} A$, where $s > r$ |
| (LE) | $\vdash P_{<s} A \to P_{\leq s} A$ |
| (DIS) | $\vdash P_{\geq r} A \wedge P_{\geq s} B \wedge P_{\geq 1} \neg(A \wedge B) \to P_{\geq \min(1, r+s)}(A \vee B)$ |
| (UN) | $\vdash P_{\leq r} A \wedge P_{<s} B \to P_{<r+s}(A \vee B)$, where $r + s \leq 1$ |

**Fig. 1.** Axioms of $\mathsf{PPJ}$

A *constant specification* is any set $\mathsf{CS}$ that satisfies

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an instance of some axiom of } \mathsf{PPJ}\}.$$

A constant specification $\mathsf{CS}$ is called:

**axiomatically appropriate:** if for every axiom instance $A$ of $\mathsf{PPJ}$, there exists a constant $c$ such that $(c, A) \in \mathsf{CS}$;

**schematic:** if for every constant $c$, the set $\{A \mid (c, A) \in \mathsf{CS}\}$ consists of all instances of several (possibly zero) axiom schemes;

**finite:** if $\mathsf{CS}$ is a finite set;
**almost schematic:** if $\mathsf{CS} = \mathsf{CS}_1 \cup \mathsf{CS}_2$ where $\mathsf{CS}_1 \cap \mathsf{CS}_2 = \emptyset$, $\mathsf{CS}_1$ is schematic and $\mathsf{CS}_2$ is finite.

Let $\mathsf{CS}$ be any constant specification. The deductive system $\mathsf{PPJ}_{\mathsf{CS}}$ is the Hilbert system obtained by adding to the axioms of $\mathsf{PPJ}$ the rules (MP), (CE), (ST) and (AN!) as given in Figure 2.

| | |
|---|---|
| | axioms of $\mathsf{PPJ}$ |
| | $+$ |
| (AN!) | $\vdash\, !^n c : !^{n-1} c : \cdots : !c : c : A$, where $(c, A) \in \mathsf{CS}$ and $n \in \mathbb{N}$ |
| (MP) | if $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$ |
| (CE) | if $\vdash A$ then $\vdash P_{\geq 1} A$ |
| (ST) | if $T \vdash A \to P_{\geq s - \frac{1}{k}} B$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to P_{\geq s} B$ |

**Fig. 2.** System $\mathsf{PPJ}_{\mathsf{CS}}$

Note that (ST) is an infinitary rule, which we need to obtain strong completeness. Observe also the difference in the definitions of rules (MP), (ST) and (CE) in Figure 2. Rule (CE) can only be applied to theorems of $\mathsf{PPJ}$ (i.e. formulas that are deducible from the empty set), whereas (MP) and (ST) can always be applied.

To introduce semantics for $\mathsf{PPJ}_{\mathsf{CS}}$, we begin with the notion of a basic evaluation, which is the cornerstone for many interpretations of justification logic [4, 13]. In the following we use $\mathcal{P}(X)$ to denote the power set of a set $X$.

**Definition 1 (Basic Evaluation).** *Let* $\mathsf{CS}$ *be a constant specification. A* basic evaluation *for* $\mathsf{CS}$*, or a basic* $\mathsf{CS}$*-evaluation, is a function* $*$ *that maps atomic propositions to truth values and maps justification terms to subsets of* $\mathcal{L}$*, i.e.*

$$* : \mathsf{Prop} \to \{\mathsf{T}, \mathsf{F}\} \quad \textit{and} \quad * : \mathsf{Tm} \to \mathcal{P}(\mathcal{L}),$$

*such that for* $u, v \in \mathsf{Tm}$*, for* $c \in \mathsf{Con}$ *and* $A, B \in \mathcal{L}$ *we have:*

1. $\big(A \to B \in u^* \text{ and } A \in v^*\big) \Longrightarrow B \in (u \cdot v)^*$
2. $u^* \cup v^* \subseteq (u + v)^*$

*3. if* $(c, A) \in \mathsf{CS}$ *then for all* $n \in \mathbb{N}$ *we have*[3]*:*

$$!^{n-1}c : !^{n-2}c : \cdots :!c : c : A \in (!^n c)^*$$

*We usually write* $t^*$ *and* $p^*$ *instead of* $*(t)$ *and* $*(p)$*, respectively.*

**Definition 2 (Algebra over a Set).** *Let* $W$ *be a non-empty set and let* $H$ *be a non-empty subset of* $\mathcal{P}(W)$*. We call* $H$ *an* algebra over $W$ *iff the following hold:*
- $W \in H$
- $U, V \in H \Longrightarrow U \cup V \in H$
- $U \in H \Longrightarrow W \setminus U \in H$

**Definition 3 (Finitely Additive Measure).** *Let* $H$ *be an algebra over* $W$ *and* $\mu : H \to [0, 1]$*. We call* $\mu$ *a* finitely additive measure *iff the following hold:*

1. $\mu(W) = 1$
2. *for all* $U, V \in H$*:*

$$U \cap V = \emptyset \quad \Longrightarrow \quad \mu(U \cup V) = \mu(U) + \mu(V)$$

**Definition 4 (Probability Space).** *A* probability space *is a triple* $\mathsf{Prob} = \langle W, H, \mu \rangle$*, where:*
- $W$ *is a non-empty set*
- $H$ *is an algebra over* $W$
- $\mu : H \to [0, 1]$ *is a finitely additive measure*

**Definition 5 (Model).** *Let* $\mathsf{CS}$ *be a constant specification. A* $\mathsf{PPJ}_{\mathsf{CS}}$-model *is a quintuple* $M = \langle U, W, H, \mu, * \rangle$ *where:*

1. $U$ *is a non-empty set of objects called worlds*
2. $W, H, \mu$ *and* $*$ *are functions, which have* $U$ *as their domain, such that for every* $w \in U$*:*
   - $\langle W(w), H(w), \mu(w) \rangle$ *is a probability space with* $W(w) \subseteq U$
   - $*_w$ *is a basic* $\mathsf{CS}$*-evaluation*[4]

The ternary satisfaction relation $\models$ is defined between models, worlds, and formulas.

[3] we agree to the convention that the formula $!^{n-1}c : !^{n-2}c : \cdots : !c : c : A$ represents the formula $A$ for $n = 0$

[4] we will usually write $*_w$ instead of $*(w)$

**Definition 6 (Truth in a $\mathsf{PPJ}_{\mathsf{CS}}$-model).** *Let* $\mathsf{CS}$ *be a constant specification and let* $M = \langle U, W, H, \mu, * \rangle$ *be a* $\mathsf{PPJ}_{\mathsf{CS}}$*-model. We define what it means for an* $\mathcal{L}$*-formula to hold in* $M$ *at a world* $w \in U$ *inductively as follows:*

$$
\begin{aligned}
M, w \models p \quad &:\Longleftrightarrow \quad p^*_w = \mathsf{T} \quad \text{for } p \in \mathsf{Prop} \\
M, w \models P_{\geq s} B \quad &:\Longleftrightarrow \quad \Big([B]_{M,w} \in H(w) \text{ and } \mu(w)\big([B]_{M,w}\big) \geq s\Big) \\
&\qquad\qquad \text{where } [B]_{M,w} = \{x \in W(w) \mid M, x \models B\} \\
M, w \models \neg B \quad &:\Longleftrightarrow \quad M, w \not\models B \\
M, w \models B \wedge C \quad &:\Longleftrightarrow \quad \big(M, w \models B \text{ and } M, w \models C\big) \\
M, w \models t : B \quad &:\Longleftrightarrow \quad B \in t^*_w
\end{aligned}
$$

**Definition 7 (Measurable Model).** *Let* $\mathsf{CS}$ *be a constant specification and let* $M = \langle U, W, H, \mu, * \rangle$ *be a* $\mathsf{PPJ}_{\mathsf{CS}}$*-model.* $M$ *is called measurable iff for every* $w \in U$ *and for every* $A \in \mathcal{L}$*:*

$$[A]_{M,w} \in H(w)$$

$\mathsf{PPJ}_{\mathsf{CS,Meas}}$ *denotes the class of* $\mathsf{PPJ}_{\mathsf{CS}}$*-measurable models.*

For a model $M = \langle U, W, H, \mu, * \rangle$, $M \models A$ means that $M, w \models A$ for all $w \in U$. Let $T \subseteq \mathcal{L}$. Then $M \models T$ means that $M \models A$ for all $A \in T$. Further $T \models A$ means that for all $M \in \mathsf{PPJ}_{\mathsf{CS,Meas}}$, $M \models T$ implies $M \models A$.

To be precise we should write $T \vdash_{\mathsf{CS}} A$ and $T \models_{\mathsf{CS}} A$ instead of $T \vdash A$ and $T \models A$, respectively, since these two notions depend on a given constant specification $\mathsf{CS}$. However, $\mathsf{CS}$ will always be clear from the context and we thus omit it.

**Definition 8 (Satisfiability).** *We say a formula* $A$ *of* $\mathcal{L}$ *is satisfiable if there exists a* $\mathsf{PPJ}_{\mathsf{CS}}$*-measurable model* $M = \langle U, W, H, \mu, * \rangle$ *and* $w \in U$ *with* $M, w \models A$*.*

We established the Deduction Theorem for $\mathsf{PJ}$ in [10]. Now we present the version for $\mathsf{PPJ}$, which can be proved in the same way.

**Theorem 1 (Deduction Theorem).** *Let* $T \subseteq \mathcal{L}$ *and* $A, B \in \mathcal{L}$*. For any constant specification* $\mathsf{CS}$ *we have:*

$$T, A \vdash B \quad \Longleftrightarrow \quad T \vdash A \to B$$

# 3 Soundness and Completeness

As usual, we can establish soundness by induction on the depth of the derivation of a formula $A$.

**Theorem 2 (Soundness).** *Let* $\mathsf{CS}$ *be any constant specification.* $\mathsf{PPJ}_{\mathsf{CS}}$ *is sound with respect to the class of* $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$*-models. I.e. for any* $A \in \mathcal{L}$ *and* $T \subseteq \mathcal{L}$ *we have:*

$$T \vdash A \quad \Longrightarrow \quad T \models A.$$

The completeness proof for $\mathsf{PPJ}_{\mathsf{CS}}$ is a combination of the completeness proof for $\mathsf{LPP}_1$ [21] and the completeness proof for $\mathsf{PJ}$ [10]. For lack of space, however, we cannot give a detailed completeness proof here. We will only present a series of definitions and lemmas (without proofs) that leads to the completeness result. First we need the notion of a $\mathsf{PPJ}_{\mathsf{CS}}$-consistent set.

**Definition 9 ($\mathsf{PPJ}_{\mathsf{CS}}$-consistent Set).** *Let* $\mathsf{CS}$ *be a constant specification and let* $T$ *be a set of* $\mathcal{L}$*-formulas.*

- $T$ *is said to be* $\mathsf{PPJ}_{\mathsf{CS}}$-consistent *iff* $T \nvdash \bot$*. Otherwise* $T$ *is said to be* $\mathsf{PPJ}_{\mathsf{CS}}$-inconsistent*.*
- $T$ *is said to be* maximal *iff for every* $A \in \mathcal{L}$ *either* $A \in T$ *or* $\neg A \in T$*.*
- $T$ *is said to be* maximal $\mathsf{PPJ}_{\mathsf{CS}}$-consistent *iff it is maximal and* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent.*

The next lemma is shown for $\mathsf{PJ}$ in [10]. The proof for $\mathsf{PPJ}_{\mathsf{CS}}$ is similar.

**Lemma 1 (Lindenbaum).** *Let* $\mathsf{CS}$ *be a constant specification. For every* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent set* $T$*, there exists a maximal* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent set* $\mathcal{T}$ *such that* $T \subseteq \mathcal{T}$*.*

**Definition 10 (Canonical Model).** *Let* $\mathsf{CS}$ *be a constant specification. The* canonical model *for* $\mathsf{PPJ}_{\mathsf{CS}}$ *is given by the quintuple* $M = \langle U, W, H, \mu, * \rangle$*, defined as follows:*

- $U = \{ w \mid w \text{ is a maximal } \mathsf{PPJ}_{\mathsf{CS}}\text{-consistent set of } \mathcal{L}\text{-formulas} \}$
- *for every* $w \in U$ *the probability space* $\langle W(w), H(w), \mu(w) \rangle$ *is defined as follows:*
  1. $W(w) = U$

2. $H(w) = \{(A)_M \mid A \in \mathcal{L}\}$ *where* $(A)_M = \{x \mid x \in U, A \in x\}$
3. *for all* $A \in \mathcal{L}$, $\mu(w)\big((A)_M\big) = \sup_s \{P_{\geq s}A \in w\}$

– *for every* $w \in W$ *the basic* CS*-evaluation* $*_w$ *is defined as follows:*
1. *for all* $p \in \mathsf{Prop}$*:*

$$p^*_w = \begin{cases} \mathsf{T} & \text{if } p \in w \\ \mathsf{F} & \text{if } \neg p \in w \end{cases}$$

2. *for all* $t \in \mathsf{Tm}$*:*

$$t^*_w = \{A \mid t : A \in w\}$$

**Lemma 2.** *Let* CS *be a constant specification. The canonical model for* $\mathsf{PPJ_{CS}}$ *is a* $\mathsf{PPJ_{CS}}$*-model.*

**Lemma 3.** *Let* $M = \langle U, W, H, \mu, * \rangle$ *be the canonical model for* $\mathsf{PPJ_{CS}}$*. Then we have*

$$(\forall A \in \mathcal{L})(\forall w \in U)\big[[A]_{M,w} = (A)_M\big].$$

From Lemma 3 we get the following corollary.

**Corollary 1.** *Let* CS *be any constant specification. The canonical model for* $\mathsf{PPJ_{CS}}$ *is a* $\mathsf{PPJ_{CS,Meas}}$*-model.*

Making use of the properties of maximal consistent sets, we can establish the Truth Lemma.

**Lemma 4 (Truth Lemma).** *Let* CS *be some constant specification and let* $M = \langle U, W, H, \mu, * \rangle$ *be the canonical model for* $\mathsf{PPJ_{CS}}$*. For every* $A \in \mathcal{L}$ *and any* $w \in U$ *we have:*

$$A \in w \iff M, w \models A.$$

Finally, we get the completeness theorem as usual.

**Theorem 3 (Strong Completeness for PPJ).** *Let* CS *be a constant specification, let* $T \subseteq \mathcal{L}$ *and let* $A \in \mathcal{L}$*. Then we have:*

$$T \models A \implies T \vdash A.$$

## 4 Decidability for a fragment of $\mathcal{L}$

Before we can show that satisfiability is decidable for all $\mathcal{L}$-formulas, we have to show that satisfiability is decidable for a subset $\mathcal{L}^{\mathsf{r}} \subseteq \mathcal{L}$ that is given by the following grammar:

$$A ::= p \mid \neg A \mid A \wedge A \mid t : B$$

where $t \in \mathsf{Tm}, p \in \mathsf{Prop}$, and $B \in \mathcal{L}$.

The key fact about $\mathcal{L}^{\mathsf{r}}$ is that the truth of an $\mathcal{L}^{\mathsf{r}}$-formula $A$ at a world $w$ in a $\mathsf{PPJ}_{\mathsf{CS}}$-model $M = \langle U, W, H, \mu, * \rangle$ only depends on the basic $\mathsf{CS}$-evaluation $*_w$.

Hence we can use the notation $* \models A$ if $A$ is a formula of $\mathcal{L}^{\mathsf{r}}$ and $*$ is a basic evaluation. We find that $A$ is satisfiable (in the sense of $\mathsf{PPJ}_{\mathsf{CS}}$) if and only if there exists a basic evaluation $*$ such that $* \models A$.

Therefore, we can use an extension of the usual decision procedure for the basic justification logic $\mathsf{J}$, see [11, 12, 20], to decide satisfiability for formulas of $\mathcal{L}^{\mathsf{r}}$.

**Theorem 4.** *Let* $\mathsf{CS}$ *be a decidable almost schematic constant specification. For any formula $A$ of the restricted language $\mathcal{L}^{\mathsf{r}}$, it is decidable whether $A$ is satisfiable.*

For lack of space, we only give a proof sketch of the above theorem. As in the decidability proof for $\mathsf{J}$, we make use of schematic variables so that we can represent a schematic constant specification in a finite way. A key step in the decidability proof is then to compute a most general unifier for schematic formulas. This is the step that needs some major adaptations for our probabilistic setting.

Consider, for example, the scheme ($\mathsf{WE}$) given by $P_{\leq r}A \to P_{<s}A$. It has three schematic variables: $A$ for formulas and $r, s$ for rational numbers. Note that there is also a side condition, $s > r$, of which the unification algorithm has to take care. Hence in addition to constructing a substitution, the unification algorithm also has to build up a system of linear inequalities for the rational variables. For instance, in order to unify $P_{\geq r}A$ and $P_{\geq s}B$ the algorithm has to unify $A$ and $B$ and to equate $r$ and $s$, i.e. it adds $r = s$ to the linear system. In the end, the constructed substitution only is a most general unifier if the linear system is satisfiable.

Of course, one has to take care of the syntactic abbreviations when representing axioms. That means, the scheme (WE) actually is $P_{\geq 1-r}\neg A \to \neg P_{\geq s}A$ with the side condition $s > 1-r$ (note that the implication again is an abbreviation).

Another complication are constraints of the form

$$l = \min(1, r+s) \tag{2}$$

that originate from the scheme (DIS). Obviously, (2) is not linear. However, for a system $C$ of linear inequalities, we find that

$$C \cup \{l = \min(1, r+s)\}$$

has a solution if and only if

$$C \cup \{l = r+s, r+s \leq 1\} \text{ or } C \cup \{l = 1, r+s > 1\}$$

has a solution. Thus we can reduce solving a system involving (2) to solving several linear systems.

## 5 Decidability of $\mathsf{PPJ}_{\mathsf{CS}}$

**Definition 11 (Subformulas).** *The set of subformulas* $\mathsf{subf}(A)$ *of an* $\mathcal{L}$*-formula* $A$ *is recursively defined by:*

$$\begin{aligned}
\mathsf{subf}(p) &:= \{p\} \\
\mathsf{subf}(P_{\geq s}B) &:= \{P_{\geq s}B\} \cup \mathsf{subf}(B) \\
\mathsf{subf}(\neg B) &:= \{\neg B\} \cup \mathsf{subf}(B) \\
\mathsf{subf}(B \wedge C) &:= \{B \wedge C\} \cup \mathsf{subf}(B) \cup \mathsf{subf}(C) \\
\mathsf{subf}(t : B) &:= \{t : B\} \cup \mathsf{subf}(B)
\end{aligned}$$

**Definition 12.** *Let* $A \in \mathcal{L}$ *and assume that* $\mathsf{subf}(A) = \{A_1, \dots, A_k\}$. *The set* $\mathsf{subfCon}(A)$ *contains all sets of the form* $\{\pm A_1, \dots, \pm A_k\}$, *where* $\pm A_i$ *is either* $A_i$ *or* $\neg A_i$.

*Elements of* $\mathsf{subfCon}(A)$ *are interpreted conjunctively. That is for* $C \in \mathsf{subfCon}(A)$, *we simply write* $M, w \models C$ *instead of* $M, w \models \bigwedge C$. *Hence* $M, w \models C$ *means that all elements of* $C$ *are true at* $w$ *in* $M$. *Accordingly, we say that* $C$ *is satisfiable if the formula* $\bigwedge C$ *is so.*

*We define the mapping* $\mathsf{j}$ *on sets* $C$ *of* $\mathcal{L}$*-formulas by:*

$$\mathsf{j}(C) := C \cap \mathcal{L}^{\mathsf{r}}.$$

Before proving that $\mathsf{PPJ}_{\mathsf{CS}}$ is decidable we need to establish some auxiliary lemmata.

**Lemma 5.** *Let* $M = \langle U, W, H, \mu, * \rangle \in \mathsf{PPJ}_{\mathsf{CS,Meas}}$ *and let* $A \in \mathcal{L}$. *Let* $B \in \mathsf{subf}(A)$, *let* $C \in \mathsf{subfCon}(A)$ *and let* $w \in U$. *Assume that* $M, w \models C$. *Then we have:*

$$M, w \models B \quad \Longleftrightarrow \quad B \in C.$$

*Proof.* We prove the two directions of the lemma separately:

$\Longleftarrow$: From $B \in C$ and $M, w \models C$ we immediately get $M, w \models B$.

$\Longrightarrow$: Since $B$ is a subformula of $A$, we have either $B \in C$ or $\neg B \in C$. If $\neg B \in C$, then we would have $M, w \models \neg B$, i.e. $M, w \not\models B$, which contradicts the fact that $M, w \models B$. Thus, we conclude $B \in C$. □

**Lemma 6.** *Let* $\mathsf{CS}$ *be a constant specification and let* $A \in \mathcal{L}$. *Then* $A$ *is satisfiable if and only if there exists a set* $Y = \{B_1\ ,\ \dots\ ,\ B_n\} \subseteq \mathsf{subfCon}(A)$ *such that all of the following conditions holds:*

1. *for some* $i \in \{1, \dots, n\}$, $A \in B_i$.
2. *for every* $1 \le i \le n$, $\mathsf{j}(B_i)$ *is satisfiable.*
3. *for every* $1 \le i \le n$, *there are variables* $x_{ij}$ *with* $1 \le j \le n$, *such that the following system of linear inequalities is satisfiable:*

$$\sum_{j=1}^{n} x_{ij} = 1$$

$$(\forall 1 \le j \le n)\big[x_{ij} \ge 0\big]$$

$$\textit{for every } P_{\ge s}C \in B_i, \sum_{\{j \mid C \in B_j\}} x_{ij} \ge s$$

$$\textit{for every } \neg P_{\ge s}C \in B_i, \sum_{\{j \mid C \in B_j\}} x_{ij} < s$$

*Proof.* We prove the two directions of the lemma separately:

$\Longrightarrow$: Let $M = \langle U, W, H, \mu, * \rangle \in \mathsf{PPJ}_{\mathsf{CS,Meas}}$. Assume that $A$ is satisfiable in some world of $M$.

Let $\approx$ denote a binary relation over $U$ such that for all $w, x \in U$ we have:

$$w \approx x \quad \text{if and only if} \quad \big(\forall B \in \mathsf{subf}(A)\big)\big[M, w \models B \Leftrightarrow M, x \models B\big].$$

It is easy to see that $\approx$ is an equivalence relation. Let $K_1, \ldots, K_n$ be the equivalence classes of $\approx$. For every $i \in \{1, \ldots, n\}$ we choose some $w_i \in K_i$. For every $i \in \{1, \ldots, n\}$ some subformulas of $A$ hold in the world $w_i$ and some do not. So for every $i \in \{1, \ldots, n\}$ there exists a $B_i \in \mathsf{subfCon}(A)$ such that $M, w_i \models B_i$. For $i \neq j$ we have $B_i \neq B_j$ since $w_i$ and $w_j$ belong to different equivalence classes. Let $Y = \{B_1, \ldots, B_n\}$. It remains to show that the conditions in the statement of the lemma hold:

1. Let $w \in U$ be such that $M, w \models A$. The world $w$ belongs to some equivalence class of $\approx$, which is represented by $w_i$. Thus $M, w_i \models A$. By Lemma 5 we find $A \in B_i$, i.e. condition 1 holds.
2. For every $1 \leq i \leq n$ we have $M, w_i \models B_i$. Because of $\mathsf{j}(B_i) \subseteq B_i$ we immediately get $M, w_i \models \mathsf{j}(B_i)$. Hence condition 2 holds.
3. Let $i \in \{1, \ldots, n\}$. We set

   $$y_{ij} = \mu(w_i)(K_j \cap W(w_i)), \text{ for every } 1 \leq j \leq n.$$

   Some calculations show that these values $y_{ij}$ satisfy the linear system in condition 3.

$\Longleftarrow$: Assume that there exists $Y = \{B_1, \ldots, B_n\} \subseteq \mathsf{subfCon}(A)$ such that conditions 1–3 hold. For every $1 \leq i \leq n$, let $*_i$ be a basic evaluation such that $*_i \models \mathsf{j}(B_i)$. We define the quintuple $M = \langle U, W, H, \mu, * \rangle$ by:

- $U = \{w_1, \ldots, w_n\}$ for some $w_1, \ldots, w_n$.
- For all $1 \leq i \leq n$ we set:
  1. $W(w_i) = U$
  2. $H(w_i) = \mathcal{P}(W(w_i))$
  3. $\mu(w_i)(V) = \sum_{\{j \mid w_j \in V\}} x_{ij}$ for every $V \in H(w_i)$
  4. $*_{w_i} = *_i$.

We can show that $M \in \mathsf{PPJ}_{\mathsf{CS,Meas}}$. However, we have to omit the proof due to lack of space.

It remains to show $M, w_i \models A$ for some $i$. We first establish

$$(\forall D \in \mathsf{subf}(A))(\forall 1 \leq i \leq n)\big[D \in B_i \iff M, w_i \models D\big] \qquad (3)$$

by induction on the structure of $D$ (again we have to omit the proof).

It holds that $A \in \mathsf{subf}(A)$. Thus, by (3) we find:

$$(\forall 1 \leq i \leq n)\big[A \in B_i \iff M, w_i \models A\big].$$

By condition 1, there exists an $i$ such that $A \in B_i$. Thus, there exists an $i$ such that $M, w_i \models A$. Hence, $A$ is $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-satisfiable. □

In the proof of Lemma 6 we construct a model with at most $2^{|\mathsf{subf}(A)|}$ worlds that satisfies $A$. Hence a corollary of Lemma 6 is that any $A \in \mathcal{L}$ is $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-satisfiable if and only if it is satisfiable in a $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-model with at most $2^{|\mathsf{subf}(A)|}$ worlds. In other words, Lemma 6 implies a small model property for $\mathsf{PPJ}_{\mathsf{CS}}$.

Moreover, Lemma 6 dictates a procedure to decide the satisfiability problem for $\mathsf{PPJ}_{\mathsf{CS}}$.

**Theorem 5.** *Let* $\mathsf{CS}$ *be a decidable almost schematic constant specification. The* $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$*-satisfiability problem is decidable.*

*Proof.* Let $A \in \mathcal{L}$. The formula $A$ is satisfiable if and only if for some $Y \subseteq \mathsf{subfCon}(A)$ all conditions in the statement of Lemma 6 hold. Since $\mathsf{subfCon}(A)$ is finite, it suffices to show that for every $Y \subseteq \mathsf{subfCon}(A)$ the conditions 1–3 in the statement of Lemma 6 can be effectively checked:

- Decidability of condition 1 is trivial.
- Decidability of condition 2 follows from Theorem 4.
- In condition 3 we have to check for the satisfiability of a set of linear inequalities, which is a well-known decidable problem [18].

We conclude that the satisfiability problem for $\mathsf{PPJ}_{\mathsf{CS}}$ is decidable. □

## 6 Application to the Lottery Paradox

Kyburg's famous lottery paradox [16] goes as follows. Consider a fair lottery with 1000 tickets that has exactly one winning ticket. Now assume a proposition is believed if and only if its degree of belief is greater than 0.99. In this setting it is rational to believe that ticket 1 does not win, it is rational to believe that ticket 2 does not win, and so on. However, this entails that it is rational to believe that no ticket wins because rational belief is closed under conjunction. Hence it is rational to believe that no ticket wins and that one ticket wins.

$\mathsf{PPJ}_{\mathsf{CS}}$ makes the following analysis of the lottery paradox possible. First we need a principle to move from degrees of belief to rational belief (this formalizes what Foley [8] calls *the Lockean thesis*): we suppose that for each term $t$, there exists a term $\mathsf{pb}(t)$ such that

$$t : (P_{>0.99}A) \rightarrow \mathsf{pb}(t) : A. \tag{4}$$

Let $w_i$ be the proposition *ticket $i$ wins*. For each $1 \leq i \leq 1000$, there is a term $t_i$ such that $t_i : (P_{=\frac{999}{1000}} \neg w_i)$ holds. Hence by (4) we get

$$\mathsf{pb}(t_i) : \neg w_i \quad \text{for each } 1 \leq i \leq 1000. \tag{5}$$

Now if $\mathsf{CS}$ is axiomatically appropriate, then

$$s_1 : A \wedge s_2 : B \rightarrow \mathsf{con}(s_1, s_2) : (A \wedge B) \tag{6}$$

is a valid principle (for a suitable term $\mathsf{con}(s_1, s_2)$). Hence by (5) we conclude that

$$\text{there exists a term } t \text{ with } t : (\neg w_1 \wedge \cdots \wedge \neg w_{1000}), \tag{7}$$

which leads to a paradoxical situation since it is also believed that one of the tickets wins.

In $\mathsf{PPJ}_{\mathsf{CS}}$ we can resolve this problem by restricting the constant specification such that (6) is valid only if $\mathsf{con}(s_1, s_2)$ does not contain two different subterms of the form $\mathsf{pb}(t)$. Then the step from (5) to (7) is no longer possible and we can avoid the paradoxical belief.

This analysis is inspired by Leitgeb's [17] solution to the lottery paradox and his *Stability Theory of Belief* according to which *it is not permissible to apply the conjunction rule for beliefs across different contexts*. Our proposed restriction of (6) is one way to achieve this in a formal system. A related and very interesting question is whether one can interpret the above justifications $t_i$ as stable sets in

Leitgeb's sense. Of course, our discussion of the lottery paradox is very sketchy but we think that probabilistic justification logic provides a promising approach to it that is worth further investigations.

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