Non-Wellfounded Proof Theory for Interpretability Logic*

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Abstract. Sasaki [7] has presented a cut-elimination procedure for IK4, i.e., interpretability logic without Löb's axiom. We show that his main idea can also be used to obtain cut-elimination for the full interpretability logic IL. To achieve this, we introduce a traditional Gentzen-style sequent calculus for IL and a non-wellfounded version of it. We then develop a general proof theory for non-wellfounded systems of this type, which makes a simple cut-elimination argument possible. Our results provide a useful basis for further research; in particular, they allow us to establish uniform interpolation for IL.

Keywords: Proof theory · Interpretability logic · Cut elimination

Introduction

This paper is concerned with the proof theory of interpretability logic IL ([13]), i.e., the extension of provability logic with a binary modality formalizing interpretability. We introduce two new calculi for IL: a wellfounded Gentzen calculus GIL and a non-wellfounded local progress calculus $\mathrm{G}^{\infty}\mathrm{IL}$. We show proof-theoretically the equivalence of these two calculi and also their equivalence to the usual Hilbert-style calculus for IL. Further, we develop a general proof theory for local progress calculi, which makes it possible to obtain a simple cut elimination result, which can be transferred from the non-wellfounded to the finitary calculus. Our procedure is displayed in Figure1.

In that figure, we included a cyclic calculus $\mathrm{G}^{\circ}\mathrm{IL}$. Its equivalent to the non-wellfounded calculus $\mathrm{G}^{\infty}\mathrm{IL}$ can be shown via an auxiliary calculus $\mathrm{G}^{\mathrm{slim}}\mathrm{IL}$. This cyclic system can be used to establish uniform interpolation for IL. We leave the translations of $\mathrm{G}^{\circ}\mathrm{IL}$ and $\mathrm{G}^{\mathrm{slim}}\mathrm{IL}$ and the proof of uniform interpolation to be published in future work, due to lack of space.

Thus, the contributions of this paper are threefold:

1. We present a general proof theory of non-wellfounded local progress calculi. In particular, we introduce the notions of admissible, locally admissible, eliminable, and locally eliminable rules and study their relationship.

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$$\begin{tikzcd}
\mathrm{IL} \arrow[r, leftrightarrow, "\text{Thm 1}"] & \mathrm{GIL} + \mathrm{Cut} \arrow[r, "\text{Thm 2}"] & \mathrm{G}^{\infty}\mathrm{IL} + \mathrm{Cut} \arrow[d, "\text{Thm 1}"'] & \mathrm{G}^{\circ}\mathrm{IL} \arrow[dl] \\
& \mathrm{GIL} \arrow[u, "\text{Trivial}"'] & \mathrm{G}^{\infty}\mathrm{IL} \arrow[l, "\text{Thm 4}"'] \arrow[r] & \mathrm{G}^{\mathrm{slim}}\mathrm{IL} \arrow[u]
\end{tikzcd}$$

Fig. 1. The plan. Arrows without labels are omitted in this paper.

2. We present a simple syntactic cut elimination method for interpretability logic. To do so, we introduce a traditional Gentzen-style sequent calculus for IL and a non-wellfounded version of it.
3. Our non-wellfounded proofs exhibit a regular structure (i.e., they lead to cyclic proofs), which makes them a useful tool for further investigations, such as establishing uniform interpolation for IL.

Related Work. There are two directions of closely related work. The first one is non-wellfounded and cyclic proof theory. The structure and methodology of this paper has been inspired by the seminal [10]. We follow the trend started in that paper of defining a non-wellfounded Gentzen calculi from a finite one where cut elimination becomes easier to show. In relation with cut elimination in non-wellfounded and cyclic proof, there are many proposed methods. The interested reader may consult [1, 2, 4, 9, 10, 11, 12], among others.

We use our own method of cut elimination, described in detail in [12], as it simplifies the non-wellfounded cut elimination to the point of making it totally analogue to the finitary case.

Secondly, the proof theoretical study of interpretability logics. Sasaki's work [6, 8, 7] has been a fundamental reference for this paper. Part of our motivation was to simplify his approach with the use of modern tools (non-wellfounded proof theory) and build from it. More recently, [5] has also studied the proof theory of subsystems of IL.

Summary of Sections. In the next section we will introduce the basic concepts of interpretability logic and non-wellfounded proof theory needed for the rest of the paper. Section 2 will introduce the Gentzen calculi $\mathrm{G}^{\infty}\mathrm{IL}$ and GIL. Section 3 is devoted to showing the equivalence of IL and GIL + Cut. Finally, Section 4 shows the central square of Figure 1, thus providing a simple cut elimination procedure for GIL.

1 Preliminaries

1.1 Interpretability Logic

In this subsection we will define the interpretability logic that we will be working with. We will also prove that certain formulas, which will be useful to us in the next sections, are theorems of this logic.

The syntax of interpretability logic is given by

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \phi \rhd \phi,$$

where p ranges over a fixed set of propositional variables. We call formulas of this language IL-formulas. If it will be clear from the context that we are talking about IL-formula, we will just write formula instead of IL-formula. Other Boolean connectives can be defined as abbreviations as usual. $\Box\phi$ can be defined as an abbreviation, namely $\Box\phi = \neg\phi \rhd \bot$ and we set $\Diamond\phi = \neg\Box\neg\phi$. We will also use the abbreviation $\blacksquare\phi = (\phi \rhd \bot) \wedge \phi$. A formula of the form $\phi \rhd \psi$ will be called a $\rhd$-formula.

We use lower case Latin letters p, q, ..., possibly with subscripts, for propositional variables and lower case Greek letters ϕ, ψ, ..., possibly with subscripts, for IL-formulas. To avoid too many parentheses in longer formulas, we treat $\rhd$ as having higher priority than $\to$, but lower than other Boolean connectives. Unary operators $\Box$, $\Diamond$ and $\neg$ have the highest priority.

In some proofs we will use the following auxiliary definition of a size of an IL-formula.

Definition 1. *The* size $|\phi|$ *of an* IL*-formula* ϕ *is defined recursively as follows:*

$$|\bot| = 0, \qquad |p| = 1, \qquad |\phi \to \psi| = |\phi \rhd \psi| = |\phi| + |\psi| + 1.$$

Note that, contrary to the usual definition, the size of $\bot$ is smaller than the size of any other formula.

We define the interpretability logic we will consider in this paper.

Definition 2. *Interpretability logic* IL *is the smallest set of* IL*-formulas that contains all the tautologies and axioms*

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,	(4) $\Box\phi \to \Box\Box\phi$,
(L) $\Box(\Box\phi \to \phi) \to \Box\phi$,	(J1) $\Box(\phi \to \psi) \to (\phi \rhd \psi)$,
(J2) $(\phi \rhd \chi) \wedge (\chi \rhd \psi) \to (\phi \rhd \psi)$,	(J3) $(\phi \rhd \psi) \wedge (\chi \rhd \psi) \to (\phi \vee \chi) \rhd \psi$,
(J4) $\phi \rhd \psi \to (\Diamond\phi \to \Diamond\psi)$.	(J5) $\Diamond\phi \rhd \phi$

and is closed under modus ponens and necessitation:

$$\frac{\phi \to \psi \quad \phi}{\psi}, \qquad \frac{\phi}{\Box\phi}.$$

Sometimes we will be referring to (L) axiom as Löb axiom.

In the following lemma we will put together some basic properties of IL. These results will be used in some proofs in the remainder of this paper.

Lemma 1. *Let* ϕ, ψ *be formulas and* Σ *be a non-empty finite multiset of formulas. Then*

1. IL $\vdash \phi \to \psi$ *implies* IL $\vdash \phi \rhd \psi$.
2. *(Löb's rule in* IL*)* IL $\vdash \psi \wedge \bigwedge(\Sigma \rhd \bot) \to \bigvee \Sigma$ *implies* IL $\vdash \psi \rhd \bigvee \Sigma$.
3. IL $\vdash \phi \rhd \blacksquare\phi$.

1.2 Non-wellfounded Proof Theory

We introduce the basic concepts of (non-wellfounded) proof theory that we are going to use. The details can be found in [12]. We start with the definition of a non-wellfounded finitely branching tree, from now own simply called tree.

Definition 3. *A tree with labels in A is a function T such that*

1. $\mathrm{Dom}(T) \subseteq \mathbb{N}^{<\omega}$ *closed under prefixes and* $\mathrm{Im}(T) \subseteq A$.
2. *For each $w \in \mathrm{Dom}(T)$ there is an unique k, called the* arity *of w, such that $wi \in \mathrm{Dom}(T)$ if and only if $i < k$.*

The elements of $\mathrm{Dom}(T)$*, also denoted as* $\mathrm{Node}(T)$*, are called* nodes *of T.*

Given a tree T an (infinite) branch *is an infinite sequence $b = \langle b_i \rangle_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$, $b{\restriction}i \in \mathrm{Node}(T)$.*

Basics of Local Progress Calculi. We use upper case Greek letters Γ, Δ, Σ, Γ', Δ', ..., possibly with subscripts, for finite multisets of formulas. The expression $\Gamma \rhd \bot$ denotes the multiset $\{\phi \rhd \bot \mid \phi \in \Gamma\}$. By a sequent, we mean an ordered pair $\langle \Gamma, \Delta \rangle$ usually denoted as $\Gamma \Rightarrow \Delta$. We use upper case Latin letters S, S', ..., possibly with subscripts, for sequents. Inside sequents, we will write Γ, Δ to mean $\Gamma \cup \Delta$ and ϕ, Γ or Γ, ϕ to mean $\{\phi\} \cup \Gamma$, as usual. Further, we will write expressions like $(\Gamma, \phi, \Delta) \rhd \bot$ to mean $(\Gamma \rhd \bot) \cup \{\phi \rhd \bot\} \cup (\Delta \rhd \bot)$. Also, we will write sequences of sequents like $S_n, \dots, S_0$ as $[S_i]_{n \dots i \dots 0}$.

Definition 4. *An n-*ary rule *is a set of $n+1$-tuples $\langle S_0, \dots, S_n \rangle$ where each S_i is a sequent. The elements of a rule are called its* instances.

A local-progress sequent calculus *is a pair $G = (\mathcal{R}, L)$ where*

1. *$\mathcal{R}$ is a set of rules.*
2. *L is a function such that given a n-ary rule R and a rule instance $\langle S_0, \dots, S_n \rangle$ returns a subset of $\{0, \dots, n-1\}$, called* progressing premises*. L is called the* progressing function.

Definition 5. *Let G be a local-progress sequent calculus. A* preproof *π in G is a non-wellfounded tree, whose nodes are anotated by a sequent and a rule of G, that is generated by the rules of G. In other words, for any n-ary node w of π we have that $\langle S_0, \dots, S_{n-1}, S \rangle \in R$, where R is the rule at w, S is the sequent at w and S_i is the sequent at wi (the i-th successor of w).*

Given a preproof π in G and an infinite branch b in π we will say that b progress at *i iff $b_{i+1} \in L_R(S_0, \dots, S_{n-1}, S)$ where the node $b{\restriction}i$ is n-ary, R is the rule at node $b{\restriction}i$, S is the sequent at node $b{\restriction}i$ and S_j is the sequent at node $(b{\restriction}i)j$ for $j < n$. A preproof π in G is said to be a* proof *in G iff for any infinite branch b of π the set $\{i \in \mathbb{N} \mid b$ progresses at $i\}$ is infinite.*

A local-progress calculus is said to be wellfounded *iff its local progress function is the constant function always returning $\varnothing$. Given a local progress calculus G and a rule R not in $\mathcal{R}$ we will define the local-progress calculus $G + R$ by adding the rule R to the calculus and extending the local progress function such that no premise of an instance of R is a progressing premise.*

The Method of Translations. In [12] we developed a method to prove translations between local progress calculi, i.e., to provide functions transforming proofs of one calculus into proofs (not necessarily of the same sequent) in another calculus. Here, we will define informally the concepts and methods, the interested reader should consult [12] for more details.

The idea goes as follows. Given a proof π in a local progress calculus G we can define an partition of its nodes, the elements of the partition will be called *local fragments*. Two nodes will belong to the same local fragment if the smallest path between them does not goes through progress. Here, with passing through progress we mean going form the premise to the conclusion of a rule instance, or from conclusion to premise, such that the premise is progressing in the rule instance. Thanks to the condition that any infinite branch progresses infinitely often, it is easy to see that each local fragment will be a finite tree, in other words, this slices the non-wellfounded tree into (possibly infinitely many) finite trees. Figure 2 is an example on how the slicing can look in this setting, where each triangle represents a local fragment.

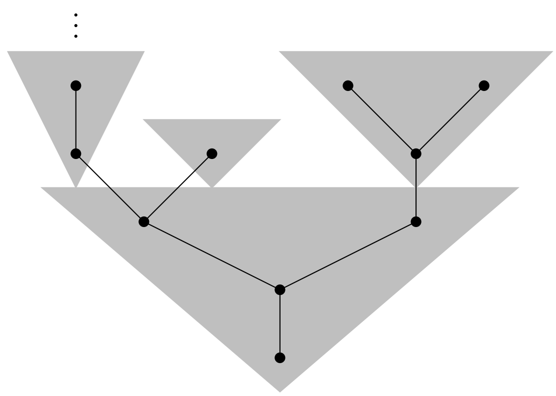

Fig. 2. Structure of proofs in local progress calculi

The bottom-most local fragment, i.e., the one to which the root belongs to, is called the *main local fragment*. We define the *local height* of a proof π, denoted as $\mathrm{lhg}(\pi)$, as the height of its main local fragment (which is a finite tree, so indeed it has a height).

Finally, the translation method goes as follows. To define a function from local-progress Gentzen calculus G to local-progress Gentzen calculus G', it suffices to provide a function (called *corecursive step*) that, given a proof π in G, returns:

1. a local fragment in G', i.e., a finite tree generated by the rules of G' where everly leaf is either axiomatic or a progressing premise and every progressing premise is a leaf;

2. for each non-axiomatic leaf (of the local fragment) with sequent S, a proof of S in G.

Then, the desired translation function is obtained by extending this corecursive step via corecursion. The procedure is displayed in Figure 3.

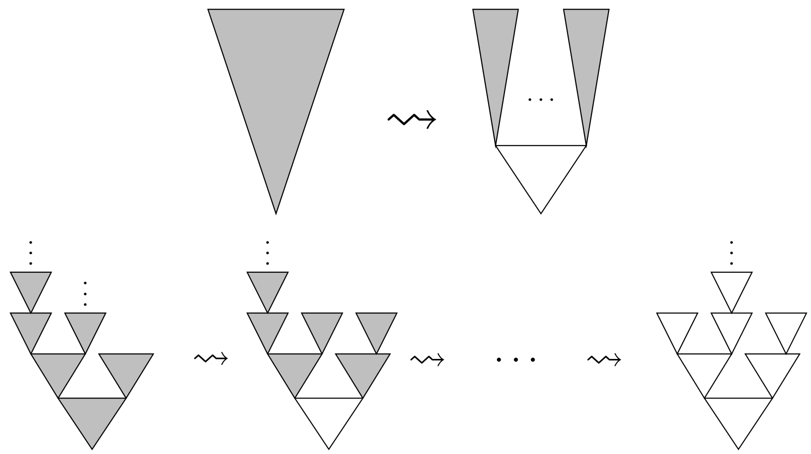

Fig. 3. Corecursive step function (top) and its extension from proofs to proofs (bottom). Tall gray (white) triangles represent proofs in G (G') and short gray (white) triangles represent local fragments in G (G').

Properties of Rules. Finally we introduce some properties of rules and proofs that will be fundamental to show cut elimination.

Definition 6. *Let R be an n-ary rule, G be a local progress Gentzen calculus and π a proof in $G + R$. We say that*

1. *R is* admissible *in G iff for any instance $\langle S_0, \dots, S_{n-1}, S\rangle$ of the rule R, $G \vdash S_0, \dots, G \vdash S_{n-1}$ implies that $G \vdash S$.*
2. *R is* invertible *iff for each $i < n$, the rule*

$$R_i^{-1} = \{(S_n, S_i) \mid \textit{Exists } S_0, \dots, S_{i-1}, S_{i+1}, \dots S_{n-1}.\ (S_0, \dots, S_n) \in R\}$$

is admissible. In words, if each of the rules which says that from the conclusion you can infer the premises is admissible.

3. *R is* eliminable *in G iff for any sequent S if $G + R \vdash S$ then $G \vdash S$.*
4. *π is* locally R-free *iff it contains no instances of R in its main local fragment.*
5. *R is* locally admissible *in G iff for any instance $\langle S_0, \dots, S_{n-1}, S\rangle$ of the rule if $G \vdash S_0, \dots, G \vdash S_{n-1}$ with locally R-free proofs, then there is a locally R-free proof of $G \vdash S$.*
6. *R is* locally eliminable *iff for any S, if $G \vdash S$ then there is a locally R-free proof in G of S.*

All the previous properties can be understood as asserting the existence of a proof π from the assumption that some proofs $\tau_0, \ldots, \tau_{n-1}$ exist. Let P be a property of proofs, we say that any of the properties above holds preserving *P if, adding the extra assumption that $\tau_0, \ldots, \tau_{n-1}$ fulfill P, then π also fulfills P.*

Note that the usual proof of admissibility implying eliminability does not hold for non-wellfounded proofs, as it requires an induction on the height of a non-wellfounded proof (which does not exist). For that reason we introduce the new notions of local height and local eliminability. The main application of the method of translations is the following lemma, which relates these new notions to eliminability.

Lemma 2. *For any local progress sequent calculi, the following holds*

R eliminable iff R locally eliminable iff R locally admissible.

Proof. That R is eliminable trivially implies that R is locally admissible. To show that R locally admissible implies R locally eliminable it suffices to do an induction in the local height. Finally, to show that R locally eliminable implies that R is eliminable it suffices to apply the method of translations using local eliminability to define a corecursive step.

2 Sequent Calculi for IL

In this section we introduce a two sequent systems for GIL. Let us introduce an useful convention for describing the rules of these systems. In case $X \subseteq \mathbb{N}$ we will define the sets

$$\varPhi_X := \{\phi_i \mid i \in X\} \qquad \text{and} \qquad \varPsi_X := \{\psi_i \mid i \in X\}.$$

In particular X will always been an interval like (i, j), $[i, j]$ or $[i, j)$.

$$\frac{}{p, \varGamma \Rightarrow p, \varDelta}\ \text{ax} \qquad \frac{}{\bot, \varGamma \Rightarrow \varDelta}\ \bot\text{L}$$

$$\frac{\varGamma \Rightarrow \varDelta, \phi \quad \psi, \varGamma \Rightarrow \varDelta}{\phi \to \psi, \varGamma \Rightarrow \varDelta} \to \text{L} \qquad \frac{\phi, \varGamma \Rightarrow \varDelta, \psi}{\varGamma \Rightarrow \varDelta, \phi \to \psi} \to \text{R}$$

$$\frac{[\psi_i, (\psi_i, \varPhi_{[0,i)}, \phi) \rhd \bot \Rightarrow \varPhi_{[0,i)}, \phi]_{m\ldots i\ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \varGamma \Rightarrow \psi_m \rhd \phi, \varDelta}\ \rhd_{\text{IL}}$$

$$\frac{[\psi_i, (\varPhi_{[0,i)}, \phi) \rhd \bot \Rightarrow \varPhi_{[0,i)}, \phi]_{m\ldots i\ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \varGamma \Rightarrow \psi_m \rhd \phi, \varDelta}\ \rhd_{\text{IK4}}$$

$$\frac{\varGamma \Rightarrow \varDelta, \chi \quad \chi, \varGamma \Rightarrow \varDelta}{\varGamma \Rightarrow \varDelta}\ \text{Cut}$$

Fig. 4. Sequent rules

We introduce two different sequent systems for IL.

Definition 7. *We define the sequent calculus* GIL *as the wellfounded calculus given by the rules of Figure 4 without rules* $\rhd_{\mathrm{IK4}}$ *and* Cut*.*

We define the sequent calculus $\mathrm{G}^{\infty}\mathrm{IL}$ *as the local progress sequent calculus given by the rules of Figure 4 without rules* $\rhd_{\mathrm{IL}}$ *and* Cut*. Progress only occurs at the premises of* $\rhd_{\mathrm{IK4}}$*.*

In the rules ax, $\bot L$, $\to L$ and $\to R$ of Figure 4 the explicitly displayed formula in the conclusion is called the *principal formula*. In $\rhd_{\mathrm{IL}}$ and $\rhd_{\mathrm{IK4}}$ the formula $\psi_m \rhd \phi$ is called *principal*, and multisets of formulas Γ and Δ are called the *weakening part* of these rules. The explicitly displayed formula in the Cut rule is called the cut formula. We named the rule $\rhd_{\mathsf{IK4}}$ of Figure 4 since it was inspired by $\rhd_{\mathsf{IK4}}$ for system $\mathsf{IK4}$ in [7] and could be use to define a Gentzen calculus for $\mathsf{IK4}$.

The calculus GIL is inspired from the calculus for IK4 in [7]. It provides a simplification of the calculus defined there, as we are capable of give a much more concrete shape to the modal rule. However, we notice a peculiar property of our system: the premises depend on an ordering of the $\rhd$-formulas of the conclusion. This implies that the same conclusion could have been obtained in multiple ways, depending on the ordering chosen. The necessity of an order comes from the axiom (J2) of IL.

The following lemma will be used in many proofs in the rest of this paper, as usual it is proven by induction on the size of ϕ. When we use this lemma in a proof we will simply write Ax just as we write ax for the rule in Figure 4.

Lemma 3. *Let* ϕ *be a formula. Then in* GIL *and in* $\mathrm{G}^{\infty}\mathrm{IL}$ *we have that*

$$\vdash \phi, \Gamma \Rightarrow \phi, \Delta.$$

We state the eliminability of some rules that will be useful, they are proved by showing admissibility or local admissibility (depending on the system) and admissibility or local admissibility is shown by induction on the height or local height, respectively.

Lemma 4. *The rules*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathrm{Wk} \qquad \frac{\Gamma \Rightarrow \Delta, \bot}{\Gamma \Rightarrow \Delta}\ \bot\mathrm{R}$$

are eliminable in GIL(+Cut) *and in* GIL(+Cut)*. In addition*

1. *In* GIL(+Cut) *they are eliminable preserving height.*
2. *In* $\mathrm{G}^{\infty}\mathrm{IL}$(+Cut) *they are eliminable preserving local height and local* Cut-*freeness.*

Lemma 5. *The rules* $\to L$ *and* $\to R$ *are invertible in* GIL(+Cut)*, preserving height; and in* $\mathrm{G}^{\infty}\mathrm{IL}$(+Cut)*, preserving local height and local* Cut-*freeness.*

Notice that in $\mathrm{G}^{\infty}\mathrm{IL}$(+Cut) we added a preservation of local height and local cut-freeness. It is not hard to see, that if we show local admissibility preserves these properties while extending local admissibility to eliminability this preservation remains. For preservation of height in GIL(+Cut) a similar argument applies.

Lemma 6. *The rule*

$$\frac{\phi, \Sigma \rhd \bot \Rightarrow \Sigma}{\Sigma \rhd \bot, \Gamma \Rightarrow \phi \rhd \bot, \Delta}\ \text{Nec}$$

is admissible in GIL(+Cut) *and in* G^∞IL(+Cut)

Proof. We show it for GIL, the other proof being similar. Assume $\pi \vdash \phi, \Sigma \rhd \bot \Rightarrow \Sigma$ in GIL(+Cut) and let us enumerate $\Sigma = \{\phi_0, \ldots, \phi_{m-1}\}$. Then, the desired proof for GIL is

$$\dfrac{\dfrac{\overset{\pi}{\phi, \Phi_{[0,m)} \rhd \bot \Rightarrow \Phi_{[0,m)}}}{\phi, (\phi, \Phi_{[0,m)}, \bot) \rhd \bot \Rightarrow \Phi_{[m,0)}, \bot}\ \text{Wk} \quad \dfrac{}{\bot, \ldots \Rightarrow \ldots}\ \bot\text{L} \quad \ldots}{\Sigma \rhd \bot, \Gamma \Rightarrow \phi \rhd \bot, \Delta}\ \rhd_{\text{IL}}$$

where in the right-most dots we are omitting some proofs by $\bot$L.

Finally, we note some nice properties of the cut-free systems.

Proposition 1. *Any preproof of* G^∞IL *is a proof of* G^∞IL.

Proof. The rules $\to$ L and $\to$ R reduce the size of the sequent (which is just the sum of the sizes of each formula ocurrence in it). So any infinite branch in a preproof must have infinitely many instances of $\rhd_{\text{IK4}}$.

Due to the shape of the rules we need to slightly change the usual definition of subformula set.

Definition 8. *Let ϕ be a formula. We define the set* $\mathrm{Sub}_\rhd(\phi)$ *as follows: A*

$$\begin{aligned}
&\mathrm{Sub}_\rhd(p) = \{p\}, \qquad \mathrm{Sub}_\rhd(\bot) = \{\bot\},\\
&\mathrm{Sub}_\rhd(\phi \to \psi) = \{\phi \to \psi\} \cup \mathrm{Sub}_\rhd(\phi) \cup \mathrm{Sub}_\rhd(\psi),\\
&\mathrm{Sub}_\rhd(\phi \rhd \psi) = \{\phi \rhd \psi, \phi \rhd \bot, \psi \rhd \bot, \bot\} \cup \mathrm{Sub}_\rhd(\phi) \cup \mathrm{Sub}_\rhd(\psi).
\end{aligned}$$

If Γ is a multiset, $\mathrm{Sub}_\rhd(\Gamma) = \bigcup\{\mathrm{Sub}_\rhd(\phi) \mid \phi \in \Gamma\}$*; and if $S = (\Gamma \Rightarrow \Delta)$ is a sequent, then* $\mathrm{Sub}_\rhd(S) = \mathrm{Sub}_\rhd(\Gamma \cup \Delta)$.

Proposition 2 (Subformula property). *Let $\pi \vdash S$ in* G^∞IL *or in* GIL *and ϕ be a formula occurring in π. Then $\phi \in \mathrm{Sub}_\rhd(S)$.*

3 Equivalence of IL and GIL + Cut

We show the equivalence of Hilbert style proofs in IL and sequent proofs in the system GIL+Cut. First we remember the interpretation of sequents as formulas.

Definition 9. *Given a sequent $S = \Gamma \Rightarrow \Delta$, we define $S^\sharp = \bigwedge \Gamma \to \bigvee \Delta$.*

Lemma 7. *Let* $\mathrm{IL} \vdash \phi$*, then* $\mathrm{GIL} + \mathrm{Cut} \vdash \Rightarrow \phi$.

Proof. By induction on the length of the Hilbert-style proof of ϕ. The case of classical propositional tautology is trivial, the proofs in GIL of the modal axioms are easy to construct. For modus ponens case it suffices to use Lemma 5 and Cut. For necessitation case it suffices to use Lemma 6.

The converse of the previous lemma is a simple consequence of the following.

Theorem 1. *For any sequent S, $\mathrm{IL} \vdash S^\sharp$ if and only if $\mathrm{GIL} + \mathrm{Cut} \vdash S$.*

Proof. Let $S = (\Gamma \Rightarrow \Delta)$. Using Lemma 7, we have that $\mathrm{IL} \vdash S^\sharp$ implies $\mathrm{GIL} + \mathrm{Cut} \vdash \Rightarrow \bigwedge \Gamma \to \bigvee \Delta$. Then, using invertibility of $\to \mathrm{L}$, $\to \mathrm{R}$ and admissibility of $\bot\mathrm{R}$, we obtain $\mathrm{GIL} + \mathrm{Cut} \vdash \Gamma \Rightarrow \Delta$.

For the other direction, let $\pi \vdash S$ in $\mathrm{GIL} + \mathrm{Cut}$. We proceed by induction on the height of π and cases in the last rule of π. The cases where the last rule of π is either $\mathrm{Ax}, \bot\mathrm{L}, \to \mathrm{L}, \to \mathrm{R}, \mathrm{Cut}$ follow from simple propositional tautologies. So we focus on the $\rhd_{\mathrm{IL}}$ case. Then π is of shape

$$\dfrac{\left[\begin{matrix}\pi_i \\ \psi_i \rhd \bot, \psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{matrix}\right]_{m \ldots i \ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \Delta, \psi_m \rhd \phi} \rhd_{\mathrm{IL}}.$$

By the induction hypothesis we get

$$\mathrm{IL} \vdash (\psi_i \rhd \bot) \wedge \psi_i \wedge \bigwedge (\Phi_{[0,i)} \rhd \bot) \wedge (\phi \rhd \bot) \to \bigvee \Phi_{[0,i)} \vee \phi, \qquad \text{for } i \leq m,$$

so by Lemma 1 we have $\mathrm{IL} \vdash ((\psi_i \rhd \bot) \wedge \psi_i) \rhd \bigvee \Phi_{[0,i)} \vee \phi$, for $i \leq m$. Using Lemma 1 again with (J2) we have $\mathrm{IL} \vdash \psi_i \rhd \left(\bigvee \Phi_{[0,i)} \vee \phi\right)$, for $i \leq m$. By induction in i we show that $\mathrm{IL} \vdash (\bigwedge_{i<m} \phi_i \rhd \psi_i) \to \psi_i \rhd \phi$, so assume $\mathrm{IL} \vdash (\bigwedge_{i<m} \phi_i \rhd \psi_i) \to \psi_j \rhd \phi$, for $j < i$. Using (J3) we get $\mathrm{IL} \vdash (\bigwedge_{i<m} \phi_i \rhd \psi_i) \to (\bigvee_{i<m} \psi_i) \rhd \phi$ and then $\mathrm{IL} \vdash (\bigwedge_{i<m} \phi_i \rhd \psi_i) \to (\bigvee \Phi_{[0,i)}) \rhd \phi$. Also $\mathrm{IL} \vdash \phi \rhd \phi$, so we get $\mathrm{IL} \vdash (\bigwedge_{i<m} \phi_i \rhd \psi_i) \to (\bigvee \Phi_{[0,i)} \vee \phi) \rhd \phi$. But $\mathrm{IL} \vdash \psi_i \rhd \bigvee \Phi_{[0,i)} \vee \phi$ so by the use of (J2) we conclude the desired $\mathrm{IL} \vdash (\bigwedge_{i<m} \phi_i \rhd \psi_i) \to \psi_i \rhd \phi$.

4 Equivalence of GIL(+Cut) and G^∞IL(+Cut)

In this section we will provide a cut elimination method for GIL by translating to G^∞IL and back. Thanks to G^∞IL being a local progress calculus, we can show cut eliminability by just proving local admissibility. On the other hand, the translation to and back from the local progress calculus will be defined by recursion.

4.1 From GIL + Cut to G^∞IL + Cut

In this subsection we will prove that anything provable in GIL + Cut is also provable in G^∞IL + Cut.

Lemma 8. *We have that $\mathrm{G}^\infty\mathrm{IL} \vdash \phi \rhd \psi, \psi \rhd \chi \Rightarrow \phi \rhd \chi$ and $\mathrm{G}^\infty\mathrm{IL} \vdash \Rightarrow \phi \rhd \blacksquare \phi$.*

Theorem 2. *Let S be a sequent. If* $\mathrm{GIL} + \mathrm{Cut} \vdash S$*, then* $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut} \vdash S$*.*

Proof. Let $\pi \vdash S$ in GIL + Cut. We proceed by induction on the height of the proof π and case analysis in the last rule applied. The only interesting case is when the last rule is $\rhd_{\mathrm{IL}}$, so π is of shape

$$\dfrac{\left[\begin{array}{c}\pi_i \\ \psi_i, (\psi_i, \Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{array}\right]_{m\ldots i\ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \Delta, \psi_m \rhd \phi} \rhd_{\mathrm{IL}}$$

So by induction hypothesis we get a proof π_i' in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ proving the same sequent as π_i, for $i \leq m$. First, define a proof τ_i in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ for $i \leq m$ as

$$\dfrac{\begin{array}{c}\pi_i' \\ \psi_i, (\psi_i, \Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{array}}{\blacksquare\psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi} \wedge\mathrm{L}.$$

Then define the proof ρ in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ as

$$\dfrac{\left[\begin{array}{c}\tau_i \\ \blacksquare\psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{array}\right]_{m\ldots i\ldots 0}}{\{\phi_i \rhd \blacksquare\psi_i\}_{i<m}, \Gamma \Rightarrow \Delta, \blacksquare\psi_m \rhd \phi} \rhd_{\mathrm{IK4}}.$$

We have the following proofs ρ_i in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ for $i < m$

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow \psi_i \rhd \blacksquare\psi_i}\ \mathrm{Lm}\ 8}{\phi_i \rhd \psi_i \Rightarrow \phi_i \rhd \blacksquare\psi_i, \psi_i \rhd \blacksquare\psi_i}\ \mathrm{Wk} \qquad \dfrac{}{\phi_i \rhd \psi_i, \psi_i \rhd \blacksquare\psi_i \Rightarrow \phi_i \rhd \blacksquare\psi_i}\ \mathrm{Lm}\ 8}{\phi_i \rhd \psi_i \Rightarrow \phi_i \rhd \blacksquare\psi_i}\ \mathrm{Cut}$$

and the following proof ρ_m in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow \psi_m \rhd \blacksquare\psi_m}\ \mathrm{Lm}\ 8}{\blacksquare\psi_m \rhd \phi \Rightarrow \psi_m \rhd \phi, \psi_m \rhd \blacksquare\psi_m}\ \mathrm{Wk} \qquad \dfrac{}{\psi_m \rhd \blacksquare\psi_m, \blacksquare\psi_m \rhd \phi \Rightarrow \psi_m \rhd \phi}\ \mathrm{Lm}\ 8}{\blacksquare\psi_m \rhd \phi \Rightarrow \psi_m \rhd \phi.}\ \mathrm{Cut.}$$

The desired proof is obtain by applying cuts with the proofs ρ and ρ_i for $i \leq m$.

4.2 From $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ to $\mathrm{G}^\infty\mathrm{IL}$

In this subsection we will prove the cut elimination theorem for $\mathrm{G}^\infty\mathrm{IL}$.

Lemma 9. *We have that* Ctr *is eliminable in* $\mathrm{G}^\infty\mathrm{IL}(+\mathrm{Cut})$*. Additionally, eliminating* Ctr *preserves local* Cut*-freeness.*

Proof. We are going to show that Ctr is locally admissible (see Lemma 2). The construction of local admissibility does not introduce any new cuts, obtaining the preservativity condition.

We proceed by induction in the local height of the proof and cases in the last rule applied. The only interesting case is where π is of shape

$$\dfrac{\begin{bmatrix} \pi_i \\ \psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi \end{bmatrix}_{m\ldots i\ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \Delta, \psi_m \rhd \phi} \rhd_{\mathrm{IK4}}$$

and the formulas we desired to contract do occur on the left hand side and not occur in Γ. So there are $j < k < m$ such that $\phi_i \rhd \psi_i = \phi_k \rhd \psi_k$ and we want to show that the sequent $\{\phi_i \rhd \psi_i\}_{i<m, i\neq k}, \Gamma \Rightarrow \Delta, \psi_m \rhd \phi$ is provable. For each $i > k$ define the proof ρ_i in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Ctr}$ as

$$\dfrac{\dfrac{\begin{array}{c} \pi_i \\ \psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi \\ \hdashline \psi_i, (\Phi_{[k+1,i)}, \phi_k, \Phi_{[0,k)}, \phi) \rhd \bot \Rightarrow \Phi_{[k+1,i)}, \phi_k, \Phi_{[0,k)}, \phi \end{array}}{\psi_i, (\Phi_{[k+1,i)}, \Phi_{[0,k)}, \phi) \rhd \bot \Rightarrow \Phi_{[k+1,i)}, \phi_k, \Phi_{[0,k)}, \phi} \mathrm{Ctr}}{\psi_i, (\Phi_{[k+1,i)}, \Phi_{[0,k)}, \phi) \rhd \bot \Rightarrow \Phi_{[k+1,i)}, \Phi_{[0,k)}, \phi} \mathrm{Ctr}$$

where in order to apply Ctr we used that $\phi_k = \phi_j \in \Phi_{[0,k)}$. Then, the desired proof (which is trivially locally Ctr-free), is

$$\dfrac{\rho_m \quad \cdots \quad \rho_{k+1} \quad \pi_{k-1} \quad \cdots \quad \pi_0}{\{\phi_i \rhd \psi_i\}_{i<m, i\neq k}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta} \rhd_{\mathrm{IK4}}$$

where $\rhd_{\mathrm{IK4}}$ has been applied with ordering $\phi_0 \rhd \psi_0, \ldots, \phi_{k-1} \rhd \psi_{k-1}, \phi_{k+1} \rhd \psi_{k+1}, \ldots, \phi_{m-1} \rhd \psi_{m-1}$ and principal formula $\psi_m \rhd \phi$.

Theorem 3 (Local Cut-admissibility). *Assume we have proofs $\pi \vdash \Gamma \Rightarrow \Delta, \chi$ and $\tau \vdash \chi, \Gamma \Rightarrow \Delta$ in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ which are locally Cut-free. Then there is $\rho \vdash \Gamma \Rightarrow \Delta$ in $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut}$ which is locally Cut-free.*

Proof. By induction on the lexicographic order of the pairs $\langle |\chi|, \mathrm{lhg}(\pi) + \mathrm{lhg}(\tau) \rangle$, i.e., the size of the formula and the sum of the local height of the premises.

The only interesting case is when both proofs end in an application of $\rhd_{\mathsf{IK4}}$, the cut formula is principal in π and occurs in the ordering used in τ. Then π and τ are of the following shape:

$$\dfrac{\begin{bmatrix} \pi_i \\ \psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi \end{bmatrix}_{m\ldots i\ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma_0 \Rightarrow \Delta_0, \psi_m \rhd \phi} \rhd_{\mathrm{IK4}}$$

$$\dfrac{\begin{bmatrix} \tau_i \\ \psi'_i, (\Phi'_{[0,i)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,i)}, \phi' \end{bmatrix}_{n\ldots i\ldots 0}}{\{\phi'_j \rhd \psi'_j\}_{j<n}, \Gamma_1 \Rightarrow \Delta_1, \psi'_n \rhd \phi'} \rhd_{\mathrm{IK4}},$$

where $\psi_m \rhd \phi = \phi'_k \rhd \psi'_k = \chi_0 \rhd \chi_1$ for some $k < n$ and

$$\big(\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma_0 \Rightarrow \Delta_0\big) = \big(\{\phi'_j \rhd \psi'_j\}_{j<n, j\neq k}, \Gamma_1 \Rightarrow \Delta_1, \psi'_n \rhd \phi'\big). \qquad \text{(i)}$$

Subcase 1: $\chi_0 = \bot$. We are going to define proofs $(\rho_j)_{k<j\leq n}$ such that

$$\rho_j \vdash \psi'_j, (\Phi'_{(k,j)}, \Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{(k,j)}, \Phi'_{[0,k)}, \phi'.$$

Then, the desired proof will be

$$\dfrac{\rho_n \quad \cdots \quad \rho_{k+1} \quad \tau_{k-1} \quad \cdots \quad \tau_0}{\{\phi'_j \rhd \psi'_j\}_{j<n, j\neq k}, \Gamma_1 \Rightarrow \Delta_1, \psi'_n \rhd \phi'} \rhd_{\mathrm{IK4}}$$

We notice that $\chi_0 = \bot$ implies that $\phi'_k = \bot$, so we define τ'_j for $j > k$ by applying $\bot$R from Lemma 4 to τ_j (thus delemting $\phi'_k = \bot$ from the right hand side of the sequent). We define ρ_j as

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow \bot}\ \bot\mathrm{L}}{\ldots \Rightarrow \ldots, \bot \rhd \bot}\ \rhd_{\mathrm{IK4}} \qquad \dfrac{\dfrac{\tau'_j}{\psi'_j, (\Phi'_{[0,j)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,k)}, \Phi'_{(k,j)}, \phi'}}{\psi'_j, (\Phi'_{(k,j)}, \bot, \Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{(k,j)}, \Phi'_{[0,k)}, \phi'}}{\psi'_j, (\Phi'_{(k,j)}, \Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{(k,j)}, \Phi'_{[0,k)}, \phi'}\ \mathrm{Cut}$$

Subcase 2: $\chi_0 \neq \bot$. Let us write $\Sigma = \{\phi_i \rhd \psi_i\}_{i<m}$ and $\Sigma' = \{\phi'_j \rhd \psi'_j\}_{j<n, j\neq k}$. Define $\Gamma_2 := \Gamma_0 \setminus (\Sigma' \setminus \Sigma) = \Gamma_1 \setminus (\Sigma \setminus \Sigma')$. We have, thanks to equality (i), that

$$\Gamma_2, \Sigma \cap \Sigma', \Sigma \setminus \Sigma', \Sigma' \setminus \Sigma = \{\phi_i \rhd \psi_i\}_{i<m}, \Gamma_0 = \{\phi'_j \rhd \psi'_j\}_{j<n, j\neq k}, \Gamma_1.$$

We also notice that contracting $\Gamma_2, \Sigma, \Sigma'$ we can obtain the desired sequent. Let us define proofs $(\rho_i)_{i<m}, (\rho'_j)_{j\leq n, j\neq k}$ such that

$$\rho'_j \vdash \psi'_j, (\Phi'_{(k,j)}, \Phi_{[0,m)}, \Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{(k,j)}, \Phi_{[0,m)}, \Phi'_{[0,k)}, \phi', \text{ for } k < j \leq n,$$
$$\rho_i \vdash \psi_i, (\Phi_{[0,i)}, \Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi_{[0,i)}, \Phi'_{[0,k)}, \phi', \text{ for } i < m,$$
$$\rho'_j \vdash \psi'_j, (\Phi'_{[0,j)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,j)}, \phi', \text{ for } j < k.$$

Then the desired proof (locally Cut-free) will be

$$\dfrac{\rho'_n \quad \cdots \quad \rho'_{k+1} \quad \rho_{m-1} \quad \cdots \quad \rho_0 \quad \rho'_{k-1} \quad \cdots \quad \rho'_0}{\{\phi_i \rhd \psi_i\}_{i<m}, \{\phi'_j \rhd \psi'_j\}_{j<n, j\neq k}, \Gamma_2 \Rightarrow \Delta', \psi'_n \rhd \phi'} \rhd_{\mathrm{IK4}}$$

where $\rhd_{\mathrm{IK4}}$ is applied with ordering

$$\phi'_0 \rhd \psi'_0, \ldots, \phi'_{k-1} \rhd \psi'_{k-1}, \phi_0 \rhd \psi_0, \ldots, \phi_{m-1} \rhd \psi_{m-1}, \phi'_{k+1} \rhd \psi'_{k+1}, \ldots, \phi'_{n-1} \rhd \psi'_{n-1}.$$

and main formula $\psi'_n \rhd \phi'$. Then the desired proof will by obtained by applying contraction, i.e., Lemma 9 to ρ as contraction preserves local Cut-freeness.

Notice that when defining $(\rho_i)_{i<m}, (\rho'_j)_{j\leq n, j\neq k}$ we can freely use Cut, since all these instance of cut will not appear in the local fragment of the desired proof.

We define ρ'_j for $j < k$ as τ_j, so we only need to define ρ'_j for $k < j \leq n$ and ρ_i for $i < m$. To define ρ'_j for $k < j \leq n$ we notice we have the following proofs:

$$\tau_j \vdash \psi'_j, (\Phi'_{(k,j)}, \chi_0, \Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{(k,j)}, \chi_0, \Phi'_{[0,k)}, \phi',$$

$$\pi_m \vdash \chi_0, (\Phi_{[0,m)}, \chi_1) \rhd \bot \Rightarrow \Phi_{[0,m)}, \chi_1, \qquad \tau_k \vdash \chi_1, (\Phi_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,k)}, \phi'.$$

Applying Lemma 6 to π_m and to τ_k we obtain proofs π'_m and τ'_k such that $\pi'_m \vdash (\Phi_{[0,m)}, \chi_1) \rhd \bot \Rightarrow \chi_0 \rhd \bot$, $\tau'_k \vdash (\Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \chi_1 \rhd \bot$. Then the desired proof ρ'_j is

$$\dfrac{\chi_1 \rhd \bot \quad \dfrac{\mathrm{wk}(\tau'_k) \qquad \chi_0 \ \dfrac{\chi_0 \rhd \bot \ \dfrac{\mathrm{wk}(\pi'_m) \quad \mathrm{wk}(\tau_j)}{\psi'_j, (\Phi, \chi_1) \rhd \bot \Rightarrow \Phi, \chi_0, \chi_1}\,\mathrm{Cut} \qquad \mathrm{wk}(\pi_m)}{\psi'_j, (\Phi, \chi_1) \rhd \bot \Rightarrow \Phi, \chi_1}\,\mathrm{Cut}}{\psi'_j, \Phi \rhd \bot \Rightarrow \Phi, \chi_1}\,\mathrm{Cut} \qquad \chi_1 \qquad \mathrm{wk}(\tau_k)}{\psi'_j, \Phi \rhd \bot \Rightarrow \Phi}\,\mathrm{Cut}$$

where we denoted $\Phi'_{(k,j)}, \Phi_{[0,m)}, \Phi'_{[0,k)}, \phi'$ as Φ and annotated the cut formula at the left of the rule application.

All that is left is to define proofs ρ_i for $i < m$. We remember that we have the following proofs:

$$\pi_i \vdash \psi_i, (\Phi_{[0,i)}, \chi_1) \rhd \bot \Rightarrow \Phi_{[0,i)}, \chi_1 \qquad \tau_k \vdash \chi_1, (\Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,k)}, \phi'.$$

Applying Lemma 6 we obtain $\tau'_k \vdash (\Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \chi_1 \rhd \bot$. Then the desired proof ρ_i is defined as

$$\dfrac{\chi_1 \ \dfrac{\chi_1 \rhd \bot \ \dfrac{\mathrm{wk}(\tau'_k) \quad \mathrm{wk}(\pi_i)}{\psi_i, \Phi \rhd \bot \Rightarrow \Phi, \chi_1}\,\mathrm{Cut} \qquad \mathrm{wk}(\tau_k)}{\psi_i, \Phi \rhd \bot \Rightarrow \Phi}\,\mathrm{Cut}}{}$$

where we denoted $\phi_{[0,i)}, \phi'_{[0,k)}, \phi'$ as Φ and annoted the cut formula at the left of the rule application.

Corollary 1. *If* $\mathrm{G}^\infty\mathrm{IL} + \mathrm{Cut} \vdash S$*, then* $\mathrm{G}^\infty\mathrm{IL} \vdash S$*.*

Proof. By Lemma 2 and Theorem 3.

4.3 From G^∞IL to GIL

Theorem 4. *For any Λ finite set of formulas, we have that* $\mathrm{G}^\infty\mathrm{IL} \vdash \Gamma \Rightarrow \Delta$ *implies* $\mathrm{GIL} \vdash \Lambda \rhd \bot, \Gamma \Rightarrow \Delta$*.*

Proof. Let $\pi \vdash \Gamma \Rightarrow \Delta$ in G^∞IL. By induction on the lexicographical order $\big\langle |\,\mathrm{Sub}_\rhd(\Gamma \cup \Delta) \setminus \Lambda|, \mathrm{lhg}(\pi)\big\rangle$ and the case analysis in the last rule of π. The only interesting case is when the last rule of π be $\rhd_{\mathrm{IK4}}$. So π is of shape

$$\dfrac{\left[\begin{array}{c}\pi_i \\ \psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{array}\right]_{m\ldots i\ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta}\,\rhd_{\mathrm{IK4}}$$

and let us denote the conclusion of π_i as S_i and the conclusion of π as S. We want to show that $\mathrm{GIL} \vdash \Lambda \rhd \bot, \{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta$. For $i \le m$ we define proofs $\tau_i \vdash \psi_i, (\psi_i, \Phi_{[0,i)}, \Lambda, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \Lambda, \phi$ by cases.

Case 1. If $\psi_i \in \Lambda$ then we define τ_i as

$$\dfrac{}{\psi_i, (\psi_i, \Phi_{[0,i)}, \Lambda, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \Lambda, \phi}\ \mathrm{Ax}$$

since the formula ψ_i appears on both sides of this sequent.

Case 2. If $\psi_i \notin \Lambda$ then, since $\psi_i \in \mathrm{Sub}_{\rhd}(S_i)$ and $\mathrm{Sub}_{\rhd}(S_i) \subseteq \mathrm{Sub}_{\rhd}(S)$, we have $|\,\mathrm{Sub}_{\rhd}(S_i) \setminus (\Lambda \cup \{\psi_i\})| < |\,\mathrm{Sub}_{\rhd}(S_i) \setminus \Lambda| \leq |\,\mathrm{Sub}_{\rhd}(S) \setminus \Lambda|$. So by induction hypothesis applied to π_i with set $\Lambda \cup \{\psi_i\}$ we obtain a proof π_i' in GIL such that $\pi_i' \vdash \psi_i, (\psi_i, \Phi_{[0,i)}, \Lambda, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi$. We define τ_i applying Wk to π_i' so $\tau_i \vdash \psi_i, (\psi_i, \Phi_{[0,i)}, \Lambda, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \Lambda, \phi$ in GIL. Let $\Lambda = \{\chi_0, \ldots, \chi_{n-1}\}$. We define ρ_j for $j < n$ as the following proof in GIL

$$\dfrac{}{\bot, (\bot, \Lambda_{[0,j)}, \phi) \rhd \bot \Rightarrow \Lambda_{[0,j)}, \phi}\ \bot\mathrm{L}.$$

Then, the desired proof, is

$$\dfrac{\tau_m \quad \cdots \quad \tau_0 \quad \rho_{n-1} \quad \cdots \quad \rho_0}{\Lambda \rhd \bot, \{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta}\ \rhd_{\mathrm{IL}}$$

where the last rule was applied with the ordering $\chi_0 \rhd \bot, \ldots, \chi_{n-1} \rhd \bot, \phi_0 \rhd \psi_0, \ldots, \phi_{m-1} \rhd \psi_{m-1}$ and principal formula $\psi_m \rhd \phi$.

By Theorem 2, Theorem 1 and Theorem 4 (where we take Λ to be an empty set) we obtain the cut elimination for the system GIL.

Corollary 2. *Let S be a sequent. If* $\mathrm{GIL} + \mathrm{Cut} \vdash S$*, then* $\mathrm{GIL} \vdash S$*.*

5 Conclusion

We defined two new sequent calculi for IL, a wellfounded and a local progress (non-wellfounded) one. Both have a nice subformula property appering first in [7] for IK4 (IL without Löb's axiom), but with a much more concrete modal rule which simplifies their proof-theoretic treatment.

Local progress proof theory, with our addition of local admissibility, allows us to show cut elimination for IL without any complications that usually appear in (wellfounded) GL, making it quite similar to cut elimination in simpler systems such as K4 (or more concretely IK4).

Finally, with the help of these systems (in particular, using cyclic proofs) we have been capable of proving uniform interpolation for IL (to appear somewhere else). As far as the authors know, this result was not known to this date.

We leave it as future work to study extensions of IL, in particular ILP should be easy to handle. However, the logics ILW and ILM should provide bigger challenges. Particularly, ILM is known to lack Craig interpolation ([3]). This hints at the inexistence of a nice sequent calculi for it.

Bibliography

[1] Matteo Acclavio, Gianluca Curzi, and Giulio Guerrieri. *Infinitary cut-elimination via finite approximations (extended version)*. 2024. arXiv: 2308.07789 [cs.LO]. URL: https://arxiv.org/abs/2308.07789.

[2] Bahareh Afshari and Johannes Kloibhofer. "Cut Elimination for Cyclic Proofs: A Case Study in Temporal Logic". In: *Proceedings Twelfth International Workshop on Fixed Points in Computer Science*. Electronic Proceedings in Theoretical Computer Science, to appear.

[3] Carlos Areces, Eva Hoogland, and Dick de Jongh. "Interpolation, Definability and Fixed Points in Interpretability Logics". In: *Advances in Modal Logic*. Ed. by Marcus Kracht et al. CSLI Publications, 1998, pp. 53–76.

[4] Anupam Das and Damien Pous. "Non-Wellfounded Proof Theory For (Kleene+Action) (Algebras+Lattices)". In: *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*. Ed. by Dan R. Ghica and Achim Jung. Vol. 119. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018, 19:1–19:18. ISBN: 978-3-95977-088-0. DOI: 10.4230/LIPIcs.CSL.2018.19. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2018.19.

[5] Sohei Iwata, Taishi Kurahashi, and Yuya Okawa. "The persistence principle over weak interpretability logic". In: *Mathematical Logic Quarterly* 70.1 (2024), pp. 37–63. DOI: https://doi.org/10.1002/malq.202200020.

[6] Katsumi Sasaki. "A Cut-Free Sequent System for the Smallest Interpretabili ty Logic". In: *Studia Logica* 70 (2002), pp. 353–372. DOI: https://doi.org/10.1023/A:1015150314504.

[7] Katsumi Sasaki. "A sequent system for a sublogic of the smallest interpretability logic". In: *Academia. Mathematical sciences and information engineering : journal of the Nanzan Academic Society* 3 (2003), pp. 1–17. DOI: https://doi.org/10.15119/00000018.

[8] Katsumi Sasaki. "A sequent system for the interpretability logic with the persistence axiom". In: *Academia. Mathematical sciences and information engineering : journal of the Nanzan Academic Society* 2 (2002), pp. 25–34. DOI: https://doi.org/10.15119/00000117.

[9] Alexis Saurin. "A Linear Perspective on Cut-Elimination for Non-wellfounded Sequent Calculi with Least and Greatest Fixed-Points". In: *Automated Reasoning with Analytic Tableaux and Related Methods*. Ed. by Revantha Ramanayake and Josef Urban. Cham: Springer Nature Switzerland, 2023, pp. 203–222. ISBN: 978-3-031-43513-3.

[10] Yury Savateev and Daniyar Shamkanov. "Non-Well-Founded Proofs for the Grzegorczyk Modal Logic". In: *The Review of Symbolic Logic* 14 (Apr. 2018). DOI: 10.1017/S1755020319000510.

[11] Daniyar Shamkanov. *On structural proof theory of the modal logic K+ extended with infinitary derivations*. 2023. arXiv: 2310.10309 [math.LO]. URL: https://arxiv.org/abs/2310.10309.

[12] Borja Sierra Miranda, Thomas Studer, and Lukas Zenger. "Coalgebraic Proof Translations of Non-Wellfounded Proofs". In: *Advances in Modal Logic*. Ed. by Agata Ciabattoni, David Gabelaia, and Igor Sedlar. Vol. 15. College Publications, 2024, pp. 527–548.

[13] Albert Visser. "Interpretability Logic". In: *Mathematical Logic*. Ed. by Petio Petrov Petkov. Boston, MA: Springer US, 1990, pp. 175–209. ISBN: 978-1-4613-0609-2. DOI: 10.1007/978-1-4613-0609-2_13. URL: https://doi.org/10.1007/978-1-4613-0609-2_13.